Opuscula Mathematica • Vol. 29 • No. 3 • 2009

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APPROXIMATION METHODS FOR A CLASS OF DISCRETE WIENER-HOPF EQUATIONS

Abstract. In this paper, we consider approximation methods for operator equations of the form

$$Au + Bu = f,$$

where A is a discrete Wiener-Hopf operator on l_p $(1 \leq p < \infty)$ which symbol has roots on the unit circle with arbitrary multiplicities (not necessary integers). Conditions on perturbation B and f are given in order to guarantee the applicability of projection-iterative methods. Effective error estimates, and simultaneously, decaying properties for solutions are obtained in terms of some smooth spaces.

Keywords: projection methods, iterative methods, discrete Wiener-Hopf equations, Toeplitz operators.

Mathematics Subject Classification: 65J10, 65Q05.

1. INTRODUCTION

In [9] projection-iterative methods were developed for operator equations (considered in Banach space $l_p(\mathbb{N})$ $(1 \leq p < \infty)$) of the

$$Au + Bu = f, \tag{1.1}$$

where A is a Toeplitz operator (or, in other terminology, a discrete Wiener-Hopf operator), B is considered as a perturbation of A, and f is a given element. The methods were adopted for the general situation of equations in which A need not be a Fredholm operator, that means that the symbol $A(z)$ of A vanishes on some points of the unit circle. However, in [9] there was exclusively considered the case in which the roots those belong to the unit circle can be only of integer multiplicities. The purpose of this work is to extend the results of [9] to the general case of arbitrary multiplicities (not necessary to be integer positive numbers) for the roots of $A(z)$ lying on the unit circle. This case is also interested by itself and in applications (in this respect, see

[15,16] and [20]), but, on the other hand additional difficulties occur in its analysis. We would like to mention the works [1,2,11,14] and [15] for the standard texts on analysis of Toeplitz operators and [11,16–18] and [20] in which (purely) projection methods for Wiener-Hopf equations are developed. We note also [19] for an overview on recent results on this topic.

The paper is organized as follows: in Section 2 (Preliminaries) we recall projection-iterative methods proposed and developed in [9] for abstract operator equations. These methods are applied in Section 3 to discrete Wiener-Hopf equations mentioned above. In Section 4 we illustrate the obtained results by considering some concrete examples for which error estimations with optimal constants are given.

2. PROJECTION-ITERATIVE METHODS. PRELIMINARIES

In this section we consider the projection-iterative methods developed in [9]. For a convenience, we present the approximation procedure and the main result about solution existence, convergence of methods and effective error estimation. We discuss the following class of perturbed operator equations

$$Au + Bu = f, \tag{2.1}$$

where A and B are linear bounded operators in a Banach space $\mathbb{E}$, and f is a given element in $\mathbb{E}$. We assume that the operator A is invertible on the left, i.e. $A^{(-1)}A = I$. It should be stressed, however, that in general the operator $A^{(-1)}$ may be unbounded but $\mathrm{Dom}(A^{(-1)}) \supset \mathrm{Ran}(A)$ ($\mathrm{Dom}(A^{(-1)})$ denotes the domain of $A^{(-1)}$ and $\mathrm{Ran}(A)$ is the range of A). Moreover, we require that $f \in \mathrm{Ran}(A)$ and $Bu \in \mathrm{Ran}(A)$.

Next we introduce a family of operators (in general unbounded) L_τ ($\tau \geq 0$) with the property that for each $\tau \geq 0$ the operator L_τ is one-to-one, i.e. $\mathrm{Ker}L_\tau = 0$, and $L_0 = I$ (cf. [4], Assumption 4). On the domain $D_\tau = \mathrm{Dom}(L_\tau)$ of the operator L_τ ($\tau \geq 0$) we introduce a new norm

$$|u|_\tau = \|L_\tau u\|, \qquad u \in D_\tau.$$

($\|\cdot\|$ stands for the norm of the space $\mathbb{E}$). The norm $|u|_\tau$ turns the linear manifold D_τ into a normed space, which we denote by $\mathbb{E}_\tau$. Clearly $\mathbb{E}_0 = \mathbb{E}$.

Without any loss of generality we assume the monotonicity of the norms $|\cdot|_\tau$ with respect to the parameter τ, i.e. for $\tau' \geq \tau \geq 0$ there holds

$$|u|_\tau \leq |u|_{\tau'}, \qquad u \in \mathbb{E}_{\tau'}. \tag{2.2}$$

The above condition implies in particular that in case L_τ is a closed operator the space $\mathbb{E}_\tau$ is complete.

We will give conditions on A and B such that the corresponding equation (2.1) can be reduced to an equation of the form

$$u - Tu = g, \tag{2.3}$$

for which the projection-iterative method may be applicable. To this end, we introduce several assumptions, which connect the family (L_τ) with the operators A and B.

(i) There exists a number $m > 0$ and a constant $a > 0$ independent of τ such that for every $\tau \geq 0$ the estimate

$$|A^{(-1)}u|_\tau \leq a|u|_{\tau+m} \qquad (u \in \mathbb{E}_{\tau+m}) \tag{2.4}$$

holds for every $\tau \geq 0$.

(ii) There exists $\tau \geq 0$ such that

$$|A^{(-1)}Bu|_\tau \leq c(\tau)|u|_\tau \qquad (u \in \mathbb{E}_\tau) \tag{2.5}$$

with $0 \leq c(\tau) < 1$.

(iii) There exists an absolute constant $b > 0$ such that

$$|A^{(-1)}Bu|_\tau \leq b|u|_{\tau-\epsilon} \qquad (u \in \mathbb{E}_{\tau-\epsilon}) \tag{2.6}$$

if $\tau \geq \epsilon > 0$.

A prototype of the above assumptions **(i)**–**(iii)** was introduced in [5] (see also [6]).

Let assumptions **(i)** and **(iii)** be satisfied. If $f \in \mathbb{E}_{\tau_0}$ with $\tau_0 \geq m$, then each solution of the equation (2.1) belongs to $\mathbb{E}_\tau$ for $\tau \leq \tau_0 - m$. In addition, if **(ii)** is also satisfied with $\tau \leq \tau_0 - m$, then for each $f \in \mathbb{E}_{\tau_0}$ $(\tau_0 \geq m)$ the equation (2.1) can only have a unique solution u and $u \in \mathbb{E}_\tau$ for every $\tau \leq \tau_0 - m$ (see [9]).

