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ANALYSIS OF CLEMENTS' METHOD FOR CAPABILITY INDICES ESTIMATION

1. INTRODUCTION

In a practical approach the statistical methods in quality management are first of all: statistical process control (SPC), measurement system analysis (MSA), acceptance sampling techniques and process and product reliability assessment. Statistical methods take particular place in a process improvement; in this scope the following statistical instruments are applied: hypotheses and test procedures, analysis of variance (ANOVA), regression and correlation analysis, design of experiment (DOE) etc. [1–4].

One of the fundamental tasks of SPC is the assessment of capability of a process/ machine relating to client's expectations. In this scope many different capability indices are applied in practice, e.g. Cp, Cpk, Pp, Ppk, Cm, Cmk, Tp, Tpk [5– 10].

This work concerns the comparative analysis of capability indices determination in case of the distribution unlike the normal one.

2. CAPABILITY PROCESS ESTIMATION IN CASE OF DISTRIBUTION UNLIKE THE NORMAL ONE

It is stressed widely that SPC methods are situated in an "avoiding loss" convention – information achieved by these methods we use "on-line" in order to assure a desirable performance of the process (instead of concentrate on process effects that not always are desirable and expected, we focus on the process that generates these results!).

In case of distributions unlike the normal a natural tolerance – i.e. the range in which practically all possible realization of the process should be contained – we define as [6, 8, 10]:

$$T_n = x_{0.99865} - x_{0.00135} \tag{1}$$

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where:

$x_{0.00135}$ – value meeting the condition: $P(x < x_{0.00135}) = 0.00135$,

$x_{0.99865}$ – value meeting the condition $P(x < x_{0.99865}) = 0.99865$ (i.e. $P(x >= x_{0.99865}) = 0.00135$) (Fig. 1).

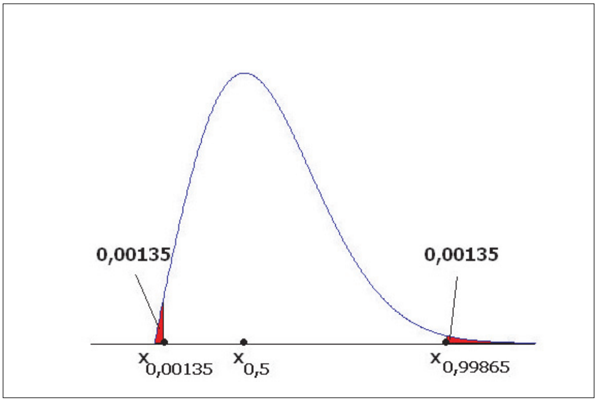

Fig. 1. *Non-normal distribution; definition of natural tolerance*

Let's give attention that the definition (1) guarantees exactly the same as the 6s range (*s* – standard deviation) in case of a normal distribution, i.e. within the range determined by a natural tolerance practically all possible realizations of the process should be contained, and participation of the realization outside the range determined by natural tolerance limits equals 0.0027 (i.e. 2700 ppm).

Additionally, let $x_{0.5}$ means the median ($P(x < x_{0.5}) = 0.5$) which, in case of an asymmetric distribution, will be a measure of midpoint of grouping of values (of course, in case of symmetric distributions e.g. a normal distribution there is the equality $x_{0.5} = \bar{x}$, where: $\bar{x}$ – mean value).

Based on the values defined above, i.e. $x_{0.00135}$, $x_{0.5}$, $x_{0.99865}$ process capability indices *Cp*, *Cpk* we define as below (Tab. 1) [6, 8].

Table 1. Capability indices for non-normal distribution – calculation formulae

Specification limits	potential capability Cp	real capability Cpk
double-sided specification limits	$\frac{USL - LSL}{x_{0,99865} - x_{0,00135}}$ (2)	$\min\left(\frac{x_{0,5} - LSL}{x_{0,5} - x_{0,00135}}; \frac{USL - x_{0,5}}{x_{0,99865} - x_{0,5}}\right)$ (3)
lower specification limit only	N / A	$\frac{x_{0,5} - LSL}{x_{0,5} - x_{0,00135}}$ (4)
upper specification limit only	N / A	$\frac{USL - x_{0,5}}{x_{0,99865} - x_{0,5}}$ (5)
Symbols: USL – upper specification limit LSL – lower specification limit N / A – not available		

