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PERIODIC CONTROL OF THE BAR WITH POINT MASS

SUMMARY

The following article discusses the usage of optimization theory with quasi-periodic control to form a vibrating bar with point mass that is situated on the bar's axis. The maximum principle was changed and used to calculate a numerical sample. The principle uses the new condition for optimisation containing several constituent hamiltonians. The hamiltonians facilitated the calculation of the quasi-periodic controls. The result of the calculation was a numerical sample, which specified how the quasi-periodic cross-section of the bar depends on the mass and the point of the mass position along the bar's axis.

Keywords: *quasi-periodic control, longitudinal vibration, bar, point mass*

STEROWANIE OKRESOWE PRĘTEM Z MASĄ PUNKTOWĄ

W artykule zaprezentowano zastosowanie teorii optymalizacji z quasi-okresowym sterowaniem do kształtowania drgającego podłużnie pręta z masą punktową umieszczoną na jego osi. Do obliczeń wykorzystana została zmieniona zasada maksimum, w której występuje nowy warunek optymalności z sumą hamiltonianów składowych pozwalający wyznaczyć quasi-okresowe sterowania. W pracy przeliczony został numeryczny przykład, w którym określono, w jaki sposób quasi-okresowy przykrój poprzeczny pręta zależy od wielkości masy i punktu jej przyłożenia na osi tego pręta.

Słowa kluczowe: *quasi-okresowe sterowanie, drgania podłużne, pręt, masa punktowa*

1. ASSUMPTIONS AND DEFINITIONS

In order to solve the problem with the periodic control we can apply the modified maximum principle (Piekarski 2006; Nizioł and Piekarski 2007). In this one the new optimality condition has form

$$H_{opt}(x_q) = \max_{u \in U} \sum_{q=1}^{N} H^q(x_q) \tag{*}$$

Partial hamiltonians H^q, q = 1, 2, ..., N are defined for each subintervals of interval [0, L] in which the control functions are periodical. For N = 1 condition (*) is the Pontriagin's optimality condition.

The basic aim of this chapter is to introduce notions which are relevant to the further analysis. The whole interval of the independent variable x is divided into two subintervals by means of point L_1.

$$0 < L_1 < L.$$

In the case of the optimization with the quasi-periodic control we must assume that the following lengths are various:

$$\begin{aligned} \Delta L_1 &= L_1 - L_0 = L_1, \qquad L_0 = 0 \\ \Delta L_2 &= L_2 - L_1 = L - L_1, \;\; L_2 = L \end{aligned} \tag{1}$$

In the subintervals q, the variable x is denoted by x_q

$$x_q \in [L_q, L_{q-1}] \tag{2}$$

or

$$\begin{aligned} &x_q = \Delta L_q \tau + L_{q-1}, \\ &0 \le \tau \le 1, \end{aligned} \qquad q = 1, 2 \tag{3}$$

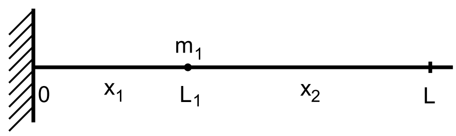

Fig. 1. Cantilever bar with the point mass

The system of state equations for the longitudinal vibration bar (Fig. 1) has the form:

$$\begin{aligned} \frac{d}{dx_q} W_1(x_q) &= \frac{1}{EF(x_q)} W_2(x_q) \\ \frac{d}{dx_q} W_2(x_q) &= -\rho\omega^2 F(x_q) W_1(x_q) \end{aligned} \tag{4}$$

where $W_1(x_q)$ and $W_2(x_q)$ are the displacements and forces into subintervals (1) and (2), $F(x_q)$ is the cross-section of the bar on the point x_q.

