Opuscula Mathematica • Vol. 29 • No. 1 • 2009

Semyon B. Yakubovich

BEURLING'S THEOREMS AND INVERSION FORMULAS FOR CERTAIN INDEX TRANSFORMS

Abstract. The familiar Beurling theorem (an uncertainty principle), which is known for the Fourier transform pairs, has recently been proved by the author for the Kontorovich-Lebedev transform. In this paper analogs of the Beurling theorem are established for certain index transforms with respect to a parameter of the modified Bessel functions. In particular, we treat the generalized Lebedev-Skalskaya transforms, the Lebedev type transforms involving products of the Macdonald functions of different arguments and an index transform with the Nicholson kernel function. We also find inversion formulas for the Lebedev-Skalskaya operators of an arbitrary index and the Nicholson kernel transform.

Keywords: Beurling theorem, Kontorovich-Lebedev transform, Lebedev-Skalskaya transforms, Fourier transform, Laplace transform, modified Bessel functions, uncertainty principle, the Nicholson function.

Mathematics Subject Classification: 44A10, 44A15, 33C10.

1. INTRODUCTION

Let $\mathbb{R}_+ = (0, \infty)$ and the cosine Fourier transform of a Lebesgue integrable function $f(y) \in L_1(\mathbb{R}_+; dy)$ be defined as usual by

$$(F_c f)(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(y) \cos xy dy. \tag{1}$$

Beurling's theorem [3] says that if

$$\int\limits_{\mathbb{R}_+} \int\limits_{\mathbb{R}_+} |f(y)(F_c f)(x)| e^{xy} dx dy < \infty, \tag{2}$$

then $f = 0$. Recently (cf. [15]) we have proved an analog of Beurling's theorem for the Kontorovich-Lebedev transform [4, 9, 10]

$$K_{ix}[f] = \int_0^\infty K_{ix}(y) f(y) dy, \ x > 0, \tag{3}$$

which is associated with the modified Bessel function $K_\mu(z)$ [2] as the kernel. The latter function is a fundamental solution of the differential equation

$$z^2 \frac{d^2 u}{dz^2} + z \frac{du}{dz} - (z^2 + \mu^2) u = 0$$

and can be represented by the integrals of the Fourier and Mellin types [7, Vol. I], [9, 10], respectively:

$$K_\mu(x) = \int_0^\infty e^{-x \cosh u} \cosh \mu u \ du, \tag{4}$$

$$K_\mu(x) = \frac{1}{2} \left(\frac{x}{2} \right)^\mu \int_0^\infty e^{-t - \frac{x^2}{4t}} t^{-\mu - 1} dt, \tag{5}$$

The modified Bessel function reveals the following asymptotic behavior [2]

$$K_\mu(z) = \left(\frac{\pi}{2z} \right)^{1/2} e^{-z} [1 + O(1/z)], \qquad z \to \infty, \tag{6}$$

and near the origin:

$$z^{|\mathrm{Re}\mu|} K_\mu(z) = 2^{\mu - 1} \Gamma(\mu) + o(1), \ z \to 0, \ \mu \neq 0, \tag{7}$$

$$K_0(z) = -\log z + O(1), \ z \to 0. \tag{8}$$

So, if $f(y)$ belongs to the weighted Lebesgue space $L_1(\mathbb{R}_+; K_0(y)dy)$ of those measurable functions on $\mathbb{R}_+$ for which

$$\int_0^\infty |f(y)| K_0(y) dy < \infty,$$

and

$$\int\limits_{\mathbb{R}_+} \int\limits_{\mathbb{R}_+} |f(y) K_{ix}[f]| \, K_x(y) dx dy < \infty, \tag{9}$$

then $f = 0$.

In this paper, analogous theorems and inversion formulas will be established for certain index transforms [9, 10]. Precisely, we will study the Lebedev-Skalskaya type transforms [6,12], the Lebedev transform involving a square of the Macdonald function as the kernel [5, 11], an index transform involving a product of the modified Bessel functions of different arguments [14] and an index transform with the Nicholson function [13].

