

Behrouz Kheirfam

**SENSITIVITY ANALYSIS  
IN PIECEWISE LINEAR FRACTIONAL  
PROGRAMMING PROBLEM  
WITH NON-DEGENERATE OPTIMAL SOLUTION**

**Abstract.** In this paper, we study how changes in the coefficients of objective function and the right-hand-side vector of constraints of the piecewise linear fractional programming problems affect the non-degenerate optimal solution. We consider separate cases when changes occur in different parts of the problem and derive bounds for each perturbation, while the optimal solution is invariant. We explain that this analysis is a generalization of the sensitivity analysis for *LP*, *LFP* and *PLP*. Finally, the results are described by some numerical examples.

**Keywords:** piecewise linear fractional programming, fractional programming, piecewise linear programming, sensitivity analysis.

**Mathematics Subject Classification:** 90C31.

## 1. INTRODUCTION

In practice, numerical results are subject to errors and the exact solution of the problem under consideration is not known. The results obtained by some methods, although being approximations of the solutions of the problem, could be considered as the exact results of the corresponding perturbed problem and this is the motivation to investigate the sensitivity analysis. We would like to know the effect of data perturbation on the optimal solution. Hence, the study of sensitivity analysis is of great importance. Generally, independent and simultaneous perturbations are investigated. The materials presented in the rest of this section are selected from [10].

The piecewise linear fractional programming problem (*PLFP*) is defined as follows:

$$\begin{aligned} \min \quad Z(x) &= \frac{P(\mathbf{x})}{D(\mathbf{x})} = \frac{\alpha_0 + \sum_{j=1}^n f_j(x_j)}{\beta_0 + \sum_{j=1}^n g_j(x_j)} \\ \text{s.t:} \quad \mathbf{Ax} &= \mathbf{b} \\ 0 &\leq \mathbf{x} \leq \mathbf{u}, \end{aligned} \quad (PLFP)$$

where  $f_j(x_j)$  and  $g_j(x_j)$ ,  $j = 1, 2, \dots, n$ , are continuous piecewise linear convex and concave functions, respectively, such that  $\beta_0 + \sum_{j=1}^n g_j(x_j) > 0$  for any feasible solution  $\mathbf{x}$ ,  $\mathbf{A}$  is an  $m \times n$  matrix of full row rank,  $\mathbf{b}$  is an  $m$ -vector and  $\mathbf{u}$  is an  $n$ -vector.

Let  $0 = \delta_0^j < \delta_1^j < \dots < \delta_{\tau_j}^j < \delta_{\tau_j+1}^j = u_j$  be an ascending order of the breakpoints of both  $f_j(x_j)$  and  $g_j(x_j)$ . Then within each subinterval  $[\delta_i^j, \delta_{i+1}^j]$ ,  $i = 0, 1, \dots, \tau_j$ , both  $f_j(x_j)$  and  $g_j(x_j)$  are linear functions. Therefore  $f_j(x_j)$  and  $g_j(x_j)$  can be written as

$$f_j(x_j) = c_i^j x_j + \alpha_i^j, \quad \delta_i^j \leq x_j \leq \delta_{i+1}^j; \quad i = 0, 1, 2, \dots, \tau_j, \quad (1.1)$$

and

$$g_j(x_j) = d_i^j x_j + \beta_i^j, \quad \delta_i^j \leq x_j \leq \delta_{i+1}^j; \quad i = 0, 1, 2, \dots, \tau_j, \quad (1.2)$$

for some real numbers  $c_i^j, \alpha_i^j, d_i^j$  and  $\beta_i^j$ ,  $i = 0, 1, \dots, \tau_j$ ,  $j = 1, 2, \dots, n$ .

The following lemmas determine the convexity and the concavity conditions for a continuous piecewise linear function [6].

**Lemma 1.1.** *A continuous piecewise linear function is convex if and only if its slope is nondecreasing with respect to  $x_j$ ; that is,  $c_0^j \leq c_1^j \leq \dots \leq c_{\tau_j}^j$ ,  $j = 1, 2, \dots, n$ .*

**Lemma 1.2.** *A continuous piecewise linear function is concave if and only if its slope is non-increasing with respect to  $x_j$ ; that is,  $d_0^j \geq d_1^j \geq \dots \geq d_{\tau_j}^j$ ,  $j = 1, 2, \dots, n$ .*

Let  $\mathbf{x}^0$  be an optimal solution to *PLFP*. For each  $j = 1, 2, \dots, n$ , choose an index  $j_i$  such that  $\delta_{j_i}^j \leq x_j^0 \leq \delta_{j_i+1}^j$ . Then any optimal solution to the *LFP* problem:

$$\begin{aligned} (LFP) \quad \min \quad & \frac{\alpha^* + \sum_{j=1}^n c_{j_i}^j x_j}{\beta^* + \sum_{j=1}^n d_{j_i}^j x_j} \\ \text{s.t:} \quad & \mathbf{Ax} = \mathbf{b} \\ & \delta_{j_i}^j \leq x_j \leq \delta_{j_i+1}^j, \quad j = 1, 2, \dots, n, \end{aligned}$$

is also an optimal solution to the *PLFP* where  $\alpha^* = \alpha_0 + \sum_{j=1}^n \alpha_{j_i}^j, \beta^* = \beta_0 + \sum_{j=1}^n \beta_{j_i}^j$ .

Definition of a basic feasible solution (*BFS*) for *PLFP* is introduced as follows:

Let  $\mathbf{A} = [A_{.1}, \dots, A_{.n}]$  be the coefficients matrix and  $B = \{B_1, \dots, B_m\} \subset \{1, \dots, n\}$  be a subset of the indices of the columns of the matrix  $\mathbf{A}$ , such that  $\mathbf{B} = [A_{.B_1}, \dots, A_{.B_m}]$  is a non-singular matrix with inverse  $\mathbf{B}^{-1} = [\beta_{ij}]$ . Let  $N = \{1, 2, \dots, n\} \setminus B$ . The variables  $x_{B_i}$ ,  $i = 1, \dots, m$ , are called basic variables and  $x_j$ ,  $j \in N$ , are referred to as non-basic variables. These vectors are denoted by  $\mathbf{x}_B$  and  $\mathbf{x}_N$ , respectively. Consequently, the solution  $\mathbf{x} = (\mathbf{x}_B, \mathbf{x}_N)$ , such that

$$x_j = \delta_{\nu_j}^j, \quad j \in N, \quad \nu_j \in \{0, 1, \dots, \tau_j + 1\},$$

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} - \sum_{j \in N} \mathbf{B}^{-1}A_{.j}x_j, \quad (1.3)$$

is called a basic solution. If, in addition  $0 \leq \mathbf{x}_B \leq \mathbf{u}_B$ , then  $\mathbf{x}$  is a basic feasible solution (*BFS*). Moreover, if  $x_{B_i} \in \{\delta_0^{B_i}, \delta_1^{B_i}, \dots, \delta_{\tau_{B_i}+1}^{B_i}\}$  for some  $i$ , then  $\mathbf{x}$  is a degenerate *BFS*. If  $x_{B_i} \notin \{\delta_0^{B_i}, \delta_1^{B_i}, \dots, \delta_{\tau_{B_i}+1}^{B_i}\}$  for any  $i$ , then it is a non-degenerate *BFS*.

It is showed in [10] that there exists an optimal solution of *PLFP* which is a *BFS*. The optimality criterion given by Punnen and Pandey [10] for *PLFP* using the simplex algorithm is stated as follows:

Let  $\mathbf{B}$  denote the optimal basis matrix and let  $\mathbf{x}^* = (\mathbf{x}_B^*, \mathbf{x}_N^*)$  be the corresponding non-degenerate basic feasible solution for *PLFP*. This solution will be optimal if

$$\eta_j^-(\mathbf{x}^*) = (c_{\nu_j-1}^j - \mathbf{c}_B \mathbf{B}^{-1} A_{.j}) - Z(\mathbf{x}^*)(d_{\nu_j-1}^j - \mathbf{d}_B \mathbf{B}^{-1} A_{.j}) \leq 0,$$

and

$$\eta_j^+(\mathbf{x}^*) = (c_{\nu_j}^j - \mathbf{c}_B \mathbf{B}^{-1} A_{.j}) - Z(\mathbf{x}^*)(d_{\nu_j}^j - \mathbf{d}_B \mathbf{B}^{-1} A_{.j}) \geq 0,$$

for  $j = 1, 2, \dots, n$ , where  $Z(\mathbf{x}^*)$  is the objective function value at the optimal solution  $\mathbf{x}^*$ ,  $\mathbf{c}_B$  and  $\mathbf{d}_B$  are the sub-vectors of  $\mathbf{c}$  and  $\mathbf{d}$  such that their  $i$ th coordinates corresponding to  $\mathbf{B}$  are  $c_{\mu(B_i)}^{B_i}$  and  $d_{\mu(B_i)}^{B_i}$ , respectively. If  $\nu_j = \tau_j + 1$  then  $\eta_j^+$  is defined as 0. Similarly, when  $\nu_j = 0$  then  $\eta_j^-$  is defined as 0. Note that  $\mu(B_i)$  denotes the index for which  $\delta_{\mu(B_i)}^{B_i} \leq x_{B_i}^* \leq \delta_{\mu(B_i)+1}^{B_i}$ .