If at the point L_1 of the contact of the subintervals we locate the mass m_1, then continuous conditions take form (see (Nizioł 1966)):

$$\begin{aligned} W_1(L_1 + 0) &= W_1(L_1 - 0) \\ W_2(L_1 + 0) - W_2(L_1 - 0) &= -m_1 \omega^2 W_1(L_1 - 0) \end{aligned} \tag{5}$$

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We introduce the following non-dimensional variables and constants:

$$z_q = \frac{x_q}{L}, \quad w_1(z_q) = \frac{W_1}{L}, \quad w_2(z_q) = \frac{W_2}{EF_0}, \quad \phi = \frac{F}{F_0}$$
$$l_q = \frac{L_q}{L}, \quad \varphi = \frac{\rho\omega^2 L^2}{E}, \quad \beta = \frac{m_1}{\rho L F_0} \tag{6}$$

Quantities ρ, ω are the density and vibration frequency of the bar. Quantity β is determined by the point mass m_1.

Taking into consideration the relation between (3) and (6) it is assumed that (Piekarski 2006):

$$w_k(z_q) = w_k(z_q(\tau)) = w_k^q(\tau), \quad q, k = 1, 2 \tag{7}$$

In our case we found that the cross-section of the bar is quasi-periodic, thereby we fulfil the generalized periodicity conditions (Piekarski 1994, 2006; Nizioł and Piekarski 2007)

$$F(x_1) = F(x_2) = F(\tau) \tag{8}$$

or

$$\phi(z_1) = \phi(z_2) = a(\tau) \tag{9}$$

where

$$a(\tau) = \frac{F(\tau)}{F_0} \tag{10}$$

is the basic control.

After applying (6), (7) and (9) to (3), we get

$$z_q = B^q \tau + l_{q-1} \tag{11}$$

where

$$B^q = \frac{\Delta L_q}{L} = \Delta l_q \tag{12}$$

and applying to (4), we get:

$$\frac{d}{d\tau} w_1^q = \frac{B^q}{a(\tau)} w_2^q(\tau)$$
$$\frac{d}{d\tau} w_2^q = -B^q \varphi a(\tau) w_1^q(\tau) \tag{13}$$

Conditions (5) have now the shape:

$$w_1^2(0) = w_1^1(1)$$
$$w_2^2(0) - w_2^1(1) = -\varphi\beta w_1^1(1) \tag{14}$$

For cantilever bar the state variables fulfil the following boundary conditions:

$$w_1^1(0) = 0$$
$$w_2^2(1) = 0 \tag{15}$$

2. SUBJECT OF OPTIMIZATION

Our aim is to obtain the minimum of the cost functional (the volume of the bar). From (3), (6), (9) we have

$$J = \int_0^L F(x)dx = LF_0 \left[\int_0^{l_1} \phi(z_1)dz_1 + \int_{l_1}^1 \phi(z_2)dz_2 \right] \tag{16}$$

We assume that the optimal control satisfies the normalization condition as in (Gajewski and Piekarski 1994; Gajewski and Życzkowski 1988). From (3), (10), (12)

$$\left[\int_0^{l_1} \phi(z_1)dz_1 + \int_{l_1}^1 \phi(z_2)dz_2 \right] =$$
$$= \int_0^1 a(\tau)B^1 d\tau + \int_0^1 a(\tau)B^2 d\tau = 1 \tag{17}$$

thus, from (16) it follows that (because from (6) is l = 1)

$$J = LF_0 = LF_0 \left[\int_0^{l_1} \phi(z_1)dz_1 + \int_{l_1}^1 \phi(z_2)dz_2 \right] =$$
$$= LF_0 \int_0^1 B^1 d\tau + LF_0 \int_0^1 B^2 d\tau \tag{18}$$

In order to carry out the optimization with the quasi-periodic control, we have to introduce the total Hamiltonian

$$H(\tau) = H^1 + H^2 = \sum_q H^q(\tau) \tag{19}$$

Where constituent hamiltonians H^q have the form

$$H^q = B^q \left[-LF_0 + \frac{\lambda_1^q w_2^q}{a(\tau)} - \varphi a(\tau)\lambda_2^q w_1^q + \lambda' a(\tau) \right] \tag{20}$$

The constant λ' is the Lagrange's multiplier which considers the conditions (17).