There are two methods of determination of the values $x_{0.00135}$, $x_{0.5}$, $x_{0.99865}$ [6, 10]:

– Clements' method; it is an approximate method based on values of shape parameters i.e. kurtosis and skewness; in this method there is no need to know the form of distribution of the analyzed parameter.
– Exact method requiring knowledge of a density function $f(x)$ determining a distribution of the analyzed parameter; in this case the values $x_{0.00135}$, $x_{0.5}$, $x_{0.99865}$ we determine from the relationships:

$$\int_{-\infty}^{x_{0.00135}} f(x)dx = 0.00135 \tag{6}$$

$$\int_{-\infty}^{x_{0.5}} f(x)dx = 0.5 \tag{7}$$

$$\int_{-\infty}^{x_{0.99865}} f(x)dx = 0.99865 \tag{8}$$

3. CLEMENTS' METHOD

In order to determine the capability indices (Table 1, formulae 2–5) it is necessary to determine the values $x_{0.00135}$, $x_{0.5}$, $x_{0.99865.}$ For that purpose Clements proposed a method basing on the Pearson curves [6, 8].

The course of proceeding in the Clements' method is the following:

– on the basis of database we determine a mean value ($\overline{x}$), a standard deviation (s), skewness (Sk) and kurtosis (Ku),
– values $x_{0.00135}$, $x_{0.5}$, $x_{0.99865}$ are being determined adequately from the formulae:

$$x_{0.00135} = \overline{x} - k_{0.00135} \cdot s \tag{9}$$

$$x_{0.99865} = \overline{x} + k_{0.99865} \cdot s \tag{10}$$

$$x_{0.5} = \begin{cases} \overline{x} - k_{0.5} \cdot s; & Sk > 0 \\ \overline{x} + k_{0.5} \cdot s; & Sk < 0 \end{cases} \tag{11}$$

where: $k_{0,00135}$, $k_{0,5}$, $k_{0,99865}$ – indices depending on skewness and kurtosis; read from the tables, indices Cp, Cpk are determined according to adequate expressions placed in Table 1 (formulae 2–5).

4. EXPERIMENTAL PROCEDURE

4.1. Aim, subject of investigations

The aim of testing was the evaluation of accuracy of the Clements' method. The procedure was as follows:

- for the selected theoretical distributions, at the assumed parameters, on the basis of a probability density function a mean value, standard deviation, skewness and kurtosis were calculated
- also on the basis of a probability density function, using the expressions (6), (7), (8) the values $x_{0.00135}$, $x_{0.5}$, $x_{0.99865}$ were determined (i.e. they are theoretical values)
- the values $x_{0.00135}$, $x_{0.5}$, $x_{0.99865}$ were determined again using the Clements' method (formulae 9–11)
- a comparison of the theoretical values and adequate values achieved with the Clements' method was performed.

4.2. Theoretical distributions

In the analysis carried out the following theoretical distributions have been used:

- exponential distribution; the probability density function is

$$f(x) = \lambda e^{-\lambda x}; \quad x \geq 0 \tag{12}$$

where λ – parameter of distribution

- lognormal distribution; the probability density function is

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left[-\frac{(\ln x - m)^2}{2\sigma^2}\right]; \quad x \geq 0 \tag{13}$$

where:

m – mean value,
σ – standard deviation

- gamma distribution; the probability density function is

$$f(x) = x^{k-1} \frac{\exp(-x/b)}{\Gamma(k)\, b^k}; \quad x \geq 0 \tag{14}$$

where:

k – shape parameter,
b – scale parameter

- beta distribution; the probability density function is

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1}; \quad x \in [0,1] \tag{15}$$

where:

α – shape parameter,

β – shape parameter

– Weibull distribution; the probability density function is

$$f(x) = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} e^{-(x/\lambda)^k}; \quad x \geq 0 \tag{16}$$

where:

k – shape parameter,

λ – scale parameter.

Density probability functions of selected distributions for the assumed parameters are presented in Figures 2–7. The fundamental description parameters are in Table 2.

