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# INFLUENCE OF NONLINEAR FOUNDATION ON TIMOSHENKO BEAM VIBRATION

*SUMMARY*

*The goal of this paper is presentation of the physical properties of the thick beam on nonlinear foundation with infrequent analyzed free boundary condition. Galerkin's method with two-elements basis is applied. This paper justifies this choice too.*
*Nonlinear effects such as internal resonance and discontinuous of amplitude-frequency characteristic is observed. The proper choice of parameters ensures the ratio first to second natural frequencies equal to 3. This is one of the conditions satisfying existing of the internal resonance. In the amplitude-frequency characteristic near first resonance there appear breaks and their numbers grows with the increase of nonlinearity of the foundation.*

***Keywords***: *Timoshenko's beam, internal resonance, forced vibration, nonlinear vibration, Galerkin's method*

*WPŁYW NIELINIOWEGO PODŁOŻA NA DRGANIA BELKI TIMOSHENKI*

*Celem pracy jest opis właściwości fizycznych krępej belki wspartej na nieliniowym podłożu, przy rzadko analizowanych warunkach brzegowych opisujących swobodne końce. Zastosowano metodę Galerkina opartą na dwuelementowej bazie. Praca zawiera uzasadnienie takiego wyboru.*
*Zaobserwowano efekty nieliniowe takie jak rezonans wewnętrzny oraz nieciągłości charakterystyki amplitudowo--częstotliwościowej. Odpowiedni dobór parametrów pozwala uzyskać stosunek pierwszej i drugiej częstości własnej rzędu 3, co jest jednym z warunków istnienia rezonansu wewnętrznego. Pokazano też, że na charakterystyce w pobliżu pierwszego rezonansu pojawiają się „przerwy", których liczba zwiększa się wraz ze wzrostem udziału nieliniowości w oddziaływaniu podłoża.*

***Słowa kluczowe***: *belka Timoshenki, rezonans wewnętrzny, drgania wymuszone, drgania nieliniowe, metoda Galerkina*

## 1. INTRODUCTION

In the literature concerning systems with a continuous mass distribution there are two kinds of papers: theoretical and engineering applications ones. Analytical description of dynamical behavior of such system is sometimes difficult. This is, for example, the case when boundary condition have a complicated form, which happens quite often. The difficulty further increases when the cross sectional dimensions are large, as in the case of a thick beam. In this case, both the shearing force as well as rotational inertia of cross section need to be taken into account. The number of boundary conditions also grows, since the order of differential equation is higher and the conditions themselves become more complex.

One of the usual ways to describe a thick beam is Timoshenko's model. Among the literature devoted to this model one should list (Bar 2004, Bar 2007, Bogacz *et al.* 1994, Grzyb 1993). These papers either account for nonlinear foundation of the beam or analyze the behavior of the system under moving loads.

In view of the non-existence of a general theory of solutions of nonlinear differential equations, appropriate methods are used, most often the Ritz and Galerkin method. Among others, A. Srinivasan has shown that these methods lead to satisfactory results in the nonlinear case for a beam with simple supports (Srinivasan 1995) and a fixed beam (Nizioł and Bar 2005).

Most of the existing work dealing with the dynamics of thick beams concerns the cases with one or both ends fixed or simply supported. There exist relatively few papers on the analysis of the case with both ends free. It has motivated the authors of this work to consider this topic. In real dynamical beam systems one cannot restrict oneself to the analysis of boundary conditions alone. This is the case when the beam interacts with other structural elements along its length. In case of big disproportion of masses one can model this as a beam resting on a foundation. The analysis of the influence of the nonlinear foundation on the behavior of the system, with the above mentioned boundary conditions, is the goal of this work. It has been assumed that nonlinearity is restricted only to the foundation. The linear case has also been investigated as it serves to check the correctness of the interpretation of the wider context and allows proper selection of the base functions in the Galerkin's method used in the nonlinear case.

## 2. MODEL OF THE SYSTEM

### 2.1. Thimoshenko beam

The model of the considered system is shown in Figure 1.