In what follows, $\mathbb{E}_\infty$ denotes the set

$$\mathbb{E}_\infty := \bigcap_{\tau \geq 0} \mathbb{E}_\tau.$$

Remark 2.1. Let assumptions **(i)** and **(iii)** be satisfied.

(a) If $f \in \mathbb{E}_\infty$, then each solution of the equation (2.1) belongs to $\mathbb{E}_\infty$.
(b) In addition, if assumption **(ii)** is also fulfilled, then for each $f \in \mathbb{E}_\infty$ the equation (2.1) may only have a unique solution u and $u \in \mathbb{E}_\infty$.

From the above arguments it is seen that under above assumptions, for $f \in \mathbb{E}_\tau$ with $\tau \geq m$, the equation (2.1) can be reduced to

$$u - Tu = g, \tag{2.7}$$

where

$$g = A^{(-1)}f \qquad \text{and} \qquad T = -A^{(-1)}B. \tag{2.8}$$

We are concerned with the approximate solution of the equation (2.1). The method, which we apply, can be described as follows. Let (P_n) be a sequence of bounded projections acting on X with the property that $P_n \to I$ strongly (I stands for the identity operator on X) and $\|P_n\| = 1$ for each $n = 1, 2, \ldots$ We take an arbitrary element $u_0 \in X$, and define, successively, a sequence (u_n) of elements of X by

$$u_n = T_n u_{n-1} + P_n g \qquad (n = 1, 2, \ldots), \tag{2.9}$$

where

$$T_n = P_n T P_n \qquad (n = 1, 2, \dots).$$

If for each $g \in X$ the sequence of the approximate solutions (u_n) converges in X to a solution u of the equation (2.1), then one says that the projection-iterative method is convergent in X.

We apply the projection-iterative method (2.9) to the equation (2.7) by supposing that the projections P_n on $\mathbb{E}$ commute with weights L_τ, i.e.

$$P_n L_\tau = L_\tau P_n \qquad \text{for all} \qquad n = 1, 2, \dots \qquad \text{and} \qquad \tau \geq 0.$$

The following theorem summarizes the projection-iterative methods for abstract operators.

Theorem 2.2. [9] *Let Assumptions* **(i)**, **(ii)** *and* **(iii)** *be satisfied, and let* (u_n) *be the approximating sequence determined by the process* (2.9) *for the equation* (2.7), *with the initial element* u_0 *chosen as above, i.e.* $u_0 \in \mathbb{E}_\tau$ *for* τ *as in assumption* **(ii)**. *If* $f \in \mathbb{E}_{\tau_0}$ $(\tau_0 \geq \tau + m)$, *then* (u_n) *converges in the norm of* $\mathbb{E}_\tau$ *to the solution* u *of* (2.1) *with the error estimate*

$$|u_n - u|_\tau \leq \frac{c(\tau)}{1 - c(\tau)} |u_{n-1} - u_n|_\tau + |R_n g|_\tau, \tag{2.10}$$

where $c(\tau, \epsilon)$ *is defined by* (2.5) *and*

$$R_n = (I - T_n)^{-1} S_n (I - T)^{-1},$$

where

$$S_n := -I + P_n - P_n T(I - P_n), \qquad n = 1, 2, \dots.$$

Corollary 2.3. [9] *Under Assumptions* **(i)**, **(ii)** *and* **(iii)**, *in the particular case of* $P_n = I$ $(n = 1, 2, \ldots)$, *the standard iteration*

$$u_n = T u_{n-1} + g \qquad (n = 1, 2, \dots)$$

for the equation (2.7), *where* $f \in \mathbb{E}_{\tau_0}$ $(\tau_0 \geq \tau + m$; τ *as in assumption* **(ii)**), *with an arbitrary initial element* u_0 *belonging to* $\mathbb{E}_\tau$, *converges in the norm* $\mathbb{E}_\tau$ *with the error estimate*

$$|u_n - u|_\tau \leq \Delta_n(\tau), \tag{2.11}$$

where

$$\Delta_n(\tau) := \frac{c(\tau)}{1 - c(\tau)} |u_{n-1} - u_n|_\tau \qquad (c(\tau) < 1).$$

As before, $c(\tau)$ *is determined by* (2.5). *The error in the norm of the primary space* $\mathbb{E}$ *can be estimate as follows*

$$\|u_n - u\| \leq \inf\{\Delta_n(\tau) : \, 0 \leq \tau \leq \tau_0 - m\,;\, c(\tau) < 1\},$$

In particular, if $f \in \mathbb{E}_\infty$, *then*

$$\|u_n - u\| \leq \inf\{\Delta_n(\tau) : \, \tau \geq 0\,;\, c(\tau) < 1\}.$$

(*We assume that* $\inf \emptyset = \infty$).

3. PERTURBED DISCRETE WIENER-HOPF EQUATIONS

In this section we apply the projection-iterative method recalled in Section 2 to a concrete class of discrete Wiener-Hopf equations.

We denote by V the elementary shift in $l_p(\mathbb{N})$, i.e.

$$(Vu)_n = u_{n-1} \qquad (n = 1, 2, \dots ; u_0 = 0).$$

The operator $V^{(-1)}$ defined by

$$(V^{(-1)}u)_n = u_{n+1} \qquad (n = 1, 2, \dots ; u = (u_n) \in l_p(\mathbb{N}))$$

is obviously an inverse on the left of V, i.e. $V^{(-1)}V = I$.

In what follows, $\mathbb{E}$ stands for one of the Banach spaces $l_p(\mathbb{N})$ $(1 \le p < \infty)$, and A is a Toeplitz operator defined on $\mathbb{E}$ by

$$(Au)_n = \sum_{k=1}^{\infty} a_{n-k} u_k \qquad (u = (u_n) \in \mathbb{E}), \tag{3.1}$$

where a_n $(n = 0, \pm 1, \dots)$ are complex numbers such that

$$\sum_{n=-\infty}^{\infty} |a_n| < \infty. \tag{3.2}$$

According to the theory of Toeplitz operators or, in other terms, discrete Wiener-Hopf operators (see [1, 11, 15]) the operator A can be regarded as the value of some function $A(z)$ of the operator V, i.e. $A = A(V)$. Namely, the function $A(z)$ is given on the complex unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and it is represented in the form

$$A(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \qquad z \in \mathbb{T}.$$

The function $A(z)$, $z \in \mathbb{T}$, is called the symbol of the operator A. The complex numbers a_n $(n = 0, \pm 1, \dots)$ are the Fourier coefficients of $A(z)$, so that, in case where $A(z)$ is given, they can be computed by the formula