2. THE LEBEDEV-SKALSKAYA TYPE TRANSFORMS

Let us consider the following integral operators

$$\text{Re } K_z[f] = \int_0^\infty \text{Re } K_z(y) f(y) dy, \ z = \alpha + i\tau, \ \tau \in \mathbb{R}_+, \tag{10}$$

$$\text{Im } K_z[f] = \int_0^\infty \text{Im } K_z(y) f(y) dy, \ z = \alpha + i\tau, \ \tau \in \mathbb{R}_+, \tag{11}$$

where $\alpha \in \mathbb{R}$ is a fixed parameter and by

$$\text{Re } K_z(y) = \frac{1}{2} \left[K_z(y) + K_{\bar{z}}(y) \right], \tag{12}$$

$$\text{Im } K_z(y) = \frac{1}{2i} \left[K_z(y) - K_{\bar{z}}(y) \right], \tag{13}$$

we as usual denote the real and imaginary parts, respectively, of the modified Bessel function $K_z(y)$. We call these operators the Lebedev-Skalskaya type transforms of a general complex index, which were introduced in [10, Chapter 6]. The case $\alpha = 0$ in (10) evidently corresponds to the Kontorovich-Lebedev operator (3) and $\alpha = \frac{1}{2}$ in (10), (11) leads us to the Lebedev-Skalskaya transforms [6]. Using (4), we easily find integral representations of functions (12), (13):

$$\text{Re } K_z(y) = \int_0^\infty e^{-x \cosh u} \cosh \alpha u \cos \tau u \ du, \tag{14}$$

$$\text{Im } K_z(y) = \int_0^\infty e^{-x \cosh u} \sinh \alpha u \sin \tau u \ du. \tag{15}$$

These kernels satisfy the following estimates (cf. [10, p.172]):

$$|\text{Re } K_{\alpha+i\tau}(y)| \le e^{-\delta\tau} K_\alpha(y \cos \delta), \tag{16}$$

$$|\text{Im } K_{\alpha+i\tau}(y)| \le e^{-\delta\tau} K_\alpha(y \cos \delta), \tag{17}$$

where δ is chosen in the interval $\left[0, \frac{\pi}{2}\right)$. Therefore, when $f \in L_1(\mathbb{R}_+; K_\alpha(y)dy)$, transforms (10), (11) are well defined and exist as Lebesgue integrals.

We are ready to prove the following Beurling theorem for the Lebedev-Skalskaya type transforms (10), (11).

Theorem 1. *Let $f \in L_1(\mathbb{R}_+; K_\alpha(y)dy)$, and let*

$$\int\limits_{\mathbb{R}_+} \int\limits_{\mathbb{R}_+} |f(y) \text{Re } K_{\alpha+ix}[f]| \, K_{|\alpha|+x}(y) dx dy < \infty, \quad \alpha \in \mathbb{R}, \tag{18}$$

$$\int\limits_{\mathbb{R}_+} \int\limits_{\mathbb{R}_+} |f(y) \text{Im } K_{\alpha+ix}[f]| \, K_{|\alpha|+x}(y) dx dy < \infty, \quad \alpha \in \mathbb{R} \setminus \{0\}. \tag{19}$$

Then $f = 0$.

Proof. It is sufficient to prove the theorem for the Re-transform (10) under condition (16) since for the Im-transform (11) the proof is quite similar. Evidently, we can assume that $f(y) \neq 0$ on a set of positive measure $K_\alpha(y)dy$, for otherwise there is nothing to prove. Due to the inequality $K_{|\alpha|+x}(y) \geq K_\alpha(y)$, condition (18) implies

$$\infty > \int\limits_{\mathbb{R}_+}\int\limits_{\mathbb{R}_+} |f(y)\text{Re } K_{\alpha+ix}[f]|\, K_{|\alpha|+x}(y)dxdy \geq \int\limits_{\mathbb{R}_+} |f(y)|K_\alpha(y)dy \int\limits_{\mathbb{R}_+} |\text{Re } K_{\alpha+ix}[f]|\, dx.$$

Therefore, Re $K_{\alpha+ix}[f] \in L_1(\mathbb{R}_+; dx)$. The latter condition guarantees the existence of the cosine Fourier transform of Re $K_{\alpha+ix}[f]$. We will show that

$$(F_c\text{Re } K_{\alpha+ix}[f])(\lambda) = \cosh \alpha\lambda \sqrt{\frac{\pi}{2}} \int_0^\infty e^{-y\cosh\lambda} f(y)dy. \tag{20}$$