The modified Pontryagin's maximum principle for the optimization without periodic state variables but with the quasi-periodic control is determined by Hamiltonians (19), (20). Adjoint variables $\lambda_k^q(\tau)$ balance the adjoint equations

$$\frac{d}{d\tau}\lambda_k^q = -\frac{\partial H^q}{\partial w_k^q(\tau)} \tag{21}$$

In our case the following is true:

$$\frac{d}{d\tau}\lambda_1^q = B^q \varphi a \lambda_2^q$$
$$\frac{d}{d\tau}\lambda_2^q = -\frac{B^q}{a}\lambda_1^q \tag{22}$$

If, at a contact point of subintervals, the state variables have a jump (see (Piekarski 2006))

$$w_k^2(0) - w_k^1(1) = h_k(\bar{w}^1(1)) \tag{23}$$

then the adjoint variables experience the jump too

$$\lambda_k^2(0) - \lambda_k^1(1) = -\left[\lambda_1^2(0)\frac{\partial h_1}{\partial w_k^1(1)} + \lambda_2^2(0)\frac{\partial h_2}{\partial w_k^1(1)}\right] \tag{24}$$

From (14) follows:

$$h_1 = 0, \quad h_2 = -\varphi\beta w_1^1(1) \tag{25}$$

From formulae (24) and (25) we obtain:

$$\begin{aligned} &\lambda_1^2(0) - \lambda_1^1(1) = \varphi\beta\lambda_2^2(0) \\ &\lambda_2^2(0) - \lambda_2^1(1) = 0 \end{aligned} \tag{26}$$

Boundary conditions for adjoint variables are following (from (15) and transversality conditions (Piekarski 1994, 2006)):

$$\begin{aligned} &\lambda_2^1(0) = 0 \\ &\lambda_1^2(1) = 0 \end{aligned} \tag{27}$$

The research problem is auto-adjoint. The dependences:

$$\begin{aligned} &\lambda_1^q = k w_2^q \\ &\lambda_2^q = -k w_1^q \end{aligned} \tag{28}$$

bring the system equations (26) to (13), boundary conditions (27) to (15) and continuous conditions (26) to (14). The partial hamiltonians (20) have the form

$$H^q = B^q\left[-LF_0 + \frac{k(w_2^q)^2}{a} + \varphi a k (w_1^q)^2 + \lambda' a\right] \tag{29}$$

The optimality condition with the total hamiltonian in the modified maximum principle in our case has the shape

$$\frac{\partial}{\partial a}(H^1 + H^2) = \frac{\partial}{\partial a}\sum_q H^q = 0 \tag{30}$$

From (30) we get the formula

$$a = \sqrt{\frac{C_2(\tau)}{\lambda + \varphi C_1(\tau)}} \tag{31}$$

where λ is the optional constant. Functions C_1 i C_2 have the form:

$$\begin{aligned} &C_1 = B^1(w_1^1)^2 + B^2(w_1^2)^2 \\ &C_2 = B^1(w_2^1)^2 + B^2(w_2^2)^2 \end{aligned} \tag{32}$$

Finally, in order to solve the research problem it is necessary to resolve the system equations (13) with conditions (14) and (15). The control function a is defined by (31).The constant λ is determined by the normalization condition (17) in the form

$$\int_0^1 a(\tau)d\tau = 1 \tag{33}$$

3. NUMERICAL CALCULATIONS

A problem of longitudinal vibration of the cantilever bar that was divided into two subintervals was considered as a numerical sample. The mass point is situated in the subintervals contact point l_1, $0 < l_1 < l$. Three cases of optimal bar forming were calculated – one periodical

$$L_1 = \frac{1}{2}$$

with boundary conditions (15), and two quasi-periodic ones

$$L_1 = \frac{1}{4} \text{ and } L_1 = \frac{3}{4}$$

also with the conditions (15). In the considered sample the frequency of the bar is changed in order to fulfil the condition

$$W_1^1(0) = 1.$$

In order to fulfil the condition while using the dependences (28) we have to consider that the constant k (which can be of any kind) has a value of one ($k = 1$). It does not change the formula (31), which specifies the optimal section and the boundary conditions (15), or continuity conditions (14).