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} A(e^{i\phi}) e^{-in\phi} d\phi \qquad (n = 0, \pm 1, \dots).$$

Condition (3.2) means that the symbol $A(z)$ belongs to the Wiener algebra, which we denote by W. The norm in this algebra is given by

$$\|A(z)\|_W = \sum_{n=-\infty}^{\infty} |a_n|.$$

We denote by W_+ a subalgebra of W, which elements are functions of the form

$$\sum_{n=0}^{\infty} b_n z^n, \qquad z \in \mathbb{T}.$$

Similarly, we denote by W_- subalgebra of W consisting of the elements of the form

$$\sum_{n=-\infty}^{0} b_n z^n, \qquad z \in \mathbb{T}.$$

Next we assume that the function $A(z)$ has only a finite number of zeros on $\mathbb{T}$, and each of them has an arbitrary multiplicity. Let α_j be the (pairwise distinct) roots of $A(z)$ on the unit circle $\mathbb{T}$ with multiplicities $\mu_j \geq 0$ $(j = 1, \ldots, r)$. (Note that $\overline{\alpha}_j = \alpha_j^{-1}$.) Now we denote

$$A_0(z) = A(z) \prod_{j=1}^{r} (z^{-1} - \overline{\alpha}_j)^{-\mu_j}, \qquad z \in \mathbb{T}. \tag{3.3}$$

In what follows, we assume that $A_0(z)$ belongs to the Wiener algebra W. Set

$$m := \max\{\lceil \mu_j \rceil : j = 1, \ldots, r\},$$

where for $x \in \mathbb{R}$ the symbol $\lceil x \rceil$ stands for the smallest integer greater than or equal to x. A sufficient condition for $A(z)$ to guarantee that the function $A_0(z)$ belongs to W is the following

$$\sum_{n=-\infty}^{\infty} |n^m a_n| < \infty$$

(cf. [15]). Clearly, $A_0(z) \in W$ is a continuous function on $\mathbb{T}$ such that $A_0(z) \neq 0$ $(z \in \mathbb{T})$. Let

$$\kappa = \operatorname{ind} A_0(z) = \frac{1}{2\pi} \big[\arg A_0(e^{i\phi}) \big]_{\phi=0}^{2\pi}$$

($[\ ]_{\phi=0}^{2\pi}$ means the increment of the function on the interval $[0, 2\pi]$). Then $A_0(z)$, $z \in \mathbb{T}$, can be written in the form (the Wiener-Hopf factorization, [14]):

$$A_0(z) = A_-(z) z^{\kappa} A_+(z), \tag{3.4}$$

where $A_+(z)$ and $A_-(z)$ are functions holomorphic inside and continuous up to the boundary in domains $|z| \leq 1$ and $|z| \geq 1$, respectively, and $A_+(z) \neq 0$ $(|z| \leq 1)$ and $A_-(z) \neq 0$ $(\infty \geq |z| \geq 1)$, moreover $A_+(z) \in W_+$ and $A_-(z) \in W_-$.

Note that the functions $A_-(z)$ and $A_+(z)$ can be considered as symbols of some Toeplitz operators. Let us denote them by A_- and A_+, respectively.

Thus, in view of (3.3) and (3.4), we may express the operator A in the form

$$A = R A_- V^{(\kappa)} A_+, \tag{3.5}$$

where

$$R = \prod_{j=1}^{r} (V^{(-1)} - \overline{\alpha}_j I)^{\mu_j}, \tag{3.6}$$

and where $V^{(\kappa)}$ stands for V^{κ} if $\kappa \geq 0$ and $(V^{(-1)})^{-\kappa}$ if $\kappa < 0$.

Throughout what follows, we use the notation

$$\binom{\mu}{k} = \frac{\mu(\mu-1)\dots(\mu-k+1)}{k!} \qquad \text{for} \qquad k = 1, 2, \dots$$

and $\binom{\mu}{0} = 1$. Note that an operator

$$(V^{(-1)} - \overline{\alpha} I)^{\mu} = (-\overline{\alpha})^{\mu} \sum_{k=0}^{\infty} \binom{\mu}{k} (-\alpha)^k V^{(-k)},$$

where $\alpha \in \mathbb{T}$, is one-to-one, and there exists a closed and unbounded (for $\mu > 0$) operator $(V^{(-1)} - \overline{\alpha} I)^{-\mu}$. For $u = (u_n) \in \operatorname{Ran}(V^{(-1)} - \overline{\alpha} I)^{\mu}$ we can write

$$(V^{(-1)} - \overline{\alpha} I)^{-\mu} u = (-\alpha)^{\mu} \sum_{k=0}^{\infty} \binom{\mu+k-1}{k} \overline{\alpha}^k V^{(-k)} u$$

(cf. [15, 20]).

In what that follows, we assume that $\kappa > 0$. In this case the operator A admits an inverse operator on the left $A^{(-1)}$ and via (3.5) we can write

$$A^{(-1)} u = A_+^{-1} V^{(-\kappa)} A_-^{-1} R^{-1} u, \qquad u \in \operatorname{Ran}(A). \tag{3.7}$$

Next, let B denote a bounded operator defined on the space $\mathbb{E}$ by $B = [b_{nk}]_{n,k=1}^{\infty}$, i.e.

$$(Bu)_n = \sum_{k=1}^{\infty} b_{nk} u_k \qquad (n = 1, 2, \dots; u = (u_n) \in \mathbb{E}), \tag{3.8}$$

where $b_{nk} \in \mathbb{C}$ $(n, k = 1, 2, \dots)$.

Our aim is to apply the general procedure recalled in Section 2 to the approximate solution of the operator equation

$$(A + B)u = f, \tag{3.9}$$

where f is a given element in $\mathbb{E}$, and A and B are operators defined on $\mathbb{E}$ by (3.1) and (3.8), respectively.

We note that the general case where the symbol $A(z)$ of A contains a non-constant factor $A_+(z)$ can be reduced to the case where the factor $A_+(z)$ is constant (cf. [9]). So we assume that the factor $A_+(z)$ (3.4) is a constant (and equals to one). In particular, this means that the symbol $A(z)$ has no roots in the domain $|z| \geq 1$. In other words, we study the case where the symbol $A(z)$ can be written in the form

$$A(z) = z^{\kappa} \prod_{j=1}^{r} (z^{-1} - \overline{\alpha}_j)^{\mu_j} A_-(z), \qquad z \in \mathbb{T}, \tag{3.10}$$

where $\kappa > 0$ and $A_-(z)$ is a function holomorphic inside and continuous up to the boundary in $|z| \geq 1$, and $A_-(z) \neq 0$ $(\infty \geq |z| \geq 1)$, moreover $A_-(z) \in W_-$.