The sensitivity analysis has been done for linear fractional programming [1, 2]. These results have been extended to the variations for both numerator and denominator of the objective function as well as with right-hand-side of the constraints. Then a primal-dual algorithm proposed to parametric right-hand-side analysis and this algorithm suggests a branch-bound method for integer linear programming [4]. An alternative procedure studied for multi-parametric sensitivity analysis in linear programming by the concept of a maximum volume in the tolerance region, which is bounded by a symmetrically rectangular parallelepiped and can be solved by a maximization problem [13]. For the example of linear fractional programming problem we refer the reader to the examples given in [3]. In Example 2 of [3] let the goods be two sets like (i)-beans, lentils and pea, (ii)-celery, lettuce and cabbages, the prices of which can vary in two different policies. Thus the problem is how we can manage this problem after it has been solved before the changes occur and this

leads to piecewise linear fractional problem. In [8,9], the sensitivity analysis with the maximum volume in the tolerance region is provided for *PLFP* when the variations include both numerator and denominator of the objective function, right-hand-side and the coefficients matrix.

In the present paper, sensitivity analysis investigated in [1,2] for the *PLFP* has been extended. Therefore, we consider separate cases when changes occur in different parts of the problem and derive bounds for each perturbation, while the optimal solution is invariant. Since linear programming (*LP*)[5], piecewise linear programming problems (*PLP*)[7] and linear fractional programming problems (*LFP*)[3,11,12] are all special cases of the *PLFP*, therefore a unified framework of sensitivity analysis is presented which covers almost all approaches that have appeared in the literature.

The paper is organized as follows. In Section 2, we obtain bounds for the parameter when the right hand side vector is perturbed. In Section 3 we consider the perturbation in the coefficients of the numerator of the objective function. Section 4 contains changes in the coefficients of the denominator of the objective function.

## 2. CHANGES IN *RHS* VECTOR $\mathbf{b}$

Let us replace the entry  $b_\gamma$  by  $b'_\gamma = b_\gamma + \delta$  in the *RHS* vector  $\mathbf{b} = (b_1, \dots, b_\gamma, \dots, b_m)^T$  and investigate how the optimal basis  $\mathbf{B}$ , optimal solution  $\mathbf{x}^*$  and the optimal value of objective function  $Z(\mathbf{x})$  are affected. So from (1.3) we will have

$$\begin{aligned} \bar{\mathbf{x}}_B &= \mathbf{B}^{-1}(b_1, \dots, b_\gamma + \delta, \dots, b_m)^T - \sum_{j \in N} \mathbf{B}^{-1} A_{.j} \delta_{\nu_j}^j = \\ &= \mathbf{B}^{-1} \mathbf{b} - \sum_{j \in N} \mathbf{B}^{-1} A_{.j} \delta_{\nu_j}^j + \delta \beta_{i,\gamma} = \mathbf{x}_B^* + \delta \beta_{i,\gamma}, \end{aligned}$$

where  $\beta_{i,\gamma}$  is the  $i$ th column of the matrix  $\mathbf{B}^{-1}$ . Now the  $i$ th component of  $\bar{\mathbf{x}}_B$  is given by

$$\bar{x}_{B_i} = x_{B_i}^* + \delta \beta_{i,\gamma}, \quad i = 1, \dots, m.$$

This new basic solution  $\bar{\mathbf{x}}_B$  will be feasible if

$$\delta_{\mu(B_i)}^{B_i} \leq x_{B_i}^* + \delta \beta_{i,\gamma} \leq \delta_{\mu(B_i)+1}^{B_i}, \quad i = 1, \dots, m.$$

Therefore, we obtain the following range for  $\delta$ :

$$\begin{aligned} \max \left\{ \max_{\substack{\beta_{i,\gamma} < 0 \\ 1 \leq i \leq m}} \frac{\delta_{\mu(B_i)+1}^{B_i} - x_{B_i}^*}{\beta_{i,\gamma}}, \max_{\substack{\beta_{i,\gamma} > 0 \\ 1 \leq i \leq m}} \frac{\delta_{\mu(B_i)}^{B_i} - x_{B_i}^*}{\beta_{i,\gamma}} \right\} &\leq \delta \leq \\ \leq \min \left\{ \min_{\substack{\beta_{i,\gamma} > 0 \\ 1 \leq i \leq m}} \frac{\delta_{\mu(B_i)+1}^{B_i} - x_{B_i}^*}{\beta_{i,\gamma}}, \min_{\substack{\beta_{i,\gamma} < 0 \\ 1 \leq i \leq m}} \frac{\delta_{\mu(B_i)}^{B_i} - x_{B_i}^*}{\beta_{i,\gamma}} \right\}. \end{aligned} \tag{2.1}$$

The new solution  $\bar{\mathbf{x}}$  is an optimal solution for the perturbed *PLFP* problem if

$$\eta_j^+(\bar{\mathbf{x}}) = (c_{\nu_j}^j - \mathbf{c}_B \mathbf{B}^{-1} A_{.j}) - Z(\bar{\mathbf{x}})(d_{\nu_j}^j - \mathbf{d}_B \mathbf{B}^{-1} A_{.j}) \geq 0, \quad j \in N, \tag{2.2}$$

and

$$\eta_j^-(\bar{\mathbf{x}}) = (c_{\nu_j-1}^j - \mathbf{c}_B \mathbf{B}^{-1} A_{.j}) - Z(\bar{\mathbf{x}})(d_{\nu_j-1}^j - \mathbf{d}_B \mathbf{B}^{-1} A_{.j}) \leq 0, \quad j \in N. \quad (2.3)$$

Consider formulas (2.2) and (2.3). Observe that the reduced costs  $c_{\nu_j-1}^j - \mathbf{c}_B \mathbf{B}^{-1} A_{.j}$ ,  $d_{\nu_j-1}^j - \mathbf{d}_B \mathbf{B}^{-1} A_{.j}$ ,  $c_{\nu_j}^j - \mathbf{c}_B \mathbf{B}^{-1} A_{.j}$  and  $d_{\nu_j}^j - \mathbf{d}_B \mathbf{B}^{-1} A_{.j}$  do not depend on  $\mathbf{b}$  and  $\bar{\mathbf{x}}$  directly. So, any change in  $\mathbf{b}$  may affect only the value of the objective function  $Z(\mathbf{x})$ . Hence, we have

$$Z(\bar{\mathbf{x}}) = \frac{\mathbf{c}_B \mathbf{B}^{-1} \mathbf{b}' + \sum_{j \in N} (c_{\nu_j}^j - \mathbf{c}_B \mathbf{B}^{-1} A_{.j}) \delta_{\nu_j}^j + \alpha_0}{\mathbf{d}_B \mathbf{B}^{-1} \mathbf{b}' + \sum_{j \in N} (d_{\nu_j}^j - \mathbf{d}_B \mathbf{B}^{-1} A_{.j}) \delta_{\nu_j}^j + \beta_0} = \frac{P(\mathbf{x}^*) + \delta \mathbf{c}_B \beta_{. \gamma}}{D(\mathbf{x}^*) + \delta \mathbf{d}_B \beta_{. \gamma}}. \quad (2.4)$$

By the assumption,  $D(\mathbf{x}) > 0$  for any feasible solution  $\mathbf{x}$ . Thus, to preserve this condition we need to have

$$D(\mathbf{x}^*) + \delta \mathbf{d}_B \beta_{. \gamma} > 0, \quad (2.5)$$

which implies

$$\delta \begin{cases} > \frac{-D(\mathbf{x}^*)}{\mathbf{d}_B \beta_{. \gamma}}, & \text{if } \mathbf{d}_B \beta_{. \gamma} > 0, \\ < \frac{-D(\mathbf{x}^*)}{\mathbf{d}_B \beta_{. \gamma}}, & \text{if } \mathbf{d}_B \beta_{. \gamma} < 0. \end{cases} \quad (2.6)$$

Moreover, by using (2.4) we can re-write (2.2) in the following form

$$\eta_j^+(\bar{\mathbf{x}}) = (c_{\nu_j}^j - \mathbf{c}_B \mathbf{B}^{-1} A_{.j}) - \frac{P(\mathbf{x}^*) + \delta \mathbf{c}_B \beta_{. \gamma}}{D(\mathbf{x}^*) + \delta \mathbf{d}_B \beta_{. \gamma}} (d_{\nu_j}^j - \mathbf{d}_B \mathbf{B}^{-1} A_{.j}) \geq 0. \quad (2.7)$$

From (2.5), the relation (2.7) is satisfied if

$$(c_{\nu_j}^j - \mathbf{c}_B \mathbf{B}^{-1} A_{.j})(D(\mathbf{x}^*) + \delta \mathbf{d}_B \beta_{. \gamma}) - (P(\mathbf{x}^*) + \delta \mathbf{c}_B \beta_{. \gamma})(d_{\nu_j}^j - \mathbf{d}_B \mathbf{B}^{-1} A_{.j}) \geq 0,$$

which implies

$$\delta(\Delta'_j \mathbf{d}_B \beta_{. \gamma} - \Delta''_j \mathbf{c}_B \beta_{. \gamma}) \geq -D(\mathbf{x}^*) \eta_j^+(\mathbf{x}^*), \quad j \in N,$$

where  $\Delta'_j = c_{\nu_j}^j - \mathbf{c}_B \mathbf{B}^{-1} A_{.j}$  and  $\Delta''_j = d_{\nu_j}^j - \mathbf{d}_B \mathbf{B}^{-1} A_{.j}$ .