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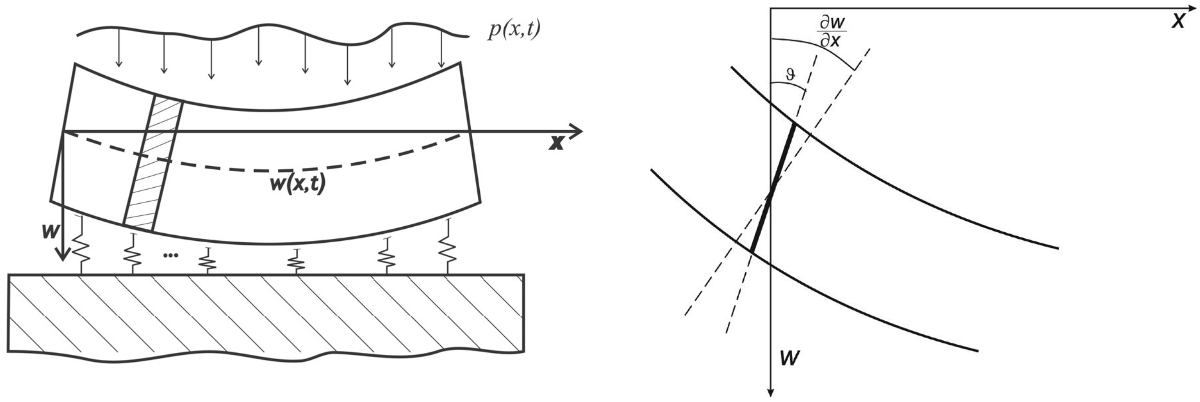


**Fig. 1.** Model of the beam resting on the elastic foundation

The boundary conditions for bending moment $M$ and the shearing force $T$ at the ends of beam can be written as follows:

$$M = EJ\vartheta_x; \qquad M(0) = M(l) = 0$$
$$T = -\frac{GA}{k}(w_x - \vartheta); \quad T(0) = T(l) = 0 \tag{1}$$

where $E$ and $G$ are Young's and Kirchhoff's modulus respectively, $k$ is share coefficient.

With the increasing of cross section beam radius the influence of dynamical and geometrical effects causes that the equation of a Bernoulli beam does not describe properly the physical state of the system. The most important of these effects is the influence of rotational cross section inertia and geometrical shear deformation of the beam cross section. In Timoshenko's beam model, contrary to Bernoulli's beam, the additional angle variable is taken into account. As a result of this the section rotation angle $\vartheta$ is an independent variable.

The most frequently considered models of the foundation are one parametric Winkler model and two parametric Winkler-Pasternak model.

In the present paper Winkler's model modified by an additional nonlinear component has been chosen. The adopted assumptions concerning the interaction between the foundation and the beam are as follows:

– the constraints are two-sided (the reaction to the compression and tension is symmetrical),
– there is no damping in the constraints and beam material ( the friction between the beam and foundation does not exist), this means that the reaction has the same direction as $w$ axis

Additionally, it has been assumed that the vibration energy dissipation takes place only in the foundation and it is proportional to the velocity of the transversal beam deflection.

In accordance with the adopted assumptions there is a reaction force of the foundation directed towards $w$, which can be expressed as:

$$q(w,\dot{w}) = -(\alpha w + \beta w^3 + \varepsilon \dot{w}) \tag{2}$$

Taking (2) into account the equations of the vibration of a Timoshenko beam resting on a foundation, forced by a harmonic force, have the following form:

$$w_{xx} - \frac{k\rho}{G} w_{tt} + \frac{k}{GA} q(w,\dot{w}) = \vartheta_x - \frac{k}{GA} p_0 \sin(\frac{\pi}{l}x)\sin(\nu t)$$
$$\vartheta_{xx} - \frac{\rho}{E}\vartheta_{tt} = -\frac{GA}{kEJ}(w_x - \vartheta) \tag{3}$$

## 2.2. Equations of motion in dimensionless variables

In the further analysis the dimensionless quantities are used:

$$\tau = \omega_0 t; \quad \xi = \frac{x}{l}; \quad \eta = \frac{w}{w_0}$$
$$\varphi = \frac{\vartheta}{\vartheta_0}; \quad \omega_0^2 = \frac{EJ\pi^4}{\rho A l^4} \tag{4}$$

The approximation of the deflection $w$ and angle $\vartheta$ has been done based on the static relations result from Euler--Bernoulli theory. It was assumed that a simply supported beam is subjected to the concentrated force acting in the middle of beam. Than the deflection is equal to $w_0 = \frac{Pl^3}{48EJ}$, and the rotation angle of cross-section at the support point is equal to $\vartheta_0 = \frac{Pl^2}{16EJ}$ (for $P = 10^7$ N static deflection $w$ = 0.57 mm).

The quantities $w_0$ and $\vartheta_0$ are connected by the formula:

$$\frac{\vartheta_0 l}{w_0} = 3.$$

Additionally the dimensionless coefficients $\tilde{k}$, $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{p}$ and $\gamma$ are introduced:

$$\tilde{k} = k\frac{E}{G}; \quad \alpha = \tilde{\alpha}EJ\frac{\pi^4}{l^4}; \quad \beta = \tilde{\beta}\frac{\alpha}{w_0^2}; \quad \tilde{p}_0 = \frac{\tilde{k}p_0}{4\pi\gamma^2 Ew_0};$$

$$\tilde{\nu} = \frac{\nu}{\omega_0}; \quad \gamma = \frac{r}{2l}; \quad 2\omega_0\zeta = \frac{\varepsilon}{\rho A}.$$