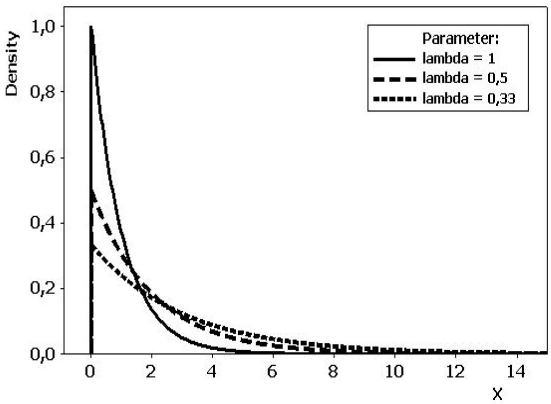

Fig. 2. *Exponential distribution; probability density function*

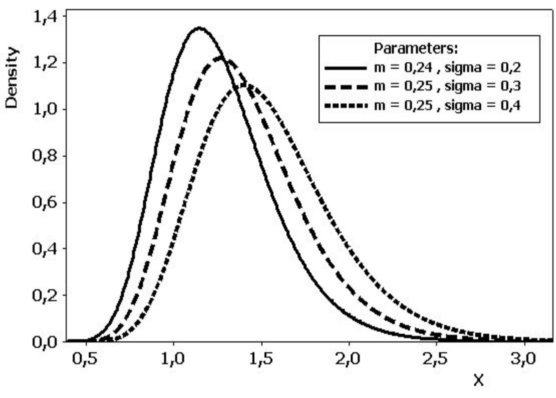

Fig. 3. *Lognormal distribution; probability density function*

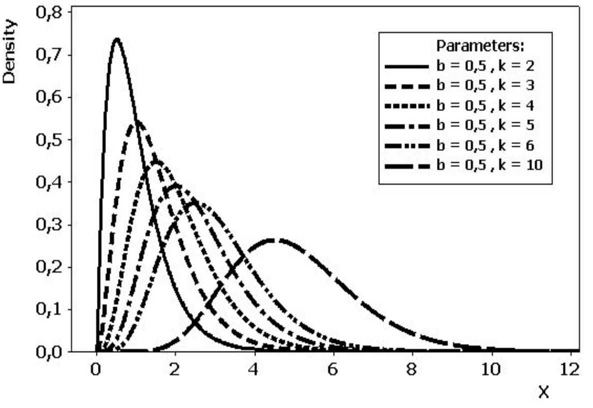

Fig. 4. *Gamma distribution; probability density function*

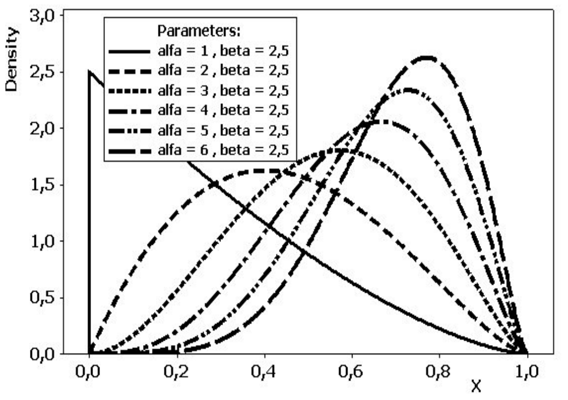

Fig. 5. *Beta distribution; probability density function*

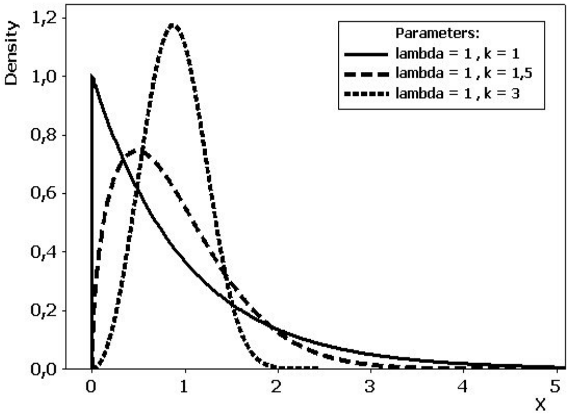

Fig. 6. *Weibull distribution; probability density function*

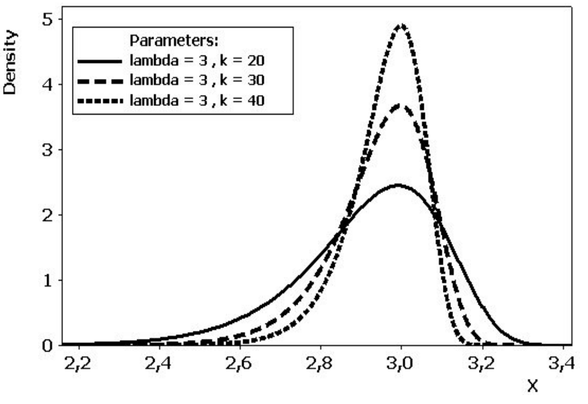

Fig. 7. *Weibull distribution; probability density function*

Table 2. Theoretical distribution – descriptive statistics

Distribution	Parameters	Descriptive Statistics				
		mean value	median	standard deviation	skewness	kurtosis
Exponential	λ=1	1.0	0.6931	1.0	2	6
	λ=0.5	1.0	1.3863	2.0	2	6
	λ=0.333	3.0	2.0794	3.0	2	6
Lognormal	m=0.25; σ=0.2	1.2840	0.2646	0.2646	0.6143	0.6784