From the latter relation we obtain

$$\begin{aligned} & \max_{j \in N} \left\{ \frac{-D(\mathbf{x}^*) \eta_j^+(\mathbf{x}^*)}{\Delta'_j \mathbf{d}_B \beta_{. \gamma} - \Delta''_j \mathbf{c}_B \beta_{. \gamma}} : \Delta'_j \mathbf{d}_B \beta_{. \gamma} - \Delta''_j \mathbf{c}_B \beta_{. \gamma} > 0 \right\} \leq \delta \leq \\ & \leq \min_{j \in N} \left\{ \frac{-D(\mathbf{x}^*) \eta_j^+(\mathbf{x}^*)}{\Delta'_j \mathbf{d}_B \beta_{. \gamma} - \Delta''_j \mathbf{c}_B \beta_{. \gamma}} : \Delta'_j \mathbf{d}_B \beta_{. \gamma} - \Delta''_j \mathbf{c}_B \beta_{. \gamma} < 0 \right\}. \end{aligned} \quad (2.8)$$

Similarly, if  $\eta_j^-(\bar{\mathbf{x}}) \leq 0$  we obtain

$$\begin{aligned} & \max_{j \in N} \left\{ \frac{-D(\mathbf{x}^*) \eta_j^-(\mathbf{x}^*)}{\bar{\Delta}'_j \mathbf{d}_B \beta_{\cdot\gamma} - \bar{\Delta}''_j \mathbf{c}_B \beta_{\cdot\gamma}} : \bar{\Delta}'_j \mathbf{d}_B \beta_{\cdot\gamma} - \bar{\Delta}''_j \mathbf{c}_B \beta_{\cdot\gamma} < 0 \right\} \leq \delta \leq \\ & \leq \min_{j \in N} \left\{ \frac{-D(\mathbf{x}^*) \eta_j^-(\mathbf{x}^*)}{\bar{\Delta}'_j \mathbf{d}_B \beta_{\cdot\gamma} - \bar{\Delta}''_j \mathbf{c}_B \beta_{\cdot\gamma}} : \bar{\Delta}'_j \mathbf{d}_B \beta_{\cdot\gamma} - \bar{\Delta}''_j \mathbf{c}_B \beta_{\cdot\gamma} > 0 \right\}, \end{aligned} \tag{2.9}$$

where  $\bar{\Delta}'_j = c_{\nu_j-1}^j - \mathbf{c}_B \mathbf{B}^{-1} A_{\cdot j}$  and  $\bar{\Delta}''_j = d_{\nu_j-1}^j - \mathbf{d}_B \mathbf{B}^{-1} A_{\cdot j}$ . Thus, we have proved the following theorem:

**Theorem 2.1.** *If  $\delta$  satisfies (2.1), (2.6), (2.8) and (2.9) then  $\bar{\mathbf{x}} = (\bar{\mathbf{x}}_B, \bar{\mathbf{x}}_N)$  where  $\bar{\mathbf{x}}_B = \mathbf{x}_B^* + \delta \beta_{\cdot\gamma}$  is an optimal solution of the perturbed PLFP problem (with  $b_\gamma \rightarrow b'_\gamma = b_\gamma + \delta$ ).*

**Remark 2.2.** Lower and upper bounds given in Theorem 2.1 are generalizations of the corresponding bounds for LP, PLP and LFP. Indeed,

1. If  $\beta_0 = 1$  and  $g_j(x_j) = 0, j = 1, 2, \dots, n$ , then the PLFP reduces to PLP and this means that  $D(\mathbf{x}^*) = 1, \Delta''_j = d_{\nu_j}^j - \mathbf{d}_B \mathbf{B}^{-1} A_{\cdot j} = 0, \eta_j^+(\bar{\mathbf{x}}) = c_{\nu_j}^j - \mathbf{c}_B \mathbf{B}^{-1} A_{\cdot j} = \Delta'_j, \eta_j^-(\bar{\mathbf{x}}) = c_{\nu_j-1}^j - \mathbf{c}_B \mathbf{B}^{-1} A_{\cdot j} = \bar{\Delta}'_j, j \in N$ , and  $Z(\bar{\mathbf{x}}) = P(\mathbf{x}^*) + \delta \mathbf{c}_B \beta_{\cdot\gamma}$ . Thus, bounds (2.1) in the current form are valid for PLP too, and restrictions (2.8) and (2.9) are not present in the bounds since  $\Delta'_j \mathbf{d}_B \beta_{\cdot\gamma} - \Delta''_j \mathbf{c}_B \beta_{\cdot\gamma} = \Delta'_j \mathbf{d}_B \beta_{\cdot\gamma} - \bar{\Delta}''_j \mathbf{c}_B \beta_{\cdot\gamma} = 0$ . Therefore, if  $\delta$  satisfies (2.1) then  $\bar{\mathbf{x}} = (\bar{\mathbf{x}}_B, \bar{\mathbf{x}}_N)$  where  $\bar{\mathbf{x}}_B = \mathbf{x}_B^* + \delta \beta_{\cdot\gamma}$  is an optimal solution of the perturbed PLP problem (when  $b_\gamma \rightarrow b'_\gamma = b_\gamma + \delta$ ).
2. If  $\beta_0 = 1, g_j(x_j) = 0$  and  $f_j(x_j), j = 1, 2, \dots, n$ , are linear functions then the PLFP reduces to LP with bounded variables. In this case, optimality conditions (2.8) and (2.9) and feasibility condition (2.1) are respectively as follows

$$\begin{aligned} \eta_j^+(\mathbf{x}^*) &= c_{\nu_j}^j - \mathbf{c}_B \mathbf{B}^{-1} A_{\cdot j} = c_j - \mathbf{c}_B \mathbf{B}^{-1} A_{\cdot j} \geq 0, \quad \text{if } x_j = 0, \\ \eta_j^-(\mathbf{x}^*) &= c_{\nu_j-1}^j - \mathbf{c}_B \mathbf{B}^{-1} A_{\cdot j} = c_j - \mathbf{c}_B \mathbf{B}^{-1} A_{\cdot j} \leq 0, \quad \text{if } x_j = u_j, \end{aligned}$$

$$\max \left\{ \max_{\substack{\beta_{i\gamma} < 0 \\ 1 \leq i \leq m}} \frac{u_{B_i} - x_{B_i}^*}{\beta_{i\gamma}}, \max_{\substack{\beta_{i\gamma} > 0 \\ 1 \leq i \leq m}} \frac{-x_{B_i}^*}{\beta_{i\gamma}} \right\} \leq \delta \leq \min \left\{ \min_{\substack{\beta_{i\gamma} > 0 \\ 1 \leq i \leq m}} \frac{u_{B_i} - x_{B_i}^*}{\beta_{i\gamma}}, \min_{\substack{\beta_{i\gamma} < 0 \\ 1 \leq i \leq m}} \frac{-x_{B_i}^*}{\beta_{i\gamma}} \right\}.$$

3. If both  $g_j(x_j)$  and  $f_j(x_j), j = 1, 2, \dots, n$ , are linear functions then the PLFP reduces to LFP and this means that  $c_{\nu_j}^j = c_j, d_{\nu_j}^j = d_j, \Delta'_j = c_j - \mathbf{c}_B \mathbf{B}^{-1} A_{\cdot j}$  and  $\Delta''_j = d_j - \mathbf{d}_B \mathbf{B}^{-1} A_{\cdot j}$ . Therefore (2.6) in the current form is valid for LFP and feasibility and optimality conditions are respectively as follows

$$\max_{\beta_{i\gamma} > 0} \frac{-x_{B_i}^*}{\beta_{i\gamma}} \leq \delta \leq \min_{\beta_{i\gamma} < 0} \frac{-x_{B_i}^*}{\beta_{i\gamma}}, \quad i = 1, 2, \dots, m,$$

$$\begin{aligned} & \max_{j \in N} \left\{ \frac{-D(\mathbf{x}^*) \eta_j(\mathbf{x}^*)}{\Delta'_j \mathbf{d}_B \beta_{\cdot \gamma} - \Delta''_j \mathbf{c}_B \beta_{\cdot \gamma}} : \Delta'_j \mathbf{d}_B \beta_{\cdot \gamma} - \Delta''_j \mathbf{c}_B \beta_{\cdot \gamma} > 0 \right\} \leq \delta \leq \\ & \leq \min_{j \in N} \left\{ \frac{-D(\mathbf{x}^*) \eta_j(\mathbf{x}^*)}{\Delta'_j \mathbf{d}_B \beta_{\cdot \gamma} - \Delta''_j \mathbf{c}_B \beta_{\cdot \gamma}} : \Delta'_j \mathbf{d}_B \beta_{\cdot \gamma} - \Delta''_j \mathbf{c}_B \beta_{\cdot \gamma} < 0 \right\}, \end{aligned}$$

where  $\eta_j(\mathbf{x}^*) = (c_j - \mathbf{c}_B \mathbf{B}^{-1} A_{\cdot j}) - Z(\mathbf{x}^*)(d_j - \mathbf{d}_B \mathbf{B}^{-1} A_{\cdot j})$ .