In order to apply the results of Section 2 we use the family of operators $(L_\tau)_{\tau\geq 0}$ (cf. [9]) defined by

$$(L_\tau u)_n = n^\tau u_n \qquad (n = 1, 2, \dots ; u \in \mathrm{Dom}(L_\tau)),$$

where

$$\mathrm{Dom}(L_\tau) = \{u \in \mathbb{E} : \sum_{n=1}^{\infty} |n^\tau u_n|^p < \infty\}.$$

For the introduced family $(L_\tau)_{\tau\geq 0}$ required conditions mentioned in Section 2 are trivially satisfied and the corresponding spaces $\mathbb{E}_\tau$, $\tau \geq 0$, are complete.

We now proceed to the verification of assumption **(i)**. To this end, we set $m = \max\{\lceil \mu_j \rceil : j = 1, \dots, r\}$. (Recall that μ_j are multiplicities of the zeros of the symbol $A(z)$ belonging to the unit circle $\mathbb{T}$).

The following auxiliary assertions will be used.

Lemma 3.1. (i) *For every $\tau \geq 0$ the operator $V^{(-1)}$ is bounded on $\mathbb{E}_\tau$, and*

$$|V^{(-1)}u|_\tau \leq |u|_\tau \qquad (u \in \mathbb{E}_\tau), \tag{3.11}$$

(ii) *For $\tau > 0$ and $\epsilon > 0$ there exists $a_0(\tau, \epsilon) > 0$ such that $a_0(\tau, \epsilon) \to 0$ as $\tau \to \infty$, and*

$$|V^{(-1)}u|_\tau \leq a_0(\tau, \epsilon)|u|_{\tau+\epsilon} \qquad (u \in \mathbb{E}_{\tau+\epsilon}), \tag{3.12}$$

(iii) *For every $0 < \mu \leq 1$, $\tau \geq 0$ and every $\alpha \in \mathbb{T}$ the estimate*

$$|(V^{(-1)} - \overline{\alpha}I)^{-\mu}u|_\tau \leq w(\tau)|u|_{\tau+1} \qquad (u \in \mathbb{E}_{\tau+1}) \tag{3.13}$$

holds with $w(\tau) > 0$ when $\mathbb{E} = l_p(\mathbb{N})$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. The assertions (i) and (ii) were proved in [9] (see also [4]). The estimate (3.13) in assertion (iii) is a kind of a Hardy type inequality with weights. Note that the inequality (3.13) is equivalent to

$$\|L_\tau(V^{(-1)} - \overline{\alpha}I)^{-\mu}L_{\tau+1}^{-1}\xi\| \leq w(\tau)\|\xi\| \qquad (\xi = (\xi_n) \in \mathbb{E}).$$

We use the classical Hardy inequality (cf. [12])

$$\sum_{n=1}^{\infty}\Big|\sum_{k=n}^{\infty}\frac{\xi_k}{k}\Big|^p \leq p^p \sum_{n=1}^{\infty}|\xi_n|^p \qquad (\xi = (\xi_n) \in \mathbb{E}).$$

Now we can estimate

$$\begin{aligned}\|L_\tau(V^{(-1)} - \overline{\alpha}I)^{-\mu}L_{\tau+1}^{-1}\xi\|^p &= \sum_{n=1}^{\infty}\left|(-\alpha)^\mu \sum_{k=0}^{\infty}\binom{\mu+k-1}{k}\overline{\alpha}^k\Big(\frac{n}{n+k}\Big)^\tau \frac{\xi_{n+k}}{n+k}\right|^p \leq \\ &\leq \sum_{n=1}^{\infty}\Big|\sum_{k=0}^{\infty}\frac{\xi_{n+k}}{n+k}\Big|^p = \sum_{n=1}^{\infty}\Big|\sum_{k=n}^{\infty}\frac{\xi_k}{k}\Big|^p \leq \\ &\leq p^p\sum_{n=1}^{\infty}|\xi_n|^p = p^p\|\xi\|^p.\end{aligned}$$

So we have $w(\tau) = p$. In fact the constant p can be replace by

$$w(\tau) = \big(1 - \frac{1}{2^\tau q}\big)^{-1}, \qquad \text{where} \qquad \frac{1}{p} + \frac{1}{q} = 1$$

(see the methods developed in [7] and [8]). □

Let us formulate a simple but important observation.

Lemma 3.2. *If Φ is a Toeplitz operator with the symbol $\Phi(z) \in W_-$ then*

$$|\Phi u|_\tau \le \|\Phi(z)\|_W |u|_\tau \qquad (u \in \mathbb{E}_\tau).$$

Proof. Let $u \in \mathbb{E}_\tau$. Since $\Phi(z) \in W_-$, the symbol of operator Φ has the form

$$\Phi(z) = \sum_{n=0}^{\infty} c_{-n} z^{-n} \qquad (z \in \mathbb{T}).$$

Hence, using the assertion (i) of Lemma 3.1, we obtain

$$|\Phi u|_\tau = |\sum_{n=0}^{\infty} c_{-n} V^{(-n)} u|_\tau \le \sum_{n=0}^{\infty} |c_{-n}||u|_\tau = \|\Phi(z)\|_W |u|_\tau. \qquad \square$$

Next we note that, by virtue of the Wiener theorem [14] (see also [1]), the Toeplitz operator A_-^{-1}, just as A_-, is upper triangular. In language of its symbol this means that $b_-(z) = 1/A_-(z) \in W_-$ and it can be represented in the form

$$b_-(z) = \sum_{n=-\infty}^{0} b_n^- z^n \qquad (z \in \mathbb{T}), \qquad \text{with} \qquad \|b_-\|_W = \sum_{n=-\infty}^{0} |b_n^-| < \infty.$$

Taking into account this fact, by Lemma 3.2 (i), we conclude that the operator A_-^{-1} is bounded in each space E_τ, and

$$|A_-^{-1} u|_\tau \le \|b_-\|_W |u|_\tau. \tag{3.14}$$

In order to estimate R^{-1} we need the following two lemmas.