Thus aquation (3) transforms into:

$$\eta_{\xi\xi} - \pi^4\tilde{k}\gamma^2\eta_{\tau\tau} - \tilde{\alpha}\tilde{k}\pi^4\gamma^2\eta - 2\tilde{k}\pi^4\gamma^2\zeta\eta_\tau -$$

$$-3\varphi_\xi - \tilde{\alpha}\tilde{\beta}\tilde{k}\pi^4\gamma^2\eta^3 = \tilde{p}_0\sin(\pi\xi)\sin(\tilde{\nu}\tau) \qquad (5)$$

$$\varphi_{\xi\xi} - \pi^4\gamma^2\varphi_{\tau\tau} + \frac{1}{3\tilde{k}\gamma^2}\eta_\xi - \frac{1}{\tilde{k}\gamma^2}\varphi = 0$$

In considering the dimensionless coefficients it is necessary to take into account that $\gamma$ and $\tilde{\alpha}$ are not independent since:

$$\alpha = \tilde{\alpha}EJ\frac{\pi^4}{l^4} = \tilde{\alpha}E\pi^5\gamma^4.$$

Therefore, when the thickness of beam changes the equivalent elasticity coefficient should be modified according to the formula:

$$\tilde{\alpha} = \tilde{\alpha}_0\left(\frac{\gamma_0}{\gamma}\right)^4.$$

## 3. LINEAR CASE

### 3.1. Frequencies and modes

Modal analysis in linear case, when damping is neglected, can be done analytically. The functions:

$$\eta_n(\xi,\tau) = X_n(\xi)\sin(\omega_n\tau + \phi_{0n});$$

$$\varphi_n(\xi,\tau) = \Phi_n(\xi)sin(\omega_n\tau + \phi_{0n})$$

form a sequence of linearly independent solutions, which determine eigenfrequencies $\omega_n$ and eigenvectors:

$$\mathbf{Y_n}^T(\xi) = [X_n(\xi), \Phi_n(\xi)] \qquad (6)$$

They satisfy equations (1) written in the dimensionless coordinates:

$$\Phi_n'(0) = \Phi_n'(1) = 0$$
$$X_n'(0) - 3\Phi_n(0) = X_n'(1) - 3\Phi_n(1) = 0 \qquad (7)$$

They were determined by Bar and are presented in the paper (Bar 2007).

Two first frequencies ($\omega_1$ , $\omega_2$) related to the modes describe rotation about mass center and translatory motion, respectively.

They are placed near each other, contrary to the Bernoulli beam, when they are equal. Therefore, the motion of a technical beam is a rigid body motion unlike for a non-slender beam which undergoes deformation. These deformations are small and differences between eigenfrequencies are negligible. The next two frequencies correspond to modes which take into account visible elastic deformation.

The modes determined above will be used as the base functions in the Galerkin method. The excitation assumed in this paper is of symmetrical character and therefore the Galerkin base should be symmetric, too. This fact allows to disregard antisymmetrical functions connected with the first and fourth eigenfrequencies. The graph below (Fig. 2) presents the shapes of modes connected with the second and third eigenfrequencies. These modes will be elements of the base for the Galerkin's method.

The modes satisfy the orthogonality condition (determined in (Bar 2007)):

$$\int_0^1 \mathbf{X_n^T N_0 X_m}\, d\xi = 0 \ \text{ for } n \neq m; \quad \mathbf{N_0} = \begin{bmatrix} 1 & 0 \\ 0 & 9\gamma^2 \end{bmatrix} \qquad (8)$$