**Example 2.3.** Consider the problem (PLFP):

$$\begin{aligned} \min \quad & Z(\mathbf{x}) = \frac{\sum_{j=1}^4 f_j(x_j)}{\sum_{j=1}^4 g_j(x_j)} \\ \text{s.t.} \quad & \begin{array}{ccccccc} 3x_1 + & & 4x_2 + & & x_3 + & & 2x_4 = & 21 \\ & x_1 + & & 3x_2 + & & x_3 + & & 3x_4 = & 13 \\ & 2x_1 + & & x_2 + & & 2x_3 + & & 3x_4 = & 14 \\ 0 \leq x_1 \leq 5, & & 0 \leq x_2 \leq 3, & & 0 \leq x_3 \leq 5, & & 0 \leq x_4 \leq 5 \end{array} \end{aligned}$$

where

$$\begin{aligned} f_1(x_1) &= \begin{cases} 3x_1, & 0 \leq x_1 \leq 1, \\ 4x_1 - 1, & 1 \leq x_1 \leq 5, \end{cases} & g_1(x_1) &= \begin{cases} 4x_1 + 1, & 0 \leq x_1 \leq 1, \\ 3x_1 + 2, & 1 \leq x_1 \leq 5, \end{cases} \\ f_2(x_2) &= \begin{cases} 2x_2 + 1, & 0 \leq x_2 \leq 1, \\ 3x_2, & 1 \leq x_2 \leq 3, \end{cases} & g_2(x_2) &= \begin{cases} 3x_2 + 1, & 0 \leq x_2 \leq 1, \\ 2x_2 + 2, & 1 \leq x_2 \leq 3, \end{cases} \\ f_3(x_3) &= \begin{cases} x_3 + 3, & 0 \leq x_3 \leq 2, \\ 2x_3 + 1, & 2 \leq x_3 \leq 3, \\ 3x_3 - 2, & 3 \leq x_3 \leq 5, \end{cases} & g_3(x_3) &= \begin{cases} 3x_3 + 1, & 0 \leq x_3 \leq 2, \\ 2x_3 + 3, & 2 \leq x_3 \leq 3, \\ x_3 + 6, & 3 \leq x_3 \leq 5, \end{cases} \\ f_4(x_4) &= \begin{cases} x_4 + 1, & 0 \leq x_4 \leq 1, \\ 2x_4, & 1 \leq x_4 \leq 3, \\ 3x_4 - 3, & 3 \leq x_4 \leq 5, \end{cases} & g_4(x_4) &= \begin{cases} 4x_4 + 1, & 0 \leq x_4 \leq 1, \\ 2x_4 + 3, & 1 \leq x_4 \leq 3, \\ x_4 + 6, & 3 \leq x_4 \leq 5. \end{cases} \end{aligned}$$

Using the simplex algorithm of Punnen and Pandey [10], the initial and the final simplex tables are given as follows (see Tab. 1 and 2).

**Table 1.** Initial simplex

$c_B$	$d_B$	$x_B$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$\bar{b}$
$M$	0	$x_5$	3	4	1	2	1	0	0	21
$M$	0	$x_6$	1	3	1	3	0	1	0	13
$M$	0	$x_7$	2	1	2	3	0	0	1	14
$\eta_j^+$			$2 - 54M$	$\frac{-7-176M}{4}$	$\frac{-11-160M}{4}$	$4 - 56M$	0	0	0	$z = \frac{5+48M}{4}$
$\eta_j^-$			0	0	0	0	0	0	0	
$x_j$			0	0	0	0	21	13	14	

Table 2. Final simplex

$c_B$	$d_B$	$x_B$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$b$
3	2	$x_2$	0	1	$-3/10$	0	$3/20$	$1/4$	$-7/20$	$3/2$
1	4	$x_4$	0	0	$1/2$	1	$-1/4$	$1/4$	$1/4$	$3/2$
4	3	$x_1$	1	0	$2/5$	0	$3/10$	$-1/2$	$3/10$	4
$\eta_j^+$			0	0	1.031	0	$M - 2.623$	$M + 1$	$M - 1.438$	$z = .885$
$\eta_j^-$			0	0	$-0.854$	0	0	0	0	
$x_j$			$32/10$	$21/10$	2	$1/2$	0	0	0	

The optimal solution is  $x^* = (32/10, 21/10, 2, 1/2, 0, 0, 0)^t$ . Here  $B = \{2, 4, 1\}$  and the matrix of the optimal basis is  $\begin{pmatrix} 4 & 2 & 3 \\ 3 & 3 & 1 \\ 1 & 3 & 2 \end{pmatrix}$  and its inverse  $\beta = \mathbf{B}^{-1} = \begin{pmatrix} 3/20 & 1/4 & -7/20 \\ -1/4 & 1/4 & 1/4 \\ 3/10 & -1/2 & 3/10 \end{pmatrix}$ .

If  $b_1 \rightarrow b'_1 = b_1 + \delta$ , then by using (2.1), (2.6), (2.8) and (2.9) we get

$$\max \{ \max\{-2\}, \max\{\frac{-22}{3}, \frac{-22}{3}\} \} \leq \delta \leq \min \{ \min\{6, 6\}, \min\{2\} \} \Rightarrow -2 \leq \delta \leq 2,$$

$$\mathbf{d}_B \beta_{.1} = \frac{1}{5} > 0 \Rightarrow \delta > \frac{-278}{\frac{1}{5}} = -139,$$

$$\Delta'_3 \mathbf{d}_B \beta_{.1} - \Delta''_3 \mathbf{c}_B \beta_{.1} = 1 > 0 \Rightarrow \delta \geq \frac{\frac{-278}{10}(1.031)}{1} = -28.66,$$

$$\bar{\Delta}'_3 \mathbf{d}_B \beta_{.1} - \bar{\Delta}''_3 \mathbf{c}_B \beta_{.1} = \frac{-3}{5} < 0 \Rightarrow \delta \geq \frac{\frac{-278}{10}(-0.854)}{\frac{-3}{5}} = -39.57.$$

Therefore, the following range is obtained for  $\delta$ ,

$$-2 \leq \delta \leq 2.$$

### 3. CHANGES IN THE COEFFICIENTS OF NUMERATOR OF THE OBJECTIVE

In this section our goal is to determine the lower and upper bounds for  $\delta$ , which guarantee that the replacement  $c_i^j \rightarrow c'^j_i = c_i^j + \delta$  does not affect the optimal basis, and the original optimal solution  $\mathbf{x}^*$  remains feasible and optimal.

By this replacement, we have to distinguish the following two cases:

Case 1.  $c_i^j \in \{c_{\nu_j}^j : \nu_j \in \{0, 1, \dots, \tau_j\}\},$

Case 2.  $c_i^j \in \{c_{\mu(B_1)}^{B_1}, \dots, c_{\mu(B_m)}^{B_m}\}.$



Case 1.  $c_i^j$  is the coefficient of a non-basic variable. Thus, this change of the coefficient does not affect feasibility of the vector  $\mathbf{x}^*$ . However, it may affect the optimal value of  $Z(\mathbf{x})$  and hence, can change the reduced costs  $\eta_j^+(\mathbf{x}^*)$  and  $\eta_j^-(\mathbf{x}^*)$ . So, by replacing  $c_{\nu_\gamma}^\gamma \rightarrow c_{\nu_\gamma}^\gamma + \delta$  we have

$$\bar{Z}(\mathbf{x}^*) = \frac{P(\mathbf{x}^*) + \delta \delta_{\nu_\gamma}^\gamma}{D(\mathbf{x}^*)}. \quad (3.1)$$

Now, the optimal solution  $\mathbf{x}^*$  of the original *PLFP* problem remains optimal for the perturbed *PLFP* problem if we have

$$\bar{\eta}_j^+(\mathbf{x}^*) = \begin{cases} (c_{\nu_j}^j - \mathbf{c}_B \mathbf{B}^{-1} A_{.j}) - \frac{P(\mathbf{x}^*) + \delta \delta_{\nu_\gamma}^\gamma}{D(\mathbf{x}^*)} (d_{\nu_j}^j - \mathbf{d}_B \mathbf{B}^{-1} A_{.j}) \geq 0, & j \neq \gamma, j \in N, \\ (c_{\nu_\gamma}^\gamma + \delta - \mathbf{c}_B \mathbf{B}^{-1} A_{.\gamma}) - \frac{P(\mathbf{x}^*) + \delta \delta_{\nu_\gamma}^\gamma}{D(\mathbf{x}^*)} (d_{\nu_\gamma}^\gamma - \mathbf{d}_B \mathbf{B}^{-1} A_{.\gamma}) \geq 0, & j = \gamma, \end{cases}$$

or

$$\begin{cases} \Delta'_j - \frac{P(\mathbf{x}^*) + \delta \delta_{\nu_\gamma}^\gamma}{D(\mathbf{x}^*)} \Delta''_j \geq 0, & j \neq \gamma, j \in N, \\ \Delta'_\gamma + \delta - \frac{P(\mathbf{x}^*) + \delta \delta_{\nu_\gamma}^\gamma}{D(\mathbf{x}^*)} \Delta''_\gamma \geq 0, & j = \gamma. \end{cases} \quad (3.2)$$