	m=0.25; σ=0.3	1.2840	0.4122	0.4122	0.9495	1.6449
	m=0.25; σ=0.4	1.2840	0.5794	0.5794	1.3219	3.2600
Gamma	b=0.5; k=2	1.0	0.83917	0.7071	1.4142	3
	b=0.5; k=3	1.5	1.33703	0.8660	1.1547	2
	b=0.5; k=4	2.0	1.83603	1.0	1.0	1.5
	b=0.5; k=5	2.5	2.33545	1.1180	0.8944	1.2
	b=0.5; k=6	3.0	2.88351	1.2247	0.8165	1
	b=0.5; k=10	5.0	4.83436	1.5811	0.6324	0.6
Beta	α=1; β=2.5	0.2857	0.24214	0.2130	0.7318	–0.2434
	α=2; β=2.5	0.4444	0.43555	0.2119	0.1614	–07661
	α=3; β=2.5	0.5454	0.55133	0.1953	–0.1241	–0.6855
	α=4; β=2.5	0.6154	0.62787	0.1776	–0.3056	–0.5062
	α=5; β=2.5	0.6667	0.68214	0.1617	–0.4340	–0.3158
	α=6; β=2.5	0.7059	0.72262	0.1478	–0.5305	–0.1362
Weibull	λ=1; k=1	1.0	0.6931	1.0	2	6
	λ=1; k=1.5	0.9027	0.7832	0.6129	1.0720	1.3904
	λ=1; k=3	0.8930	0.8850	0.3245	0.1681	–0.2705
	λ=3; k=20	2.9205	2.9455	0.1810	–0.8680	1.2672
	λ=3; k=30	2.9455	2.9636	0.1230	–0.9531	1.5841
	λ=3; k=40	2.9585	2.9726	0.0932	–0.9975	1.7630