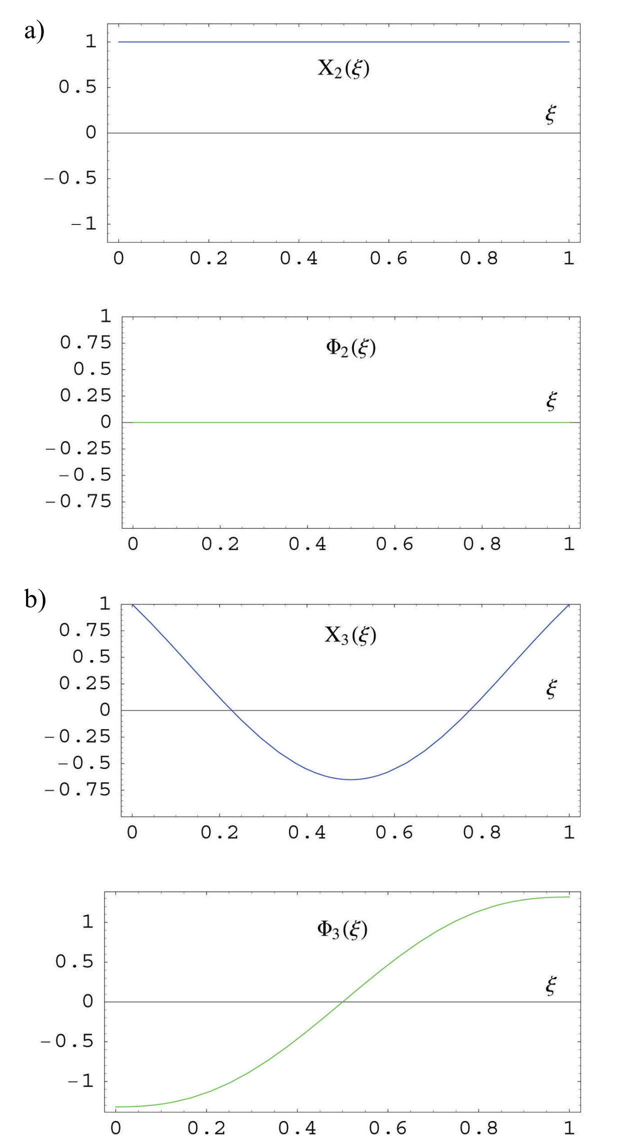


**Fig. 2.** Comparison between the second (a) and third modes (b), $\alpha = 0.2$, $\gamma = 0.104$, $\omega_2 = \sqrt{\alpha}$, $\omega_3 = 1.73$

### 3.2. Forced vibration in the linear case

It has been assumed that the excitation is harmonic and has a harmonic spatial variation. The equation of motion of the damped beam under this kind of excitation can be written in the form:

$$\eta_{\xi\xi} - \pi^4\tilde{k}\gamma^2\eta_{\tau\tau} - \pi^4\tilde{k}\gamma^2\frac{\varepsilon}{\rho A}\eta_\tau - \tilde{\alpha}\tilde{k}\pi^4\gamma^2\eta - 3\varphi_\xi =$$

$$= p_0\sin(\pi\xi)e^{i\tilde{\nu}\tau} \qquad (9)$$

$$\varphi_{\xi\xi} - \pi^4\gamma^2\varphi_{\tau\tau} + \frac{1}{3\tilde{k}\gamma^2}\eta_\xi - \frac{1}{\tilde{k}\gamma^2}\varphi = 0$$

The solutions are sought in the form:

$$\eta(\xi,\tau) = F(\xi)e^{i\tilde{\nu}\tau};$$

$$\varphi(\xi,\tau) = H(\xi)e^{i\tilde{\nu}\tau}.$$

Where $F$ and $H$ are complex functions of real argument, and $\nu \in \Re$. These functions have to satisfy the boundary conditions (7).

The motion is given by the imaginary part of the obtained solution and for example the deflection has the following form:

$$Im(Fe^{i\nu\tau}) = |F|\sin(\tilde{\nu}\tau + Arg[F]) \qquad (10)$$

The response of the system to harmonic excitation, when transient states are disregarded, is described well by means of the amplitude-frequency characteristics, which are the graph of $|F(\nu)|$. The analysis of amplitude was done at the beam middle point as well as at its ends (Fig. 3).

The maximum values of amplitudes, shown in the above graphs, correspond to eigenfrequencies: second, third and fifth. Due to small deflection the phenomena of antiresonance, which appears near $\nu = \omega_4$, is not visible.

The resonances related to the first and fourth eigenfrequencies do not appear. It is the result of the antisymmetrical character of the modes corresponding to these frequencies. Therefore, the symmetric excitation does not cause the vibration with these frequencies.

The graphs below (Fig. 4) show the argument along the beam and the shape of the deflection function $\eta(\xi, \tau)$in the selected time moments $\tau$ for the system under excitation equal to $\nu = \omega_3$.

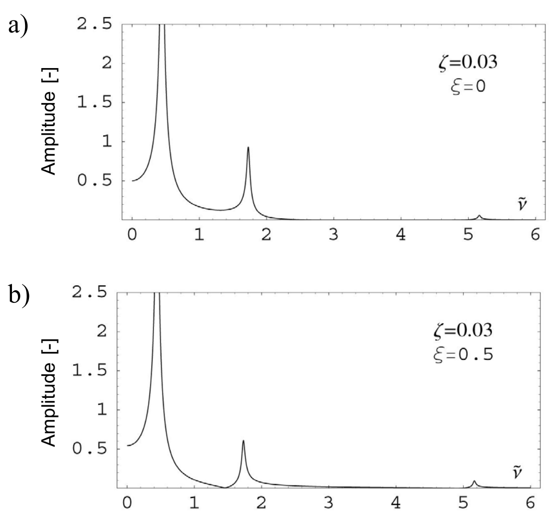


**Fig. 3.** Amplitude-frequency characteristics at the end (a) and in the middle (b) point of beam under damping $\zeta = 0.03$