From  $D(\mathbf{x}^*) > 0$ , the relation (3.2) is satisfied if

$$\begin{cases} \Delta'_j D(\mathbf{x}^*) - (P(\mathbf{x}^*) + \delta \delta_{\nu_\gamma}^\gamma) \Delta''_j \geq 0, & j \neq \gamma, j \in N, \\ (\Delta'_\gamma + \delta) D(\mathbf{x}^*) - (P(\mathbf{x}^*) + \delta \delta_{\nu_\gamma}^\gamma) \Delta''_\gamma \geq 0, & j = \gamma. \end{cases} \quad (3.3)$$

Therefore, we will get

$$\max_{\substack{j \in N \\ j \neq \gamma}} \left\{ \frac{D(\mathbf{x}^*) \eta_j^+(\mathbf{x}^*)}{\delta_{\nu_\gamma}^\gamma \Delta''_j} : \Delta''_j < 0 \right\} \leq \delta \leq \min_{\substack{j \in N \\ j \neq \gamma}} \left\{ \frac{D(\mathbf{x}^*) \eta_j^+(\mathbf{x}^*)}{\delta_{\nu_\gamma}^\gamma \Delta''_j} : \Delta''_j > 0 \right\}, \quad (3.4)$$

and

$$\delta \begin{cases} \geq \frac{-D(\mathbf{x}^*) \eta_\gamma^+(\mathbf{x}^*)}{D(\mathbf{x}^*) - \delta_{\nu_\gamma}^\gamma \Delta''_\gamma}, & \text{if } D(\mathbf{x}^*) - \delta_{\nu_\gamma}^\gamma \Delta''_\gamma > 0, \\ \leq \frac{-D(\mathbf{x}^*) \eta_\gamma^+(\mathbf{x}^*)}{D(\mathbf{x}^*) - \delta_{\nu_\gamma}^\gamma \Delta''_\gamma}, & \text{if } D(\mathbf{x}^*) - \delta_{\nu_\gamma}^\gamma \Delta''_\gamma < 0. \end{cases} \quad (3.5)$$

Similarly, if  $\bar{\eta}_j^-(\mathbf{x}^*) \leq 0$ , we will get

$$\max_{\substack{j \in N \\ j \neq \gamma}} \left\{ \frac{D(\mathbf{x}^*) \eta_j^-(\mathbf{x}^*)}{\delta_{\nu_\gamma}^\gamma \bar{\Delta}''_j} : \bar{\Delta}''_j > 0 \right\} \leq \delta \leq \min_{\substack{j \in N \\ j \neq \gamma}} \left\{ \frac{D(\mathbf{x}^*) \eta_j^-(\mathbf{x}^*)}{\delta_{\nu_\gamma}^\gamma \bar{\Delta}''_j} : \bar{\Delta}''_j < 0 \right\}, \quad (3.6)$$

and

$$\delta \begin{cases} \leq \frac{-D(\mathbf{x}^*) \eta_\gamma^-(\mathbf{x}^*)}{D(\mathbf{x}^*) - \delta_{\nu_\gamma}^\gamma \bar{\Delta}''_\gamma}, & \text{if } D(\mathbf{x}^*) - \delta_{\nu_\gamma}^\gamma \bar{\Delta}''_\gamma > 0, \\ \geq \frac{-D(\mathbf{x}^*) \eta_\gamma^-(\mathbf{x}^*)}{D(\mathbf{x}^*) - \delta_{\nu_\gamma}^\gamma \bar{\Delta}''_\gamma}, & \text{if } D(\mathbf{x}^*) - \delta_{\nu_\gamma}^\gamma \bar{\Delta}''_\gamma < 0. \end{cases} \quad (3.7)$$

Therefore, we have proved the following theorem.

**Theorem 3.1.** *If  $\delta$  satisfies (3.4), (3.5), (3.6), (3.7) and the convexity condition for  $f_j(x_j)$  holds, then optimal solution  $\mathbf{x}^*$  of the original PLFP problem is also an optimal solution of the perturbed PLFP problem (where  $c_{\nu_\gamma}^\gamma \rightarrow c_{\nu_\gamma}^\gamma + \delta$ ).*

**Remark 3.2.** Lower and upper bounds given in Theorem 2.1 are generalizations of the corresponding bounds for LP, PLP and LFP. Indeed,

1. If both  $f_j(x_j)$  and  $g_j(x_j)$ ,  $j = 1, 2, \dots, n$ , are linear functions then the PLFP reduces to LFP. Therefore from (3.4) and (3.6) we conclude that  $-\infty \leq \delta \leq \infty$  and from (3.5) and (3.7) it follows that  $\delta \leq -\eta_\gamma(\mathbf{x}^*)$  where  $\eta_\gamma(\mathbf{x}^*) = c_\gamma - \mathbf{c}_B \mathbf{B}^{-1} A_{\cdot\gamma} - Z(\mathbf{x}^*)(d_\gamma - \mathbf{d}_B \mathbf{B}^{-1} A_{\cdot\gamma})$ .
2. If  $\beta_0 = 1$  and  $g_j(x_j) = 0$ ,  $j = 1, 2, \dots, n$ , then the PLFP reduces to PLP. In this case, from (3.4), (3.6) and from (3.5), (3.7) we have, respectively,

$$-\infty \leq \delta \leq \infty,$$

$$-\Delta'_\gamma \leq \delta \leq -\bar{\Delta}'_\gamma.$$

Case 2.  $c'_i$  is the coefficient of a basic variable. Then the replacement  $c_{\mu(B_k)}^{B_k} \rightarrow c'_{\mu(B_k)} = c_{\mu(B_k)}^{B_k} + \delta$  affects the optimal value of  $P(\mathbf{x})$  as well as  $Z(\mathbf{x})$

$$\begin{aligned} \tilde{P}(\mathbf{x}^*) &= \mathbf{c}_B \mathbf{B}^{-1} \mathbf{b} + \delta \beta_k \cdot \mathbf{b} + \sum_{j \in N} (c_{\nu_j}^j - \mathbf{c}_B \mathbf{B}^{-1} A_{\cdot j}) \delta_{\nu_j}^j - \delta \sum_{j \in N} \beta_k \cdot A_{\cdot j} \delta_{\nu_j}^j + \alpha \\ &= P(\mathbf{x}^*) + \delta \beta_k \cdot (\mathbf{b} - \sum_{j \in N} A_{\cdot j} \delta_{\nu_j}^j). \end{aligned}$$

In addition, the replacement  $c_{\mu(B_k)}^{B_k} \rightarrow c'_{\mu(B_k)} = c_{\mu(B_k)}^{B_k} + \delta$  has an affect on the non-basic reduced costs:

$$c_{\nu_j}^j - \mathbf{c}'_B \mathbf{B}^{-1} A_{\cdot j} = c_{\nu_j}^j - \mathbf{c}_B \mathbf{B}^{-1} A_{\cdot j} - \delta \beta_k \cdot A_{\cdot j} = \Delta'_j - \delta \beta_k \cdot A_{\cdot j},$$

$$c_{\nu_{j-1}}^j - \mathbf{c}'_B \mathbf{B}^{-1} A_{\cdot j} = c_{\nu_{j-1}}^j - \mathbf{c}_B \mathbf{B}^{-1} A_{\cdot j} - \delta \beta_k \cdot A_{\cdot j} = \bar{\Delta}'_j - \delta \beta_k \cdot A_{\cdot j}.$$

Therefore, to satisfy the optimality condition, we can determine the new values  $\eta_j^+$  and  $\eta_j^-$  as

$$\begin{aligned} \tilde{\eta}_j^+(\mathbf{x}^*) &= \Delta'_j - \delta \beta_k \cdot A_{\cdot j} - \frac{P(\mathbf{x}^*) + \delta \beta_k \cdot (\mathbf{b} - \sum_{j \in N} A_{\cdot j} \delta_{\nu_j}^j)}{D(\mathbf{x}^*)} \Delta_j'' = \\ &= \frac{(\Delta'_j - \delta \beta_k \cdot A_{\cdot j}) D(\mathbf{x}^*) - \left( P(\mathbf{x}^*) + \delta \beta_k \cdot (\mathbf{b} - \sum_{j \in N} A_{\cdot j} \delta_{\nu_j}^j) \right) \Delta_j''}{D(\mathbf{x}^*)} \geq 0. \end{aligned} \tag{3.8}$$

The relation (3.8) is satisfied if

$$(\Delta'_j - \delta\beta_k A_{.j})D(\mathbf{x}^*) - \left(P(\mathbf{x}^*) + \delta\beta_k(\mathbf{b} - \sum_{j \in N} A_{.j}\delta_{\nu_j}^j)\right)\Delta''_j \geq 0,$$

which implies

$$\delta\beta_k \left(A_{.j}D(\mathbf{x}^*) + (\mathbf{b} - \sum_{j \in N} A_{.j}\delta_{\nu_j}^j)\Delta''_j\right) \leq D(\mathbf{x}^*)\eta_j^+(\mathbf{x}^*), \quad j \in N.$$