Table 3. Comparative analysis

Distribution	Parameters	Theoretical value			According to Clements			m_1	m_2	m_3
		$x_{0.00135}$	$x_{0.5}$	$x_{0.99865}$	$x_{0.00135}$	$x_{0.5}$	$x_{0.99865}$			
Exponential	λ=1	0.00135	0.69315	6.60765	0.001	0.693	6.608	0.9997	0.9999	0.9999
	λ=0.5	0.00270	1.38629	13.2153	0.002	1.386	13.216	0.9997	0.9999	0.9999
	λ=0.333	0.00405	2.07944	19.823	0.003	2.079	19.824	0.9997	0.9999	0.9999
Lognormal	m=0.25; σ=0.2	0.57695	1.28402	2.58569	0.70613	1.2840	2.3389	1.2236	1.2303	1.2339
	m=0.25; σ=0.3	0.63763	1.28402	2.85763	0.52737	1.2838	3.1563	0.8545	0.8444	0.8404
	m=0.25; σ=0.4	0.70469	1.28402	3.15817	0.40342	1.2834	4.2581	0.6583	0.6365	0.6300
Gamma	b=0.5; k=2	0.02644	0.83917	4.4501	0.02499	0.8389	4.4483	0.9985	1.0001	1.0004
	b=0.5; k=3	0.10584	1.33703	5.4348	0.10377	1.3363	5.4315	0.9989	1.0002	1.0006
	b=0.5; k=4	0.23265	1.83603	6.3402	0.233	1.836	6.338	1.0002	1.0004	1.0005
	b=0.5; k=5	0.39594	2.33545	7.1962	0.3866	2.3361	7.2038	0.9949	0.9975	0.9986
	b=0.5; k=6	0.58749	2.88351	8.0174	0.58813	2.8349	8.0147	1.0219	1.0004	0.9911
	b=0.5; k=10	1.54213	4.83436	11.0879	1.54488	4.8338	11.084	1.0010	1.0007	1.0005
Beta	α=1; β=2.5	0.00054	0.24214	0.92886	–0.0003	0.2415	0.9271	0.9989	1.0010	1.0017
	α=2; β=2.5	0.01772	0.43555	0.95635	0.01832	0.4355	0.9552	1.0016	1.0019	1.0021
	α=3; β=2.5	0.06041	0.55133	0.96826	0.06153	0.5514	0.9676	1.0021	1.0019	1.0017
	α=4; β=2.5	0.11465	0.62787	0.97501	0.11558	0.6279	0.9748	1.0017	1.0013	1.0007
	α=5; β=2.5	0.17067	0.68214	0.97938	0.17199	0.6823	0.9791	1.0023	1.0020	1.0014
	α=6; β=2.5	0.22412	0.72262	0.98244	0.22529	0.7228	0.9823	1.0021	1.0017	1.0010
Weibull	λ=1; k=1	0.00135	0.69315	6.60765	0.001	0.693	6.608	0.9997	0.9999	0.9999
	λ=1; k=1.5	0.00203	0.78322	3.52126	–0.0308	0.7824	3.5116	0.9606	0.9934	1.0032
	λ=1; k=3	0.00405	0.88500	1.8765	0.07823	0.8825	1.8541	1.0953	1.0544	1.0205
	λ=3; k=20	2.15602	2.94552	3.29704	2.15439	2.9453	3.2814	0.9982	1.0124	1.0459
	λ=3; k=30	2.407	2.96357	3.19489	2.40721	2.9636	3.1839	1.0004	1.0144	1.0497
	λ=3; k=40	2.54324	2.97264	3.14501	2.54355	2.9727	3.1367	1.0006	1.0144	1.0505

4.3. Comparative analysis

Values $x_{0.00135}$, $x_{0.5}$, $x_{0.99865}$ determined theoretically for different distributions and different parameters (according to formulae 6–8) and on the basis of the Clements' method (according to formulae 9–11) are presented in Table 3.

Because these values serve to determine the capability indices *C*p, *Cpk* (Tab. 1), additionally – in this context – the following consistency measures have been defined:

$$m_1 = \frac{x_{0.5}(theor) - x_{0.00135}(theor)}{x_{0.5}(Clements) - x_{0.00135}(Clements)} \tag{17}$$

$$m_2 = \frac{x_{0.99865}(theor) - x_{0.00135}(theor)}{x_{0.99865}(Clements) - x_{0.00135}(Clements)} \tag{18}$$

$$m_3 = \frac{x_{0.99865}(theor) - x_{0.5}(theor)}{x_{0.99865}(Clements) - x_{0.5}(Clements)} \tag{19}$$

In the meaning of these total consistency measures it is the value 1.

The values of measures m1, m2, m3 for different distributions are in Table 3.

5. SUMMARY

With the exception of the lognormal distribution the results achieved during investigation confirm a very good accuracy of the Clements' method. A very good conformity of values $x_{0.00135}$, $x_{0.5}$, $x_{0.99865}$ determined theoretically and with the Clements' method, as well as defined consistency measures (Tab. 3) indicate the truth of this statement.

In case of the lognormal distribution the operating of the Clements' method is worse (Tab. 3).

The comparative analysis performed should enable the Clements' method users its optimal application.

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