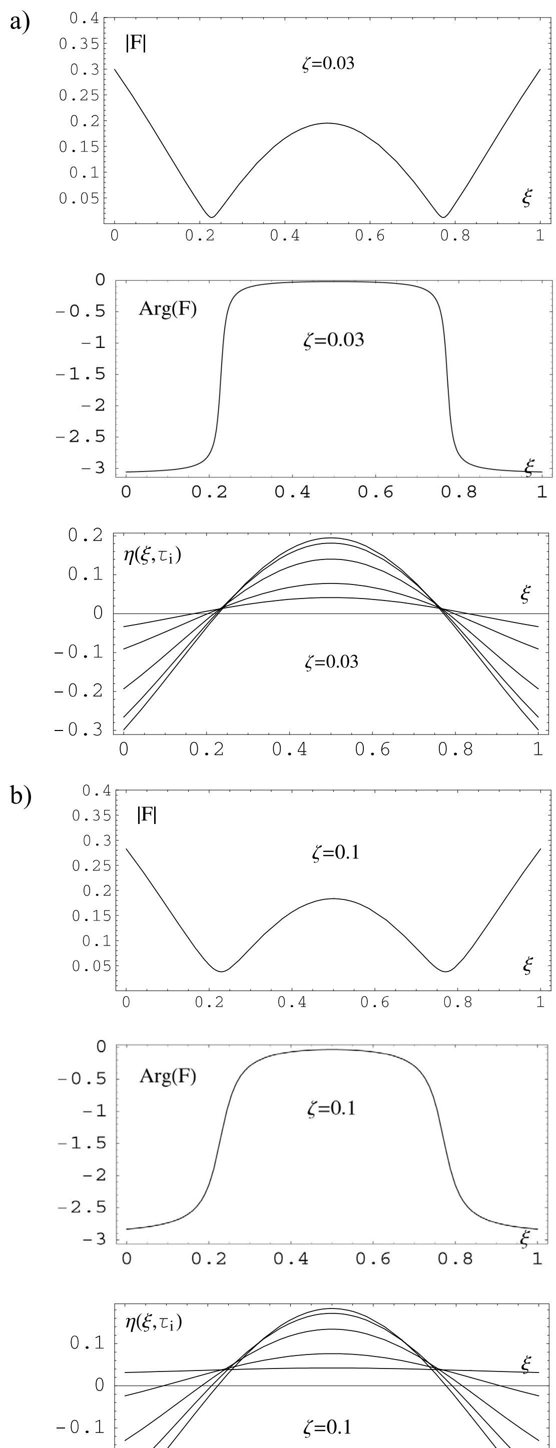


**Fig. 4.** Modulus and phase shift of the system response versus excitation and the effect of "moving antinodes" for damping $\zeta = 0.03$ and $\zeta = 0.01$ ((a) and (b) respectevely)

Damping causes the points of the beam move out of the phase. Thus, the zeros of the shape function “float”, i.e., are no longer stationary, even if the damping is minimal. The “floating” effect persist in the very short time-scale, when the damping is small. The stronger the damping is, the easier one can observe the effect. since the phase changes also occur for the strong deflected beam

The analysis of the linear system serves as an introduction to nonlinear analysis and is the justification of the choice of the base in the Galerkin method.

This is proved in the Figure 5. The shape error of the second and third modes in comparison with the steady-state displacements, for $v$ equal to $\omega_2$ and $\omega_3$ is presented. This error is about $10^{-4}$ and $10^{-3}$. It justifies the construction of the base.

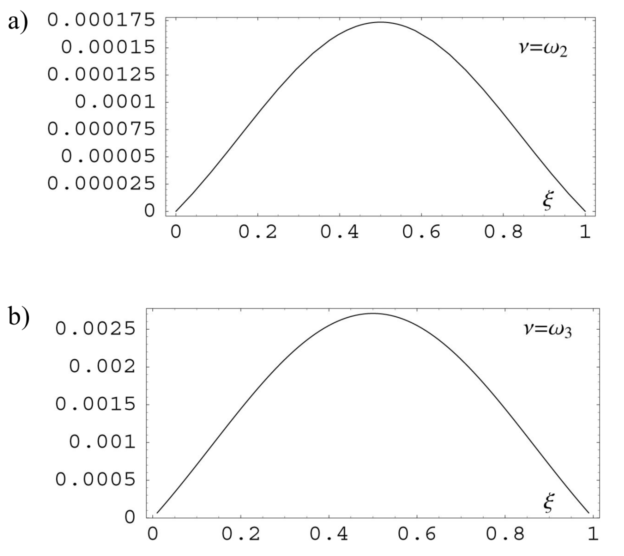


**Fig. 5.** Difference between the modes and the envelope of the beam shape in the state of second (a) and third resonance (b) for damping $\zeta = 0.03$

## 4. NONLINEAR FORCED VIBRATION

### 4.1. Choice of base and the form of the eqations

As mentioned above, as a result of the assumption of the symmetrical excitation, the two-element base, consisting of two eigenfunction, has been chosen (6). The first of these functions corresponds to the second frequency (translational) while the other function to the third one (with symmetrical deflection). The first and the fourth modes are irrelevant for the solution as they are antisymmetric.