Thus

$$\begin{aligned} & \max_{j \in N} \left\{ \frac{D(\mathbf{x}^*) \eta_j^+(\mathbf{x}^*)}{\beta_k \left(A_{.j}D(\mathbf{x}^*) + (\mathbf{b} - \sum_{j \in N} A_{.j}\delta_{\nu_j}^j)\Delta''_j\right)} : \beta_k \left(A_{.j}D(\mathbf{x}^*) + (\mathbf{b} - \sum_{j \in N} A_{.j}\delta_{\nu_j}^j)\Delta''_j\right) < 0 \right\} \\ & \leq \delta \leq \\ & \min_{j \in N} \left\{ \frac{D(\mathbf{x}^*) \eta_j^+(\mathbf{x}^*)}{\beta_k \left(A_{.j}D(\mathbf{x}^*) + (\mathbf{b} - \sum_{j \in N} A_{.j}\delta_{\nu_j}^j)\Delta''_j\right)} : \beta_k \left(A_{.j}D(\mathbf{x}^*) + (\mathbf{b} - \sum_{j \in N} A_{.j}\delta_{\nu_j}^j)\Delta''_j\right) > 0 \right\}. \end{aligned} \quad (3.9)$$

Similarly, if  $\tilde{\eta}_j^-(\mathbf{x}^*) \leq 0$ , we will get

$$\begin{aligned} & \max_{j \in N} \left\{ \frac{D(\mathbf{x}^*) \eta_j^-(\mathbf{x}^*)}{\beta_k \left(A_{.j}D(\mathbf{x}^*) + (\mathbf{b} - \sum_{j \in N} A_{.j}\delta_{\nu_j}^j)\bar{\Delta}''_j\right)} : \beta_k \left(A_{.j}D(\mathbf{x}^*) + (\mathbf{b} - \sum_{j \in N} A_{.j}\delta_{\nu_j}^j)\bar{\Delta}''_j\right) > 0 \right\} \\ & \leq \delta \leq \\ & \min_{j \in N} \left\{ \frac{D(\mathbf{x}^*) \eta_j^-(\mathbf{x}^*)}{\beta_k \left(A_{.j}D(\mathbf{x}^*) + (\mathbf{b} - \sum_{j \in N} A_{.j}\delta_{\nu_j}^j)\bar{\Delta}''_j\right)} : \beta_k \left(A_{.j}D(\mathbf{x}^*) + (\mathbf{b} - \sum_{j \in N} A_{.j}\delta_{\nu_j}^j)\bar{\Delta}''_j\right) < 0 \right\}. \end{aligned} \quad (3.10)$$

Therefore, we have proved the following theorem.

**Theorem 3.3.** *If  $\delta$  satisfies (3.9), (3.10) and the convexity condition for  $f_j(x_j)$  holds then optimal solution  $\mathbf{x}^*$  of the original PLFP problem is also an optimal solution of the perturbed PLFP problem (with  $c_{\mu(B_k)}^{B_k} \rightarrow c_{\mu(B_k)}^{B_k}$ ).*

**Remark 3.4.** Observe that the range obtained in Theorem 3.3 may be considered as a generalization of the corresponding range for the LFP, PLP and LP problems. Thus we have

1. If both  $f_j(x_j)$  and  $g_j(x_j)$ ,  $j = 1, 2, \dots, n$ , are linear functions then the PLFP reduces to LFP. In this case, the restrictions (3.9) and (3.10) reduce to

$$\begin{aligned} & \max_{j \in N} \left\{ \frac{D(\mathbf{x}^*) \eta_j(\mathbf{x}^*)}{\beta_k \left(A_{.j}D(\mathbf{x}^*) + \mathbf{b}\Delta''_j\right)} : \beta_k \left(A_{.j}D(\mathbf{x}^*) + \mathbf{b}\Delta''_j\right) < 0 \right\} \leq \delta \leq \\ & \leq \min_{j \in N} \left\{ \frac{D(\mathbf{x}^*) \eta_j(\mathbf{x}^*)}{\beta_k \left(A_{.j}D(\mathbf{x}^*) + \mathbf{b}\Delta''_j\right)} : \beta_k \left(A_{.j}D(\mathbf{x}^*) + \mathbf{b}\Delta''_j\right) > 0 \right\}, \end{aligned}$$

where  $\eta_j = \Delta'_j - Z(\mathbf{x}^*)\Delta''_j$ .

2. If  $\beta_0 = 1$  and  $g_j(x_j) = 0, j = 1, 2, \dots, n$ , then the *PLFP* reduces to *PLP*. Therefore the relations (3.9) and (3.10) exchange to

$$\max_{j \in N} \left\{ \frac{\Delta'_j}{\beta_{k.A.j}} : \beta_{k.A.j} < 0 \right\} \leq \delta \leq \min_{j \in N} \left\{ \frac{\Delta'_j}{\beta_{k.A.j}} : \beta_{k.A.j} > 0 \right\},$$

$$\max_{j \in N} \left\{ \frac{\bar{\Delta}'_j}{\beta_{k.A.j}} : \beta_{k.A.j} > 0 \right\} \leq \delta \leq \min_{j \in N} \left\{ \frac{\bar{\Delta}'_j}{\beta_{k.A.j}} : \beta_{k.A.j} < 0 \right\}.$$

**Example 3.5.** Consider Example 2.3. For the given optimal basis and solution we consider the following two cases:

**Non-basic index:** Let  $c_1^3 \rightarrow c_1^3 + \delta$ . In this case,  $\gamma = 3$  and  $N = \{3\}$ . Since  $\gamma \neq j \in N$ , the relations (3.4) and (3.6) are not applicable. Hence, from (3.5), (3.7) and the convexity of  $f_j(x_j)$  we have

$$D(\mathbf{x}^*) - \delta_1^3 \Delta_3'' = 29 > 0 \Rightarrow \delta \geq \frac{-139(1.031)}{29} = -0.99,$$

$$D(\mathbf{x}^*) - \delta_1^3 \bar{\Delta}_3'' = 27 > 0 \Rightarrow \delta \leq \frac{-139(-0.854)}{27} = 0.88,$$

$$1 \leq 2 + \delta \leq 3 \Rightarrow -1 \leq \delta \leq 1.$$

Finally, we obtain the following bounds for  $\delta$ :

$$-0.99 \leq \delta \leq 0.88.$$

**Basic index:** Let  $c_1^1 \rightarrow c_1^1 + \delta$ . In this case by using (3.9), (3.10) and the convexity of  $f_j(x_j)$  we have

$$\beta_3.(A_{.3}D(\mathbf{x}^*) + (\mathbf{b} - A_{.3}\delta_{\nu_3}^3)\Delta_3'') = 9.2 > 0 \Rightarrow$$

$$\Rightarrow \delta \leq \min_{j \in \{3\}} \left\{ \frac{D(\mathbf{x}^*)\eta_j^+(\mathbf{x}^*)}{\beta_3.(A_{.3}D(\mathbf{x}^*) + (\mathbf{b} - A_{.3}\delta_{\nu_3}^3)\Delta_3'')} \right\} = \frac{\frac{139}{5}(1.031)}{9.2} = 3.115,$$

$$\beta_3.(A_{.3}D(\mathbf{x}^*) + (\mathbf{b} - A_{.3}\delta_{\nu_3}^3)\bar{\Delta}_3'') = \frac{62}{5} > 0 \Rightarrow$$

$$\Rightarrow \delta \geq \max_{j \in \{3\}} \left\{ \frac{D(\mathbf{x}^*)\eta_j^-(\mathbf{x}^*)}{\beta_3.(A_{.3}D(\mathbf{x}^*) + (\mathbf{b} - A_{.3}\delta_{\nu_3}^3)\bar{\Delta}_3'')} \right\} = \frac{\frac{139}{5}(-0.854)}{\frac{62}{5}} = -1.915,$$

$$3 \leq 4 + \delta \Rightarrow \delta \geq -1$$

Hence, we obtain the following bounds for  $\delta$ :

$$-1 \leq \delta \leq 3.115.$$

#### 4. CHANGES IN THE COEFFICIENTS OF THE DENOMINATOR OF THE OBJECTIVE

In this section, our goal is to determine the lower and upper bounds for  $\delta$ , which guarantee that the replacement  $d_i^j \rightarrow d_i^{\prime j} = d_i^j + \delta$  does not affect the optimal basis, and the original optimal solution  $\mathbf{x}^*$  remains feasible and optimal.

By considering this replacement, we have to distinguish the following two cases:

Case 1.  $d_i^j \in \{d_{\nu_j}^j : \nu_j \in \{0, 1, \dots, \tau_j\}\}$ ,

Case 2.  $d_i^j \in \{d_{\mu(B_1)}^{B_1}, \dots, d_{\mu(B_m)}^{B_m}\}$ .