Lemma 3.3. *If $g_1, \dots, g_n$ are functions in algebra W_-, which have no common zero on the set $\{z \in \mathbb{C} : |z| \ge 1\} \cup \{\infty\}$, then there exist functions $f_1, \dots, f_n$ in W_- such that*

$$f_1 g_1 + \ldots + f_n g_n = 1.$$

Proof. Every maximal ideal in W_- is of the form

$$M_\lambda = \{f \in W_- : f(\lambda) = 0\}$$

for some point λ in the set $\{z \in \mathbb{C} : |z| \ge 1\} \cup \{\infty\}$ (see [10]). The set of all elements of the form $f_1 g_1 + \ldots + f_n g_n$ is an ideal. If it does not contain 1, it is a proper ideal and must be contained in a maximal ideal. (cf. [13], p. 88). □

Lemma 3.4. *If R is given by* (3.6) *then*

$$\operatorname{Ran}(R)=\bigcap_{j=1}^{r}\operatorname{Ran}(V^{(-1)}-\overline{\alpha}_j I)^{\mu_j} \tag{3.15}$$

and there exist operators Φ_j with symbols $\Phi_j(z)\in W_-$ $(j=1,\dots,r)$ such that the equality

$$R^{-1}=\sum_{j=1}^{r}\Phi_j\,(V^{(-1)}-\overline{\alpha}_j I)^{-\mu_j} \tag{3.16}$$

holds on $\operatorname{Ran}(R)$.

Proof. Since the roots α_j $(j=1,\dots,r)$ of $A(z)$ are pairwise distinct, we can use Lemma 3.3 with

$$g_j(z)=\prod_{i=1;i\neq j}^{r}(z^{-1}-\overline{\alpha}_i)^{\mu_i}\qquad (z\in\mathbb{T};j=1,\dots,r\,).$$

So there exist functions $f_j(z)\in W_-$ $(j=1,\dots,r)$ such that

$$\sum_{j=1}^{r}f_j(z)\prod_{i=1;i\neq j}^{r}(z^{-1}-\overline{\alpha}_i)^{\mu_i}=1\qquad (z\in\mathbb{T}). \tag{3.17}$$

Thus

$$\sum_{j=1}^{r}F_j\prod_{i=1;i\neq j}^{r}(V^{(-1)}-\overline{\alpha}_i I)^{\mu_i}=I, \tag{3.18}$$

where F_j are the operators with symbols $f_j(z)$ $(j=1,\dots,r)$. Letting

$$\phi=(V^{(-1)}-\overline{\alpha}_j I)^{\mu_j}\phi_j,\qquad \text{where}\qquad \phi_j\in\mathbb{E}\qquad (j=1,\dots,r).$$

From (3.18) we get $\phi=R\psi$ for $\psi=F_1\phi_1+\ldots+F_r\phi_r$. So we have the inclusion

$$\bigcap_{j=1}^{r}\operatorname{Ran}(V^{(-1)}-\overline{\alpha}_j I)^{\mu_j}\subset\operatorname{Ran}(R).$$

The converse inclusion is obvious. Note that equation (3.18) also implies (3.16) with $\Phi_j=F_j$ $(j=1,\dots,r)$ (cf. [3]). □

Now in view of (3.13), in the case of $\mathbb{E}=l_p(\mathbb{N})$, we obtain

$$|(V^{(-1)}-\overline{\alpha}_j I)^{-\mu}u|_\tau\le d(\lceil\mu\rceil;\tau)|u|_{\tau+\lceil\mu\rceil}\qquad (u\in\mathbb{E}_{\tau+\lceil\mu\rceil}),$$

where

$$d(k;\tau):=\prod_{j=0}^{k-1}\Big(1-\frac{1}{2^{\tau+j}q}\Big)^{-1}\le p^k\qquad (k=1,2,\dots).$$

Therefore, from (3.16) we derive the following estimate (recall that $m = \max\{\lceil\mu_j\rceil : j = 1, \ldots, r\}$)

$$|R^{-1}u|_\tau \le c_1(\tau)|u|_{\tau+m} \qquad (u \in \mathbb{E}_{\tau+m}), \tag{3.19}$$

where

$$c_1(\tau) := \sum_{j=1}^{r} d(\lceil\mu_j\rceil;\tau)\|\Phi_j(z)\|_W \le \sum_{j=1}^{r} p^{\lceil\mu_j\rceil}\|\Phi_j(z)\|_W =: c_1.$$

Now, let $u \in \mathbb{E}_{\tau+m}$ $(\tau \ge 0)$. In view of (3.11), (3.14) and (3.19) we can estimate

$$\begin{aligned}|A^{(-1)}u|_\tau &\le |V^{(-\kappa)}A_-^{-1}R^{-1}u|_\tau \le |A_-^{-1}R^{-1}u|_\tau \le \\ &\le \|b_-\|_W|R^{-1}u|_\tau \le c_1(\tau)\|b_-\|_W|u|_{\tau+m},\end{aligned}$$

i.e.

$$|A^{(-1)}u|_\tau \le a(\tau)|u|_{\tau+m} \le a|u|_{\tau+m}, \qquad u \in \mathbb{E}_{\tau+m} \tag{3.20}$$

with $a(\tau) = c_1(\tau)\|b_-\|_W$ and $a = c_1\|b_-\|_W$. Thus assumption **(i)** is verified.

By similar arguments, one may also obtain the estimate

$$|A^{(-1)}u|_\tau \le a(\tau)|V^{(-\kappa)}u|_{\tau+m}, \qquad (u \in \mathbb{E}_{\tau+m}), \tag{3.21}$$

which will be used later. The constant a is the same as in (3.20).

Next, we find conditions on B such that assumptions **(ii)** and **(iii)** to be also verified. Let $u \in \mathbb{E}_\tau$, $\tau \ge 0$. By virtue of (3.21) and (3.12), we have

$$\begin{aligned}|A^{(-1)}Bu|_\tau &\le a(\tau)|V^{(-\kappa)}Bu|_{\tau+m} \le \\ &\le a(\tau)a_0(\tau,\epsilon)|V^{(-\kappa+1)}Bu|_{\tau+m+\epsilon} \le a(\tau)a_0(\tau,\epsilon)\|B_{\tau,\epsilon}\||u|_\tau,\end{aligned}$$

where

$$B_{\tau,\epsilon} = L_{\tau+m+\epsilon}V^{(-\kappa+1)}BL_\tau^{-1}.$$

Note that the operator $B_{\tau,\epsilon}$ is induced on $\mathbb{E}$ by the matrix

$$B_{\tau,\epsilon} = \left[j^{\tau+m+\epsilon}k^{-\tau}b_{j+\kappa-1\,k}\right]_{j,k=1}^{\infty}.$$