The correctness of this choice is justified by the fact, that the solutions of the linear system under excitation are equal to eigenfunctions in the neighborhood of resonance when damping is neglected. In this paper we will compare the characteristics obtained by the Galerkin method using the chosen base functions with those obtained exactly.

Since the considered system is described by two coupled nonlinear ordinary differential equations the internal resonance may occur. This phenomena was presented in (Niziołand Bar 2007) for a simply supported beam. Due to the nonlinearity proportional to the third power of deflection the internal resonance can take place when the ratio of eigenfrequencies is close to 3. It can be achieved by choosing the appropriate parameters $\tilde{\alpha}$ and $\tilde{\beta}$. According to the Galerkin method the approximate solution of equation (5) can be written in the form:

$$\begin{bmatrix} \tilde{\eta} \\ \tilde{\varphi} \end{bmatrix} = \sum_{n=1}^{2} \mathbf{Y_n}(\xi) u_n(\tau) \quad \text{where: } \mathbf{Y_n} = \begin{bmatrix} X_n \\ \Phi_n \end{bmatrix} \tag{11}$$

Using the above formulas we get:

$$\sum_{i=1}^{2} \left[ \mathbf{Y_i''} u_i - \mathbf{D_1 Y_i} \ddot{u}_i - \mathbf{D_2 Y_i} \dot{u}_i - \mathbf{D_3 Y_i} u_i - \mathbf{D_4 Y_i'} u_i \right] - \\ - \mathbf{N(Y_1, Y_2)} \cong \mathbf{f} \tag{12}$$

where:

$$\mathbf{D_1} = \pi^4 \gamma^2 \begin{bmatrix} \tilde{k} & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{D_2} = 2\tilde{k}\pi^4\gamma^2\zeta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{D_3} = \begin{bmatrix} \tilde{k}\pi^4\gamma^2\tilde{\alpha} & 0 \\ 0 & \dfrac{1}{\tilde{k}\gamma^2} \end{bmatrix}$$

$$\mathbf{D_4} = \begin{bmatrix} 0 & -3 \\ \dfrac{1}{3\tilde{k}\gamma^2} & 0 \end{bmatrix} \tag{13}$$

$$\mathbf{N} = \begin{bmatrix} \tilde{k}\pi^4\gamma^2\tilde{\alpha}\tilde{\beta}(X_1u_1 + X_2u_2)^3 \\ 0 \end{bmatrix}$$

$$\mathbf{f} = \begin{bmatrix} p_0 \sin(\pi\xi)\sin(\nu\tau) \\ 0 \end{bmatrix}$$

By **N** we denote the nonlinear component of the operator acting on the base vectors.

Applying an inner product to equations (12) we obtain the set of ordinary differential equations with unknown functions of time $u_1$ and $u_2$:

$$\begin{aligned} & a_{11}u_{1''}(\tau) + a_{21}u_{1'}(\tau) + a_{31}u_1(\tau) + a_{31}\tilde{\beta}(u_1^3(\tau) + \\ & \quad + 3a_1u_1(\tau)u_2^2(\tau) + a_9u_2^3(\tau)) = q_{01}\sin(\nu\tau) \\ & a_{21}u_{2''}(\tau) + a_{22}u_{2'}(\tau) + a_{32}u_2(\tau) + \\ & \quad + a_{31}\tilde{\beta}(3a_1u_1^2(\tau)u_2(\tau) + 3a_9u_1(\tau)u_2^2(\tau) + \\ & \quad + a_{10}u_2^3(\tau)) = q_{02}\sin(\nu\tau) \end{aligned} \tag{14}$$