Indeed, denoting by $h(\lambda)$ the right-hand side of (20), we find

$$\int\limits_{\mathbb{R}_+} |h(\lambda)|\, d\lambda \leq \sqrt{\frac{\pi}{2}} \int\limits_{\mathbb{R}_+}\int\limits_{\mathbb{R}_+} e^{-y\cosh\lambda}|f(y)|\cosh\alpha\lambda dyd\lambda = \sqrt{\frac{\pi}{2}} \int\limits_{\mathbb{R}_+} |f(y)|K_\alpha(y)dy < \infty.$$

So $h \in L_1(\mathbb{R}_+; d\lambda)$ and $(F_c h)(x)$ can be now easily calculated by using (14) and Fubini's theorem. Thus we obtain

$$\begin{aligned}(F_c h)(x) &= \int_0^\infty \cos x\lambda \int_0^\infty e^{-y\cosh\lambda} f(y)\cosh\alpha\lambda dyd\lambda = \\ &= \int_0^\infty \text{Re } K_{\alpha+ix}(y)f(y)dy = \text{Re } K_{\alpha+ix}[f].\end{aligned}$$

Since Re $K_{\alpha+ix}[f] \in L_1(\mathbb{R}_+; dx)$, the inversion theorem for the cosine Fourier transform gives $(F_c\text{Re } K_{\alpha+ix}[f])(\lambda) = h(\lambda)$ and we establish equality (20).

Let us verify Beurling condition (2) for the pair Re $K_{\alpha+ix}[f], (F_c\text{Re } K_{\alpha+ix}[f])(\lambda)$. There is

$$\begin{aligned}&\int\limits_{\mathbb{R}_+}\int\limits_{\mathbb{R}_+} |\text{Re } K_{\alpha+ix}[f](F_c\text{Re } K_{\alpha+it}[f])(\lambda)|e^{x\lambda}dx\, d\lambda \leq \\ &\quad \leq \sqrt{2\pi} \int\limits_{\mathbb{R}_+}\int\limits_{\mathbb{R}_+} |\text{Re } K_{\alpha+ix}[f]|\cosh x\lambda \int_0^\infty e^{-y\cosh\lambda}|f(y)|\cosh\alpha\lambda dy\, dx\, d\lambda \leq \\ &\quad \leq \sqrt{2\pi} \int\limits_{\mathbb{R}_+}\int\limits_{\mathbb{R}_+} |f(y)\text{Re } K_{\alpha+ix}[f]|\, K_{|\alpha|+x}(y)dxdy < \infty.\end{aligned}$$

Thus Re $K_{\alpha+ix}[f] = 0$. Combined with (20), the latter condition yields

$$\int_0^\infty e^{-y\cosh\lambda} f(y)dy = 0,\ \lambda \in \mathbb{R}_+ \tag{21}$$

for any $f \in L_1(\mathbb{R}_+; K_\alpha(y)dy)$. We will show that in this case $f = 0$. In fact, choosing any $z_0 > 1$ we treat the left-hand side of equality (21) as the Laplace integral $(Lf)(\cosh\lambda)$, where

$$(Lf)(z) = \int_0^\infty e^{-yz} f(y)dy, \tag{22}$$

which is zero via (21) at least at the countable set of points satisfying the condition $z_n = \cosh\lambda_n = z_0 + jn,\ j > 0, n = 0, 1, 2, \dots$. Moreover, since for $f \in L_1(\mathbb{R}_+; K_\alpha(y)dy)$ (see (6), (7)), there is

$$\int_0^\infty e^{-y\cosh\lambda_n} |f(y)|dy < \infty,\ n = 0, 1, 2, \dots,$$

then by virtue of [1, Chapter I] we get $f(y) = 0$ for almost all $y \in \mathbb{R}_+$, i.e., $f = 0$ in the Lebesgue sense. In the same manner, we can verify Beurling condition (2) for the pair $\mathrm{Im}\ K_{\alpha+ix}[f], (F_c\mathrm{Im}\ K_{\alpha+ix}[f])$ under condition (19). Theorem 1 is proved. □