It is seen that the operator $B_{\tau,\epsilon}$ is, generally speaking, unbounded in the basic space $\mathbb{E}$, and even if it is bounded, its norm may increase as the parameter τ increases. However, if we let

$$b_{jk} = 0 \qquad \text{for} \qquad j > k + \kappa - 1$$

and require that the operator

$$B_{m+\epsilon} = \left[j^{m+\epsilon}|b_{j+\kappa-1\,k}|\right]_{j,k=1}^{\infty} \tag{3.22}$$

is bounded on the space $\mathbb{E}$, then we have

$$\|B_{\tau,\epsilon}\| \le \|B_{m+\epsilon}\|.$$

In this way we obtain conditions for which estimates

$$|A^{(-1)}Bu|_\tau \le c(\tau,\epsilon)|u|_\tau \tag{3.23}$$

hold with

$$c(\tau,\epsilon) = a(\tau)a_0(\tau,\epsilon)\|B_{m+\epsilon}\|.$$

Clearly, for a fixed $\epsilon > 0$, $c(\tau,\epsilon) \to 0$ as $\tau \to +\infty$. This ensures that assumption **(ii)** is satisfied.

It turns out that under the same conditions assumption **(iii)** is also satisfied. Indeed, let $u \in \mathbb{E}_{\tau-\epsilon}$, $\tau \ge \epsilon > 0$. Then, again by virtue of (3.21) and by (3.11), we have

$$|A^{(-1)}Bu|_\tau \le a(\tau)|V^{(-\kappa+1)}Bu|_{\tau+m} \le a(\tau)\|L_{\tau+m}V^{(-\kappa+1)}BL^{-1}_{\tau-\epsilon}\||u|_{\tau-\epsilon}.$$

As before, we show that under conditions $b_{jk} = 0$ for $j > k+\kappa-1$ if the operator $B_{m+\epsilon} = \left[j^{m+\epsilon}|b_{j+\kappa-1\,k}|\right]_{j,k=1}^{\infty}$ is bounded on $\mathbb{E}$, then the operator $L_{\tau+m}V^{(-\kappa+1)}BL^{-1}_{\tau-\epsilon}$ is bounded on $\mathbb{E}$ and its norm does not exceed $\|B_{m+\epsilon}\|$. Therefore

$$|A^{(-1)}Bu|_\tau \le b|u|_{\tau-\epsilon} \qquad (\tau \ge \epsilon > 0) \tag{3.24}$$

with $b = a\|B_{m+\epsilon}\|$. Assumption **(iii)** is verified.

According to Section 2, for a given element in $\mathbb{E}_\tau$ with τ enough large, the equation (3.9) can be reduced to an equation of the form (2.7), namely

$$u - Tu = g, \tag{3.25}$$

where $g = A^{(-1)}f$ and $T = -A^{(-1)}B$.

We apply the projection-iterative method (2.9) to the equation (3.25) by taking for P_n canonical projections in the space $\mathbb{E}$, that is

$$P_nu = (u_1,\dots,u_n,0,0,\dots) \qquad (u = (u_n) \in \mathbb{E}).$$

Evidently, $P_n \to I$ strongly in $\mathbb{E}$, $\|P_n\| = 1$ and $P_nL_\tau = L_\tau P_n$ for each $n = 1,2,\dots$. Hence, the required assumptions are fulfilled and Theorem 2.2 may be applied.

We set

$$T_n = P_nTP_n, \qquad g_n = P_ng \qquad (n = 1,2,\dots).$$

Take an arbitrary element $u_0 \in \mathbb{E}_\tau$ $(\tau \ge 0)$ and define the approximating sequence (u_n) by

$$u_n = T_nu_{n-1} + g_n \qquad (n = 1,2,\dots). \tag{3.26}$$

We obtain the following result.

Theorem 3.5. *Let A and B be operators defined by* (3.1) *and* (3.8)*, respectively. Assume that the symbol $A(z)$ of A can be factorized as in* (3.10)*, $\kappa > 0$ and $m = \max\{\lceil\mu_j\rceil : j = 1,\dots,r\}$. Furthermore, assume that for any $f \in \mathbb{E}_{\tau_0}$ with sufficiently large τ_0 (for istance, $\tau_0 \ge \tau + m$; τ being as in assumption* **(ii)***) the equation* (3.9) *possesses a solution u in $\mathbb{E}$. If under the conditions $b_{jk} = 0$ for $j > k+\kappa-1$ the* *operator $B_{m+\epsilon}$ defined by* (3.22) *is bounded on* $\mathbb{E}$*, then the approximating sequence* (u_n) *determined by the process* (3.26) *converges in the norm* $\mathbb{E}_\tau$ $(\tau \le \tau_0)$ *to the solution* u *of the equation* (3.9) *for any initial approximation* u_0 *in* $\mathbb{E}_\tau$*. The error estimate is given by*

$$|u_n - u|_\tau \le \frac{c(\tau,\epsilon)}{1-c(\tau,\epsilon)}|u_{n-1} - u_n|_\tau + \frac{1+c(\tau,\epsilon)}{1-c(\tau,\epsilon)}|h_n|_\tau, \tag{3.27}$$

where $c(\tau,\epsilon)$ *is as in* (3.23) *and* $h_n = (I - P_n)(I - T)^{-1}g$*.*

Proof. That the approximating sequence (u_n) converges to the solution of the equation (3.9) follows immediately from Theorem 2.2. Furthermore, since $f \in \mathbb{E}_{\tau_0}$ with τ_0 sufficiently large, by assumption **(ii)** it follows that $g \in \mathbb{E}_\tau$ and also $h_n \in \mathbb{E}_\tau$. The error estimate (3.27) is a consequence of (2.10). □

Corollary 3.6. *In the particular case of* $P_n = I$ $(n = 1, 2, \ldots)$*, under the hypotheses of Theorem 3.5 the sequence* (u_n) *determined by the iterative process*

$$u_n = Tu_{n-1} + g \qquad (n = 1, 2, \dots)$$

converges in the norm of $\mathbb{E}_\tau$ $(\tau \le \tau_0)$ *to the solution* u *of the equation* (3.9) *for any initial approximation* u_0 *in* $\mathbb{E}_\tau$*. The error estimate is given by*

$$|u_n - u|_\tau \le \Delta_n(\tau,\epsilon), \tag{3.28}$$

where

$$\Delta_n(\tau,\epsilon) := \frac{c(\tau,\epsilon)}{1-c(\tau,\epsilon)}|u_{n-1} - u_n|_\tau \qquad (c(\tau,\epsilon) < 1)$$

with $c(\tau,\epsilon)$ *given as in* (3.23).