**Table 1.** Coefficients of equations

| Coefficient | Formula | Coefficient | Formula |
|---|---|---|---|
| $a_1$ | $\int_0^1 X_2^2 d\xi$ | $a_{10}$ | $\pi^4\gamma^2(\tilde{k}a_1 + a_2)$ |
| $a_2$ | $\int_0^1 \Phi_2^2 d\xi$ | $a_{22}$ | $2\tilde{k}\pi^4\gamma^2 a_1$ |
| $a_3$ | $\int_0^1 X_2 \Phi_{2'} d\xi$ | $a_{32}$ | $\tilde{k}\alpha\pi^4\gamma^2 a_1 + \frac{1}{\tilde{k}\gamma^2}a_2 + 3a_3 - \frac{1}{3\tilde{k}\gamma^2}a_4 - a_5 - a_6$ |
| $a_4$ | $\int_0^1 X_{2'} \Phi_2 d\xi$ | $a_{11}$ | $\tilde{k}\pi^4\gamma^2$ |
| $a_5$ | $\int_0^1 X_{2'} X_{2''} d\xi$ | $a_{21}$ | $2\zeta\tilde{k}\pi^4\gamma^2$ |
| $a_6$ | $\int_0^1 \Phi_{2'} \Phi_{2''} d\xi$ | $a_{31}$ | $\tilde{k}\alpha\pi^4\gamma^2$ |
| $a_7$ | $\int_0^1 X_2 \sin(\pi\xi) d\xi$ | $q_{01}$ | $-\frac{2p_0}{\pi X_1}$ |
| $a_9$ | $\int_0^1 X_2^3 d\xi$ | $q_{02}$ | $-\frac{p_0}{a_7}$ |
| $a_{10}$ | $\int_0^1 X_2^4 d\xi$ | | |

The first ordinary differential equation from Eqs. (14) was obtained by left-side multiplying the equation (12) by the $\mathbf{Y_1}$ vector, integrating along the $\xi$ axis and using the orthogonality conditions (8). The second equation (14) can be achieved similarly using the $\mathbf{Y_2}$ vector.

Since $X_1 = C$, $\Phi_1 = 0$ and $X_2$ is symmetrical the form of the coefficients is simple. As a results the following integrals are equal to zero.

$$\int_0^1 X_1 X_2 d\xi = 0; \quad \int_0^1 \Phi_1 \Phi_2 d\xi = 0.$$

The remaining non-zero coefficients are presented in Table 1.

### 4.2.Verification of the choice of the base in linear case

When $\tilde{\beta} = 0$ (linear case) the solution of equation (5) can be determined analytically. The frequency characteristic of this solution is compared with the characteristic of equation (11), where the time dependent functions $u_i$ are describe by equation (14) for $\tilde{\beta} = 0$.

At was shown in Figure 6 the behavior of the system in the linear case can be properly described by the approximate method. Near the lower eigenfrequency the solutions are equal, but when the excitation frequency increases the difference between the solutions grows, too. Fortunately, the relative error of the maximum value of the characteristic, connected with higher frequency, does not exceed 5%. The position of the characteristic maximum points has an error of less than 0.3%. The above considerations justify the choice of base functions in the Galerkin method. One can suppose that when taking into account the nonlinearity the use of the same method is possible and the results obtained will be in accordance with physical reality.

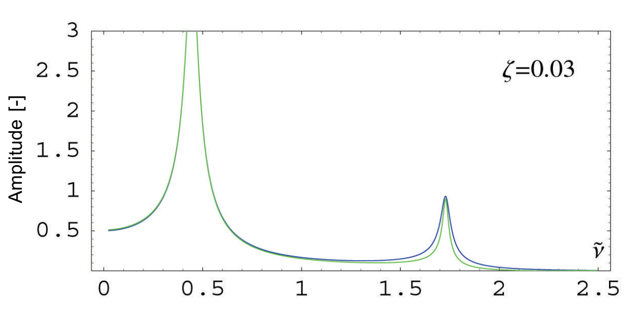


**Fig. 6.** Comparison between the frequency respond curves obtained by means of analytical and Galerkin methods in linear case for $\xi = 0.5$

### 4.3. Numerical analysis of resonant regions

One of the main research methods, applied to the identification of the mechanical system, is the determination of amplitude-frequency characteristic. This method is quite easy to apply to a linear system because of the superposition principle. It may be difficult to define a steady-state in nonlinear systems and sometimes this kind of state does not exist, for example in the case of lack of stability or for chaotic vibration.

During numerical simulations of the system described by (14) we noticed that for some parameters the steady-states occur, but the amplitudes are sensitive to small changes of frequency or the initial condition. It is connected, probably, with long settling times before the establishment of a steady-state. It results in strong amplitude dependence on the initial condition. It is shown in the Figure 7 that the change of the initial condition of 0.001 causes a noticeable jump in deflection.

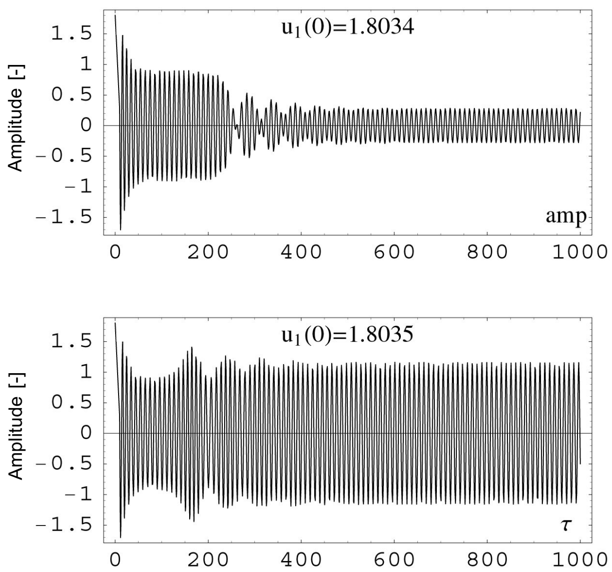


**Fig. 7.** Influence of the initial condition on the amplitude of steady-state

When determining amplitude-frequency characteristic a similar behavior was obtained while changing excitation frequencies near the first resonance.