Case 1.  $d_i^j$  is the coefficient of a non-basic variable. Thus, this change of the coefficient does not affect the feasibility of the vector  $\mathbf{x}^*$ . However, it may affect the optimal value of  $Z(\mathbf{x})$  and hence, can change the reduced costs  $\eta_j^+(\mathbf{x}^*)$  and  $\eta_j^-(\mathbf{x}^*)$ . So, by replacing  $d_{\nu_\gamma}^\gamma \rightarrow d_{\nu_\gamma}^\gamma + \delta$  we will have

$$\tilde{Z}(\mathbf{x}^*) = \frac{P(\mathbf{x}^*)}{D(\mathbf{x}^*) + \delta \delta_{\nu_\gamma}^\gamma}. \quad (4.1)$$

To preserve the strict positivity of the denominator  $D(\mathbf{x})$ , we need to have

$$D(\mathbf{x}^*) + \delta \delta_{\nu_\gamma}^\gamma > 0 \Rightarrow \delta > \frac{-D(\mathbf{x}^*)}{\delta_{\nu_\gamma}^\gamma}. \quad (4.2)$$

Now, the optimal solution  $\mathbf{x}^*$  of the original *PLFP* problem remains optimal for the perturbed *PLFP* problem if we have

$$\tilde{\eta}_j^+(\mathbf{x}^*) = \begin{cases} (c_{\nu_j}^j - \mathbf{c}_B \mathbf{B}^{-1} A_{.j}) - \frac{P(\mathbf{x}^*)}{D(\mathbf{x}^*) + \delta \delta_{\nu_\gamma}^\gamma} (d_{\nu_j}^j - \mathbf{d}_B \mathbf{B}^{-1} A_{.j}) \geq 0, & j \neq \gamma, j \in N, \\ (c_{\nu_\gamma}^\gamma - \mathbf{c}_B \mathbf{B}^{-1} A_{.\gamma}) - \frac{P(\mathbf{x}^*)}{D(\mathbf{x}^*) + \delta \delta_{\nu_\gamma}^\gamma} (d_{\nu_\gamma}^\gamma + \delta - \mathbf{d}_B \mathbf{B}^{-1} A_{.\gamma}) \geq 0, & j = \gamma, \end{cases}$$

or

$$\begin{cases} \Delta_j' - \frac{P(\mathbf{x}^*)}{D(\mathbf{x}^*) + \delta \delta_{\nu_\gamma}^\gamma} \Delta_j'' \geq 0, & j \neq \gamma, j \in N, \\ \Delta_\gamma' - \frac{P(\mathbf{x}^*)}{D(\mathbf{x}^*) + \delta \delta_{\nu_\gamma}^\gamma} (\Delta_\gamma'' + \delta) \geq 0, & j = \gamma. \end{cases} \quad (4.3)$$

From (4.2), the relation (4.3) is satisfied if

$$\begin{cases} \Delta_j' [D(\mathbf{x}^*) + \delta \delta_{\nu_\gamma}^\gamma] - P(\mathbf{x}^*) \Delta_j'' \geq 0, & j \neq \gamma, j \in N, \\ \Delta_\gamma' [D(\mathbf{x}^*) + \delta \delta_{\nu_\gamma}^\gamma] - P(\mathbf{x}^*) [\Delta_\gamma'' + \delta] \geq 0, & j = \gamma. \end{cases} \quad (4.4)$$

Therefore, we will get

$$\max_{\substack{j \in N \\ j \neq \gamma}} \left\{ \frac{-D(\mathbf{x}^*) \eta_j^+(\mathbf{x}^*)}{\delta_{\nu_\gamma}^\gamma \Delta'_j} : \Delta'_j > 0 \right\} \leq \delta \leq \min_{\substack{j \in N \\ j \neq \gamma}} \left\{ \frac{-D(\mathbf{x}^*) \eta_j^+(\mathbf{x}^*)}{\delta_{\nu_\gamma}^\gamma \Delta'_j} : \Delta'_j < 0 \right\}, \quad (4.5)$$

and

$$\delta \begin{cases} \geq \frac{-D(\mathbf{x}^*) \eta_\gamma^+(\mathbf{x}^*)}{\delta_{\nu_\gamma}^\gamma \Delta'_\gamma - P(\mathbf{x}^*)}, & \text{if } \delta_{\nu_\gamma}^\gamma \Delta'_\gamma - P(\mathbf{x}^*) > 0, \\ \leq \frac{-D(\mathbf{x}^*) \eta_\gamma^+(\mathbf{x}^*)}{\delta_{\nu_\gamma}^\gamma \Delta'_\gamma - P(\mathbf{x}^*)}, & \text{if } \delta_{\nu_\gamma}^\gamma \Delta'_\gamma - P(\mathbf{x}^*) < 0. \end{cases} \quad (4.6)$$

Similarly, if  $\tilde{\eta}_j^- \leq 0$  we will get

$$\max_{\substack{j \in N \\ j \neq \gamma}} \left\{ \frac{-D(\mathbf{x}^*) \eta_j^-(\mathbf{x}^*)}{\delta_{\nu_\gamma}^\gamma \bar{\Delta}'_j} : \bar{\Delta}'_j < 0 \right\} \leq \delta \leq \min_{\substack{j \in N \\ j \neq \gamma}} \left\{ \frac{-D(\mathbf{x}^*) \eta_j^-(\mathbf{x}^*)}{\delta_{\nu_\gamma}^\gamma \bar{\Delta}'_j} : \bar{\Delta}'_j > 0 \right\}, \quad (4.7)$$

and

$$\delta \begin{cases} \leq \frac{-D(\mathbf{x}^*) \eta_\gamma^-(\mathbf{x}^*)}{\delta_{\nu_\gamma}^\gamma \bar{\Delta}'_\gamma - P(\mathbf{x}^*)}, & \text{if } \delta_{\nu_\gamma}^\gamma \bar{\Delta}'_\gamma - P(\mathbf{x}^*) > 0, \\ \geq \frac{-D(\mathbf{x}^*) \eta_\gamma^-(\mathbf{x}^*)}{\delta_{\nu_\gamma}^\gamma \bar{\Delta}'_\gamma - P(\mathbf{x}^*)}, & \text{if } \delta_{\nu_\gamma}^\gamma \bar{\Delta}'_\gamma - P(\mathbf{x}^*) < 0. \end{cases} \quad (4.8)$$

Therefore, we have proved the following theorem:

**Theorem 4.1.** *If  $\delta$  satisfies (4.2), (4.5), (4.6), (4.7), (4.8) and  $g_j(x_j)$  is concave, then  $\mathbf{x}^*$  is an optimal solution of the perturbed PLFP problem (where  $d_{\nu_\gamma}^\gamma \rightarrow d_{\nu_\gamma}^\gamma + \delta$ ).*

Case 2.  $d_i^j$  is the coefficient of a basic variable. Thus the replacement  $d_{\mu(B_k)}^{B_k} \rightarrow d'_{\mu(B_k)} = d_{\mu(B_k)}^{B_k} + \delta$  affects the optimal value of  $D(\mathbf{x})$  as well as  $Z(\mathbf{x})$

$$\begin{aligned} \hat{D}(\mathbf{x}^*) &= \mathbf{d}_B \mathbf{B}^{-1} \mathbf{b} + \delta \beta_k \cdot \mathbf{b} + \sum_{j \in N} (d_{\nu_j}^j - \mathbf{d}_B \mathbf{B}^{-1} A_{.j}) \delta_{\nu_j}^j - \delta \sum_{j \in N} \beta_k \cdot A_{.j} \delta_{\nu_j}^j + \beta_0 = \\ &= D(\mathbf{x}^*) + \delta \beta_k \cdot (\mathbf{b} - \sum_{j \in N} A_{.j} \delta_{\nu_j}^j). \end{aligned}$$

To preserve the strict positivity of the denominator  $D(\mathbf{x})$ , we need to have

$$D(\mathbf{x}^*) + \delta \beta_k \cdot (\mathbf{b} - \sum_{j \in N} A_{.j} \delta_{\nu_j}^j) > 0. \quad (4.9)$$

Therefore, we will have

$$\delta \begin{cases} > \frac{-D(\mathbf{x}^*)}{\beta_k \cdot (\mathbf{b} - \sum_{j \in N} A_{.j} \delta_{\nu_j}^j)}, & \text{if } \beta_k \cdot (\mathbf{b} - \sum_{j \in N} A_{.j} \delta_{\nu_j}^j) > 0, \\ < \frac{-D(\mathbf{x}^*)}{\beta_k \cdot (\mathbf{b} - \sum_{j \in N} A_{.j} \delta_{\nu_j}^j)}, & \text{if } \beta_k \cdot (\mathbf{b} - \sum_{j \in N} A_{.j} \delta_{\nu_j}^j) < 0. \end{cases} \quad (4.10)$$

In addition, the replacement  $d_{\mu(B_k)}^{B_k} \rightarrow d'_{\mu(B_k)}$  has an effect on the non-basic reduced costs:

$$\begin{aligned} d_{\nu_j}^j - \mathbf{d}'_B \mathbf{B}^{-1} A_{.j} &= d_{\nu_j}^j - \mathbf{d}_B \mathbf{B}^{-1} A_{.j} - \delta \beta_k \cdot A_{.j} = \Delta_j'' - \delta \beta_k \cdot A_{.j}, \\ d_{\nu_{j-1}}^j - \mathbf{d}'_B \mathbf{B}^{-1} A_{.j} &= d_{\nu_{j-1}}^j - \mathbf{d}_B \mathbf{B}^{-1} A_{.j} - \delta \beta_k \cdot A_{.j} = \bar{\Delta}_j'' - \delta \beta_k \cdot A_{.j}. \end{aligned}$$