In all cases of the bar's optimization the mass is situated in the contact point of subintervals. Its existence is illustrated by its presence in conditions (14) of the β factor.

In order to achieve the possibility to compare the numerical results, we accept that in all cases of optimisation (for every l_1) we have

$$\beta = 0.05;\ 0.2;\ 0.5;\ 1.$$

To solve the system of state equations with imposed continuous and boundary conditions we applied the method of successive approximations using Mathematica 5 programme. For approximation no. 0 of the cross-section ($a_0 = 1$) the normalization condition (33) is tautologically fulfilled. After solving the equations and defining the constants λ_0 and ρ_0 we can find the first approximation of the basic section

$$a_1 = a_1(\tau).$$

For this section we must repeat the procedure. The process of iteration should be finished when λ and ρ is not changeable any more.

The graphical illustrations (Figs 2–7) of the results are specified below:

1) for $l_1 = 0.25$ we have $\varphi = 2.6; 2.5; 2.3; 2$ for $\beta = 0.05; 0.2; 0.5; 1$
   a) state variables in function of z diagram (Fig. 2);
   b) quasi-periodic control in function of z diagram (Fig. 3);
2) for $l_1 = 0.5$ we have $\varphi = 2.6; 2.3; 1.8; 1.3$ for $\beta = 0.05; 0.2; 0.5; 1$
   c) state variables in function of z diagram (Fig. 4);
   d) periodic control in function of z diagram (Fig. 5);
3) for $l_1 = 0.75$ we have $\varphi = 2.7; 2.1; 1.5; 0.9$ for $\beta = 0.05; 0.2; 0.5; 1$
   e) state variables in function of z diagram (Fig. 6);
   f) quasi-periodic control in function of z diagram (Fig. 7).