We can estimate the error in the norm of the primary space $\mathbb{E}$ as follows

$$\|u_n - u\| \le \inf\{\Delta_n(\tau,\epsilon) : 0 < \epsilon \le \tau \le \tau_0 - m\,;\, c(\tau,\epsilon) < 1\}.$$

In particular, if $f \in \mathbb{E}_\infty$, then

$$\|u_n - u\| \le \inf\{\Delta_n(\tau,\epsilon) : \tau \ge \epsilon > 0\,;\, c(\tau,\epsilon) < 1\}.$$

4. EXAMPLE

Let $\mathbb{E} = l_2(\mathbb{N})$ $(p = 2)$ and let us consider in this space the operator

$$A = \sum_{k=0}^{\infty} \binom{1/2}{k} V^{(-2k+1)}.$$

The matrix representation of A is the following

$$A = \begin{pmatrix} 0 & 1/2 & 0 & -1/8 & 0 & -1/16 & 0 & -5/128 & 0 & \cdots \\ 1 & 0 & 1/2 & 0 & -1/8 & 0 & -1/16 & 0 & -5/128 & \cdots \\ 0 & 1 & 0 & 1/2 & 0 & -1/8 & 0 & -1/16 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 1/2 & 0 & -1/8 & 0 & -1/16 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 1/2 & 0 & -1/8 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & 1/2 & 0 & -1/8 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1/2 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1/2 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The Toeplitz operator A is generated on the space $\mathbb{E}$ (we preserve notations used in the previous section) by the function (the symbol of A)

$$A(z) = z\sum_{k=0}^{\infty}\binom{1/2}{k}z^{-2k} = z(z^{-2}+1)^{1/2} \qquad (z \in \mathbb{T}).$$

Let B be a bounded operator of the form (3.8) and consider the following equation

$$Au + Bu = f, \qquad f \in \mathbb{E}. \tag{4.1}$$

We describe conditions under which the scheme given in the previous section may be applied to equation (4.1). The symbol $A(z)$ has two roots on the unit circle $\mathbb{T}$, and we have the factorization

$$A(z) = z(z^{-1}-i)^{1/2}(z^{-1}+i)^{1/2}, \qquad z \in \mathbb{T}. \tag{4.2}$$

We can write

$$(z^{-1}-i)^{-1/2}(z^{-1}+i)^{-1/2} = \Phi_1(z)(z^{-1}+i)^{-1/2} + \Phi_2(z)(z^{-1}-i)^{-1/2},$$

where

$$\Phi_1(z) = \frac{1}{2}\,i(z^{-1}-i)^{1/2} \qquad \text{and} \qquad \Phi_2(z) = -\frac{1}{2}\,i(z^{-1}+i)^{1/2}.$$

It is easy to see that

$$-\binom{\mu}{k}(-1)^k = \left|\binom{\mu}{k}\right| \qquad (k = 1, 2, \dots), \tag{4.3}$$

which implies

$$\sum_{k=1}^{\infty}\left|\binom{\mu}{k}\right| = 1. \tag{4.4}$$

Taking into account this fact we can compute

$$\|\Phi_1(z)\|_W = \|\Phi_2(z)\|_W = \frac{1}{2}\sum_{k=0}^{\infty}\left|\binom{1/2}{k}\right| = 1.$$

It is seen that Theorem 3.5 can be applied to the pair A and B. We have $\kappa = 1$, $m = 1$. Therefore, if the perturbation B is such that $b_{jk} = 0$ for $j > k$ and the operator defined by $B_\delta = \left[j^\delta |b_{jk}|\right]_{j,k=1}^\infty$ is bounded on $\mathbb{E}$ with $\delta > 1$, then the equation (4.1) can be reduced to

$$u - Tu = g \tag{4.5}$$

with $g = A^{(-1)} f$ and $T = -A^{(-1)} B$, where $A^{(-1)}$ is a left inverse of A. The existence of $A^{(-1)}$ follows from the representation (4.2). Actually, as in hypotheses of Theorem 3.5 we assume that the equation (4.1) possesses a solution u in $\mathbb{E}$ for $f \in \mathbb{E}_{\tau_0}$ with sufficiently large τ_0.

Similarly as in Section 3 we set

$$T_n = P_n T P_n, \qquad g_n = P_n g \qquad (n = 1, 2, \dots).$$

Take an arbitrary element $u_0 \in \mathbb{E}_\tau$ $(\tau > 0)$ and define the approximating sequence (u_n) by

$$u_n = T_n u_{n-1} + g_n \qquad (n = 1, 2, \dots). \tag{4.6}$$

By virtue of Theorem 3.5, we conclude that under above conditions on the perturbation B the approximating sequence (u_n) determined by (4.6) converges in the norm $\mathbb{E}_\tau$ $(\tau \le \tau_0)$ to the solution u of the equation (4.1). The error estimate is given by the formula (3.27), where

$$c(\tau, \epsilon) = 2w(\tau) a_0(\tau, \epsilon) \|B_{1+\epsilon}\|, \tag{4.7}$$

with $a_0(\tau, \epsilon) = \tau^\tau \epsilon^\epsilon (\tau + \epsilon)^{-\tau-\epsilon}$ as in (3.12) and $w(\tau) = \left(1 - 2^{-\tau-1}\right)^{-1}$.

In order to illustrate the above results let us consider the equation (4.1) in the case of

$$f(k) = \begin{cases} (1+\delta^2)^{1/2}, & k = 1, \\ \left(1 + (1+\delta^2)^{1/2}\right)\delta^{k-1}, & k = 2, 3, \dots. \end{cases}$$

with a $\delta \neq 0$. (Here we denote $x(n)$ for the n-th term of the sequence x.) Then

$$g(k) = \delta^k \qquad (k = 1, 2, \dots).$$

It is easily seen that $f \in \mathbb{E}_\infty$ and $g \in \mathbb{E}_\infty$. Let B be a diagonal operator of the form

$$(Bu)(k) = b(k) u(k), \qquad b(k) = \beta \rho^k \qquad (k = 1, 2, \dots),$$

where $|\rho| < 1$ and $\beta \in \mathbb{C}$. For the sake of simplicity, we put $u_0 = 0$. Then the process of the standard pure iteration gives us approximate solutions of the form

$$\begin{aligned} u_n &= \sum_{l=0}^{n-1} T^l g, \\ (T^l g)(k) &= \gamma_l (\delta \rho^l)^k \qquad (k = 1, 2, \dots), \\ \gamma_l &= (-\beta)^l \prod_{j=1}^{l} \frac{\delta \rho^j}{\left(1 + (\delta \rho^j)^2\right)^{1/2}}. \end{aligned} \tag{4.8}$$