Therefore, to satisfy the optimality condition, we can determine the new values  $\eta_j^+$  and  $\eta_j^-$  as

$$\begin{aligned} \hat{\eta}_j^+(\mathbf{x}^*) &= \Delta_j' - \frac{P(\mathbf{x}^*)}{D(\mathbf{x}^*) + \delta \beta_k \cdot (\mathbf{b} - \sum_{j \in N} A_{.j} \delta_{\nu_j}^j)} (\Delta_j'' - \delta \beta_k \cdot A_{.j}) = \\ &= \frac{\Delta_j' (D(\mathbf{x}^*) + \delta \beta_k \cdot (\mathbf{b} - \sum_{j \in N} A_{.j} \delta_{\nu_j}^j)) - P(\mathbf{x}^*) (\Delta_j'' - \delta \beta_k \cdot A_{.j})}{D(\mathbf{x}^*) + \delta \beta_k \cdot (\mathbf{b} - \sum_{j \in N} A_{.j} \delta_{\nu_j}^j)} \geq 0. \end{aligned} \quad (4.11)$$

From (4.9), the relation (4.11) is satisfied if

$$\Delta_j' (D(\mathbf{x}^*) + \delta \beta_k \cdot (\mathbf{b} - \sum_{j \in N} A_{.j} \delta_{\nu_j}^j)) - P(\mathbf{x}^*) (\Delta_j'' - \delta \beta_k \cdot A_{.j}) \geq 0.$$

Therefore, we have

$$\begin{aligned} & \max_{j \in N} \left\{ \frac{-D(\mathbf{x}^*) \eta_j^+(\mathbf{x}^*)}{\beta_k \cdot (A_{.j} P(\mathbf{x}^*) + (\mathbf{b} - \sum_{j \in N} A_{.j} \delta_{\nu_j}^j) \Delta_j')} : \beta_k \cdot (A_{.j} P(\mathbf{x}^*) + (\mathbf{b} - \sum_{j \in N} A_{.j} \delta_{\nu_j}^j) \Delta_j') > 0 \right\} \\ & \leq \delta \leq \\ & \min_{j \in N} \left\{ \frac{-D(\mathbf{x}^*) \eta_j^+(\mathbf{x}^*)}{\beta_k \cdot (A_{.j} P(\mathbf{x}^*) + (\mathbf{b} - \sum_{j \in N} A_{.j} \delta_{\nu_j}^j) \Delta_j')} : \beta_k \cdot (A_{.j} P(\mathbf{x}^*) + (\mathbf{b} - \sum_{j \in N} A_{.j} \delta_{\nu_j}^j) \Delta_j') < 0 \right\}. \end{aligned} \quad (4.12)$$

Similarly, if  $\hat{\eta}_j^-(\mathbf{x}^*) \leq 0$ , we will get

$$\begin{aligned} & \max_{j \in N} \left\{ \frac{-D(\mathbf{x}^*) \eta_j^-(\mathbf{x}^*)}{\beta_k \cdot (A_{.j} P(\mathbf{x}^*) + (\mathbf{b} - \sum_{j \in N} A_{.j} \delta_{\nu_j}^j) \bar{\Delta}_j')} : \beta_k \cdot (A_{.j} P(\mathbf{x}^*) + (\mathbf{b} - \sum_{j \in N} A_{.j} \delta_{\nu_j}^j) \bar{\Delta}_j') < 0 \right\} \\ & \leq \delta \leq \\ & \min_{j \in N} \left\{ \frac{-D(\mathbf{x}^*) \eta_j^-(\mathbf{x}^*)}{\beta_k \cdot (A_{.j} P(\mathbf{x}^*) + (\mathbf{b} - \sum_{j \in N} A_{.j} \delta_{\nu_j}^j) \bar{\Delta}_j')} : \beta_k \cdot (A_{.j} P(\mathbf{x}^*) + (\mathbf{b} - \sum_{j \in N} A_{.j} \delta_{\nu_j}^j) \bar{\Delta}_j') > 0 \right\}. \end{aligned} \quad (4.13)$$

Thus, we have the following theorem:

**Theorem 4.2.** *If  $\delta$  satisfies (4.10), (4.12), (4.13) and  $g_j(x_j)$  is concave, then  $\mathbf{x}^*$  is an optimal solution for the perturbed PLFP problem (with  $d_{\mu(B_k)}^{B_k} \rightarrow d'_{\mu(B_k)}^{B_k}$ ).*

**Example 4.3.** Consider Example 2.3. For the given optimal basis and solution, we consider the following two cases:

**Non-basic index:** Let  $d_1^3 \rightarrow d_1^3 + \delta$ . In this case,  $\gamma = 3$  and  $N = \{3\}$ . Since  $\gamma \neq j \in N$ , therefore the relations (4.5) and (4.7) are not applicable. Hence, from (4.2), (4.6), (4.8) and concavity of  $g_j(x_j)$  we will have

$$\delta > \frac{-139}{2} = -13.9,$$

$$\delta \gamma_{\nu_3} \Delta'_3 - P(\mathbf{x}^*) = -23 < 0 \Rightarrow \delta \leq \frac{-139(1.031)}{-23} = 1.246,$$

$$\delta \gamma_{\nu_3} \bar{\Delta}'_3 - P(\mathbf{x}^*) = -25 < 0 \Rightarrow \delta \geq \frac{-139(-0.854)}{-25} = -0.949,$$

$$1 \leq 2 + \delta \leq 3 \Rightarrow -1 \leq \delta \leq 1.$$

Hence, we obtain the following range for  $\delta$ :

$$-0.961 \leq \delta \leq 1.$$

**Basic index:** Let  $d_1^1 \rightarrow d_1^1 + \delta$ . In this case using (4.10), (4.12), (4.13) and the concavity of  $g_j(x_j)$  we will have

$$\beta_3(\mathbf{b} - \mathbf{A}_{.3}\delta_1^3) = \frac{16}{5} > 0 \Rightarrow \delta > \frac{-139}{\frac{16}{5}} = -8.685,$$

$$\beta_3(A_{.3}P(\mathbf{x}^*) + (\mathbf{b} - A_{.3}\delta_{\nu_3}^3)\Delta'_3) = \frac{62}{5} > 0 \Rightarrow \delta \leq \frac{-139(1.031)}{\frac{62}{5}} = -2.311,$$

$$\beta_3(A_{.3}P(\mathbf{x}^*) + (\mathbf{b} - A_{.3}\delta_{\nu_3}^3)\bar{\Delta}'_3) = \frac{46}{5} > 0 \Rightarrow \delta \leq \frac{-139(-0.854)}{\frac{46}{5}} = 2.581,$$

$$3 + \delta \leq 4 \Rightarrow \delta \leq 1.$$

Finally, the following range is obtained for  $\delta$ :

$$-2.311 \leq \delta \leq 1.$$

## 5. SUMMARY

The sensitivity analysis of optimal solutions has been presented in this paper. Three cases were considered: (i) changes in the right-hand-side vector, (ii) changes in the coefficients of the numerator of the objective function, (iii) changes in the coefficients of the denominator of the objective function. In each case the underlying theory for sensitivity analysis has been presented to obtain the bounds for each perturbation.



## REFERENCES

- [1] S.P. Aggarwal, *Parametric Linear Fractional Functionals Programming*, *Metrika* **12** (1968), 106–114.
- [2] S.P. Aggarwal, *Analysis of the Solution to a Linear Fractional Functionals Programming*, *Metrika* **16** (1970), 9–26.
- [3] E.B. Bajalinov, *Linear Fractional Programming: Theory, Methods, Applications and software*, Kluwer Academic Publishers, 2003.
- [4] G.E. Bitran, T.L. Magnanti, *Duality and sensitivity analysis with fractional objective function*, *Operations Research* **24** (1976), 675–699.
- [5] V. Chavatal, *Linear programming*, W.H. Freeman and Company, New York, 1983.
- [6] G.B. Dantzig, M.N. Thapa, *Linear Programming, 1: Introduction*, Springer, 1997.
- [7] R. Fourer, *A simplex algorithm for piecewise-linear programming I: Derivation and proof*, *Mathematical programming* **13** (1985), 204–233.
- [8] B. Kheirfam, M. Mirnia, *Multi-parametric sensitivity analysis in piecewise linear fractional programming*, *Journal of Industrial and Management Optimization* **4** (2008) 2, 343–351.
- [9] B. Kheirfam, M. Mirnia, *Approximate multi-parametric sensitivity analysis of the constraint matrix in piecewise linear fractional programming*, *AMO – Advanced Modeling and Optimization* **11** (2009) 2, 139–151.
- [10] A. Punnen, P. Pandey, *A simplex algorithm for piecewise-linear fractional programming problems*, *European Journal of Operational Research* **178** (2007), 343–358.
- [11] S. Schaible, T. Ibaraki, *Fractional programming*, *European Journal of Operational Research* **12**(1983), 247–255.
- [12] K. Swarup, *Linear fractional programming*, *Operations Research* **10** (1965), 380–387.
- [13] H.F. Wang, C.S. Huang, *Multi-parametric analysis of the maximum tolerance in a linear programming problem*, *European Journal of Operational Research* **67** (1993), 75–87.

Behrouz Kheirfam  
b.kheirfam@azaruniv.edu

Azarbijan University of Tarbiat  
Department of Mathematics  
Moallem  
Tabriz, Iran

*Received: November 9, 2008.*

*Revised: May 27, 2009.*

*Accepted: June 9, 2009.*