In the Figure 8 these characteristics are presented for increasing $\tilde{\beta}$ and the fixed damping.

The discontinuities in the plot of characteristics are visible while the nonlinearity increases.

The peak that appears on the rising part of the characteristic near the first resonance (Fig. 9) is most likely due to the internal resonance associated with the third mode.

The numerical predictions were performed with the Mathematica, usong the NDSolve method. The precision of the computation equals $10^{-16}$, but higher precision can be achieved. It was checked, that changes of the parameters WorkingPrecision -> 24 and PrecisionGoal, as well as using a RK method, does not change the calculation result.

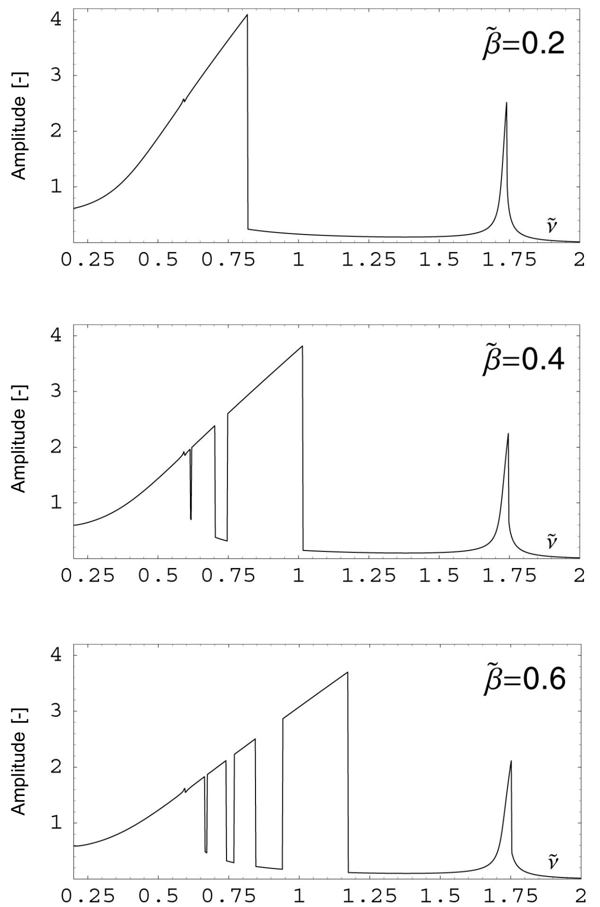


**Fig. 8.** Frequency characteristics for different values of nonlinearity

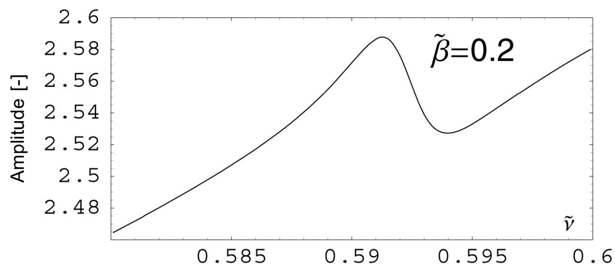


**Fig. 9.** Magnified region of frequency characteristic in the neighborhood of first resonance

## 5. CONCLUSIONS

The analysis of the frequency characteristics of a free boundary beam resting on nonlinear foundation has shown, that in the case of symmetric excitation the two elements base of the test functions in the Galekin method is satisfactory. For very large damping the method becomes inadequate.

The simulations based on the Galerkin method prove that there exist phenomena which do not occur in the linear case. For example, long lasted non-steady states occur, and the duration of transients is not adequate to damping. In a linear system the superposition of free and forced vibration excludes such a behavior.

The dependence of the amplitude of the steady state on the initial conditions is also a nonlinear effect, but it seems not to be chaotic.

The jump of the amplitude increases with the change of the initial condition and increasing the nonlinearity, but there is no relation between this phenomenon and discontinuity of amplitude-frequency characteristic. This problem remains open.

The rising part of the characteristic near the second eigenfrequency contains a small peak placed in one third of higher frequency. This phenomenon seems to be similar to internal resonance, which magnifies nonlinear effects. Particularly, it does not occur when the ratio of the second and third eigenfrequencies is chosen in another way.

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