The error in the above approximation is given by the formula

$$|u_n - u|_\tau \leq \Delta_n(\tau, \epsilon) := \frac{c(\tau, \epsilon)}{1 - c(\tau, \epsilon)} |T^{n-1} g|_\tau \qquad (c(\tau, \epsilon) < 1),$$

where $c(\tau, \epsilon)$ is given by (4.7). It is not difficult to see that for a fixed τ, $\Delta_n(\tau, \epsilon)$ has minimum for

$$\epsilon_\tau := \frac{1}{2}\left(s - 1 + \sqrt{(s-1)^2 + 4s\tau}\right), \qquad \text{where} \qquad s := \ln \frac{1}{|\rho|}.$$

Consequently,

$$|u_n - u|_\tau \leq \Delta_n(\tau) := \Delta_n(\tau, \epsilon_\tau).$$

Since $f \in \mathbb{E}_\infty$, it can be estimated the error in the norm of the primary space $\mathbb{E}$ as follows

$$\|u_n - u\| \leq \inf\{\Delta_n(\tau) : \tau \geq 0\}.$$

In turn, the norm $\|B_{1+\epsilon}\|$ in formula (4.7) can be estimated as

$$\|B_{1+\epsilon}\| = \sup_j |j^{1+\epsilon} b(j)| \leq \sup_{x \geq 0} \left(|\beta||\rho|^x x^{1+\epsilon}\right) = |\beta| \left(\frac{1+\epsilon}{e \ln \frac{1}{|\rho|}}\right)^{1+\epsilon}.$$

Table 1. *Natural logarithm of $|u_n - u|_\tau$ with optimal constants τ and ϵ for various iterations*

n	τ	ϵ	$\ln \Delta_n(\tau, \epsilon)$
1	3.110	1.323	2.1
2	3.586	1.431	-0.6
4	4.780	1.677	-6.5
8	8.073	2.217	-25.3
16	15.616	3.140	-95.0
32	31.298	4.507	-366.3
64	63.017	6.457	-1439.9
128	126.747	9.221	-5714.3

In Table 1 the natural logarithm of error estimations with optimal constants τ and ϵ for iterations $n = 1, 2, 4, 8, 16, 32, 64, 128$ is presented. The calculations were done using constants $\beta = 3$, $\delta = 0.4$ and $\rho = 0.5$.

Acknowledgments

The author would like to extend his gratitude to Professor P.A. Cojuhari for his support and inspiration.

REFERENCES

[1] A. Böttcher, B. Silbermann, *Analysis of Toeplitz Operators*, Springer-Verlag, Berlin, 1990.

[2] A. Böttcher, B. Silbermann, *Introduction to Large Truncated Toeplitz Matrices*, Universitext, Springer-Verlag, New York, 1998.

[3] P.A. Cojuhari, *Discrete spectrum of a perturbed Wiener-Hopf integral operator*, Investigations in Functional Analysis and Differential Equations **149** (1984), 69–82 [in Russian].

[4] P.A. Cojuhari, *The absence of eigenvalues for operators that are close to operators generated by infinite-dimensional Jacobi matrices*, Izv. Akad. Nauk Moldav. SSR Mat. (1990) 2, 15–21 [in Russian].

[5] P.A. Cojuhari, *The absence of eigenvalues in a perturbed discrete Wiener-Hopf operator*, Izv. Akad. Nauk Moldav. SSR Mat. (1990) 3, 26–35 [in Russian].

[6] P.A. Cojuhari, *On the spectrum of singular nonselfadjoint differential operators. Operator extensions, interpolation of functions and related topics*, Oper. Theory Adv. Appl., 61, Birkhäuser, Basel, 1993, 47–64.

[7] P.A. Cojuhari, *Generalized Hardy type inequalities and some applications to spectral theory. Operator theory, operator algebras and related topics*, Theta Found., Bucharest, 1997, 79–99.

[8] P.A. Cojuhari, *Hardy type inequalities for abstract operators*, Buletinul A.S a R.M Matematica 2(33) (2000), 79–84.

[9] P.A. Cojuhari, M.A. Nowak, *Projection-iterative methods for a class of difference equations*, Integral Equations and Operator Theory **64** (2009), 155–175.

[10] I. Gelfand, D. Raikov, G. Shilov, *Commutative normed rings*, Chelsea Publishing Co., New York, 1964.

[11] I. Gohberg, I.A. Feldman, *Convolution equations and projection methods for their solution*, Transl. of Math. Monographs, vol. 41., Amer. Math. Soc., Providence, 1974.

[12] G.H. Hardy, J.E. Littlewood, G.Pólya, *Inequalities*, Cambridge University Press, Cambridge, 1952.

[13] K. Hoffman, *Banach spaces of analytic functions*, Prentice Hall, Englewood Cliffs, N.J., 1962.

[14] M.G. Krein, *Integral equations on the half-line with a kernel depending on the difference of the arguments*, Uspehi Mat. Nauk **13** (1958) 5 (83), 3–120 [in Russian].

[15] Prössdorf S., *Einige Klassen singulärer Gleichungen*, Akademie Verlag, Berlin, 1974.

[16] Prössdorf S., Silbermann B., *Ein Projektionsverfahren zur Lösung abstrakter singulärer Gleichungen vom nicht normalen Typ und einige seiner Anwendungen*, Math. Nachr. **61** (1974), 133–155.

[17] Prössdorf S., Silbermann B., *Projektionsverfahren und die näherungsweise Lösung singulärer Gleichungen*, Teubner-Texte zur Mathematik, Leipzig, 1977.

[18] Prössdorf S., Silbermann B., *Numerical analysis for integral and related operator equations*, Birkhäuser Verlag, Basel, 1991.

[19] Roch S., *Finite sections of band-dominated operators*, American Mathematical Society, Providence, 2008.

[20] Silbermann B., *Ein Projektionsverfahren für einen diskreten Wiener-Hopfschen Operator, dessen Koeffizientensymbole Nullstellen nicht ganzzahliger Ordnung besitzen*, Math. Nachr. **74** (1976), 191–199.

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Received: October 15, 2008.
Accepted: January 13, 2009.