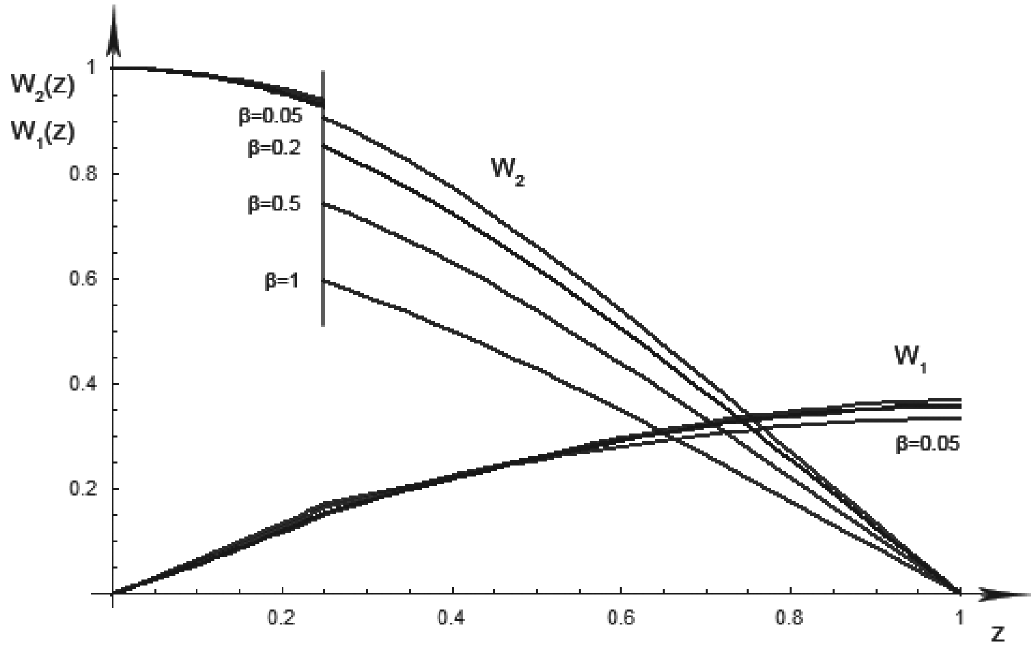

Fig. 2. Dependence of displacements and forces of z for different β

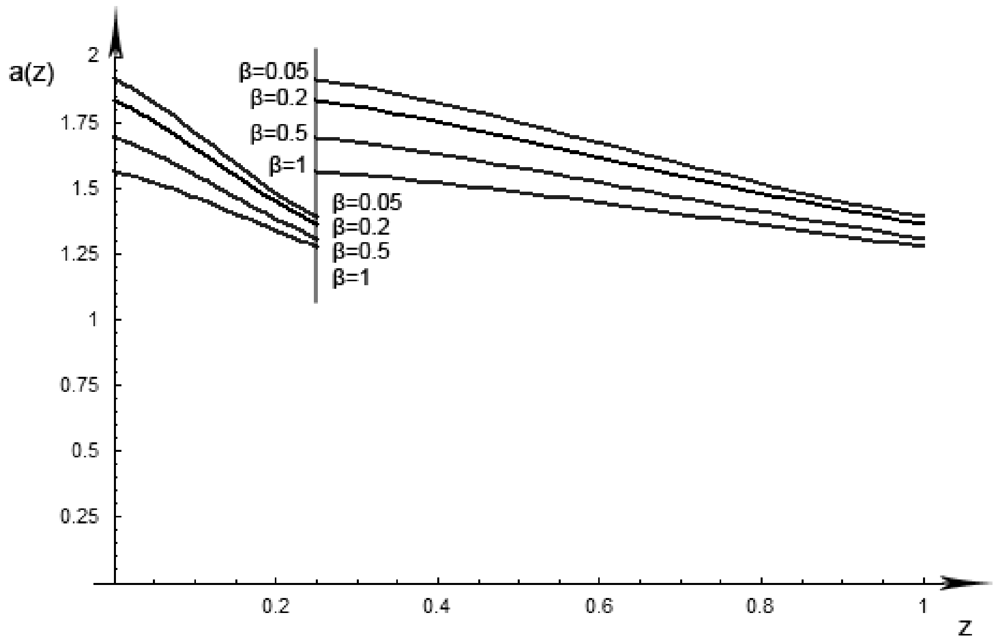

Fig. 3. Shapes of quasi-periodic basic controls for different β

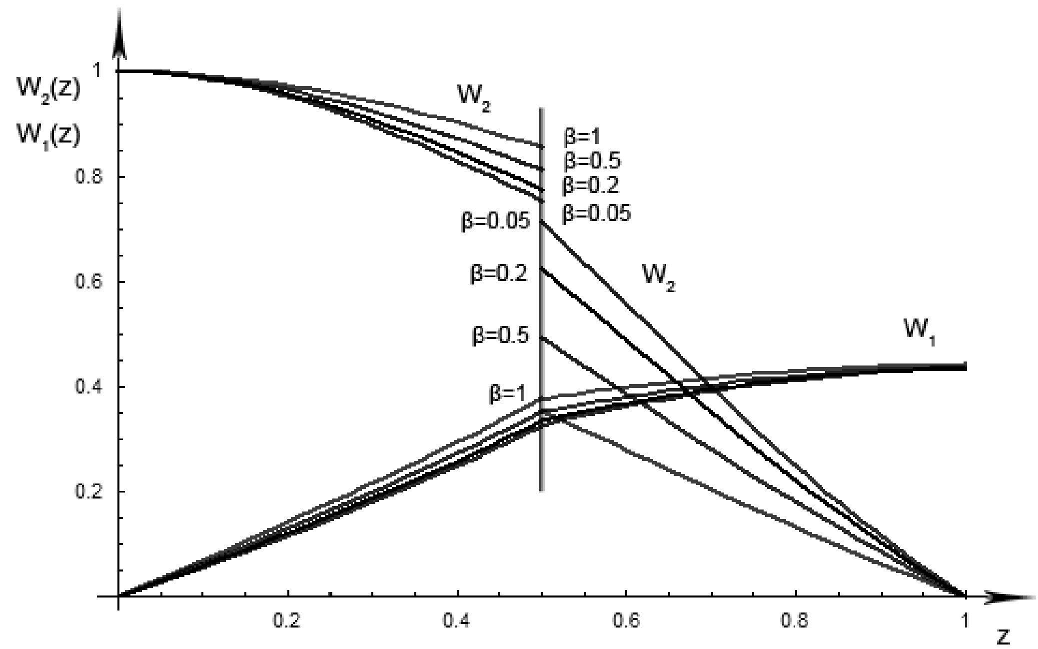

Fig. 4. Dependence of displacements and forces of z for different β

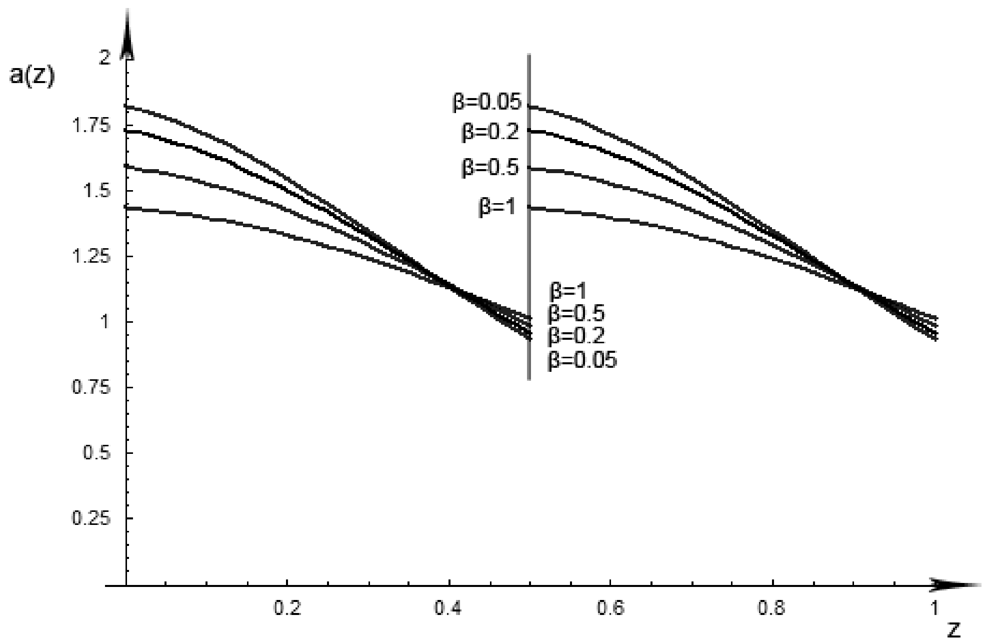

Fig. 5. Shapes of periodic basic controls for different β

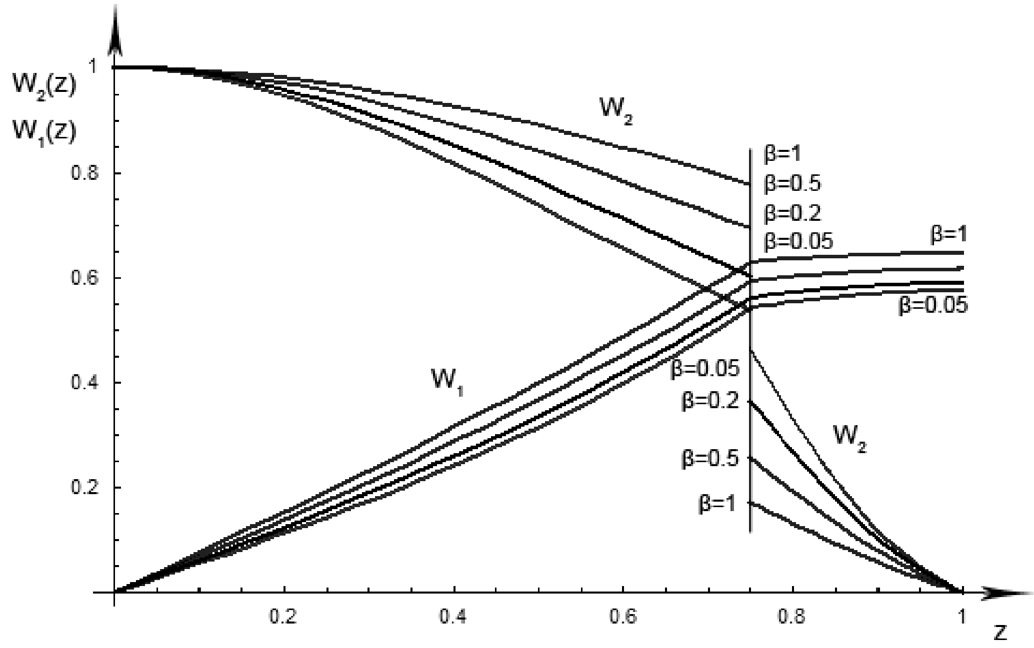

Fig. 6. Dependence of displacements and forces of *z* for different β

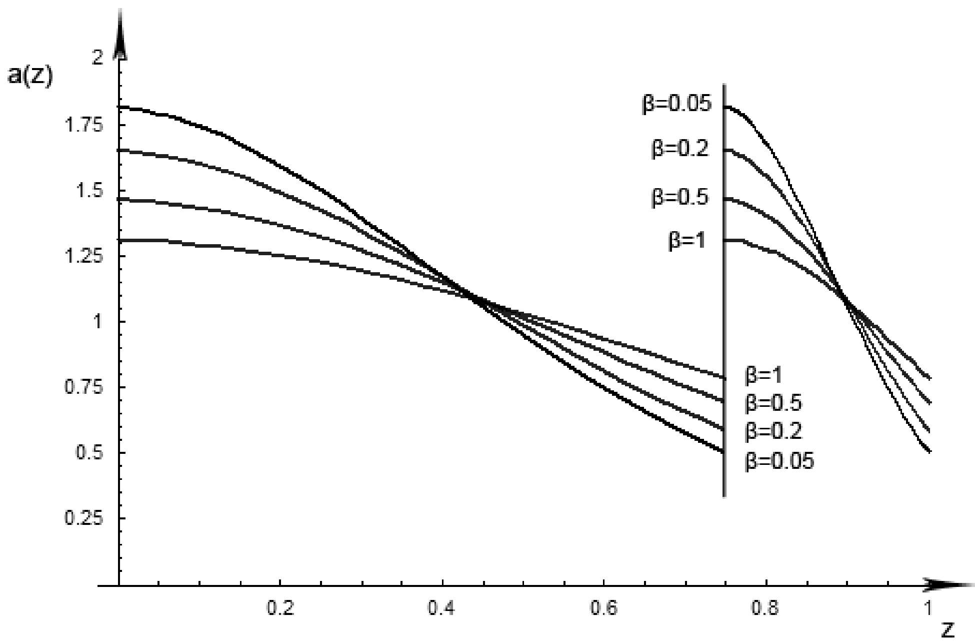

Fig. 7. Shapes of quasi-periodic basic controls for different β

4. CONCLUSIONS

Taking all of the above points into consideration, all illustrated diagrams show that the position of the mass points on the beam and the mass value have a significant influence on longitudinal forces, which act on the beam. Furthermore, the mass point position influences the quasi-periodic control shapes (sections of the beam). In all cases the mass is situated in the contact point of the subintervals. Diagrams (Figs 2, 4 and 6) show that in the first subinterval, for different masses there is no significant change in the forces for $l_1 = 0.25$. However, when l_1 is increasing the difference between forces is increasing too.

In the second subinterval, the difference between forces is large for every l_1. When l_1 and â are increasing, the values of the change in the force are increasing very strongly too. The displacements are not changing very strongly when β is changing. Diagrams (Figs 3, 5 and 7) show graphical illustration of quasi-periodic cross-sections of the beam. For every l_1 and β the difference between controls is quite significant in every subinterval. It is easily noticed that from a certain point of l_1 optimal control functions intersect depending on the mass.

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