However, when conditions (18), (19) fail, integral equations (10), (11) may have nonzero solutions. When $\alpha = \frac{1}{2}$, these solutions were found in [6] as inversion formulas of the Lebedev-Skalskaya transforms given by

$$f(x) = \frac{4}{\pi^2} \int_0^\infty \cosh\pi\tau \mathrm{Re}\ K_{\frac{1}{2}+i\tau}(x) \mathrm{Re}\ K_{\frac{1}{2}+i\tau}[f]d\tau, \tag{23}$$

$$f(x) = \frac{4}{\pi^2} \int_0^\infty \cosh\pi\tau \mathrm{Im}\ K_{\frac{1}{2}+i\tau}(x) \mathrm{Im}\ K_{\frac{1}{2}+i\tau}[f]d\tau, \tag{24}$$

respectively. Here we will find analogs of (23), (24) for a general α by using Sneddon's operational method [8, Chapter 6] recently applied in [16] to solve integral equations from a certain class, which generalize Kontorovich-Lebedev equation (3). Indeed, if $f \in L_1(\mathbb{R}_+; K_\alpha(y\cos\delta)dy) \subseteq L_1(\mathbb{R}_+; K_\alpha(y)dy),\ \delta \in \left(0, \frac{\pi}{2}\right)$, then via (16) we deduce the estimate of the Re-transform (10)

$$|\mathrm{Re}\ K_{\alpha+i\tau}[f]| \le e^{-\delta\tau} \int_0^\infty K_\alpha(y\cos\delta)|f(y)|dy,$$

which gives $\mathrm{Re}\ K_{\alpha+i\tau}[f] \in L_1(\mathbb{R}_+; d\tau)$. Taking into account (1), equality (20) immediately implies

$$\frac{2}{\pi} \int_0^\infty \mathrm{Re}\ K_{\alpha+i\tau}[f] \frac{\cos\tau u}{\cosh\alpha u} d\tau = \int_0^\infty e^{-y\cosh u} f(y)dy. \tag{25}$$

But the Re-transform (10) can be continued on $\mathbb{R}$ as an even function with respect to τ. Moreover, assuming that $\alpha \neq 0$, we apply the Fourier transform with respect to u to both sides of (25), and we change the order of integration by Fubini's theorem. Taking into account (4) and calculating an elementary integral, we finally arrive at the equality

$$\frac{1}{2\alpha} \int_{-\infty}^\infty \frac{\mathrm{Re}\ K_{\alpha+i\tau}[f]}{\cosh\left(\frac{\pi}{2\alpha}(x-\tau)\right)} d\tau = \int_0^\infty K_{ix}(y) f(y)dy. \tag{26}$$

If we show that the left-hand side of (26) belongs to $L_1(\mathbb{R}; |x|e^{\pi|x|/2}dx)$, then by virtue of the inversion theorem for the Kontorovich-Lebedev transform (3) (see [9, 15]), at each Lebesgue point of f, we obtain

$$f(y) = \frac{1}{2\pi^2\alpha y}\int_{-\infty}^{\infty} xe^{\pi x}K_{ix}(y)\int_{-\infty}^{\infty}\frac{\text{Re } K_{\alpha+i\tau}[f]}{\cosh\left(\frac{\pi}{2\alpha}(x-\tau)\right)}d\tau dx. \tag{27}$$

The latter fact can be verified assuming that Re $K_{\alpha+i\tau}[f] \in L_1(\mathbb{R}; e^{\frac{\pi}{2|\alpha|}|\tau|}d\tau)$, $\alpha \in (-1,1)\setminus\{0\}$. So

$$\begin{aligned}
&\int_{-\infty}^{\infty}|x|e^{\pi|x|/2}\left|\int_{-\infty}^{\infty}\frac{\text{Re } K_{\alpha+i\tau}[f]}{\cosh\left(\frac{\pi}{2\alpha}(x-\tau)\right)}d\tau\right|dx \le \\
&\le 2\int_{-\infty}^{\infty}|x|e^{\pi|x|/2}\int_{-\infty}^{\infty}|\text{Re } K_{\alpha+i\tau}[f]|\,e^{-\frac{\pi}{2|\alpha|}|x-\tau|}d\tau dx \le \\
&\le 2\int_{-\infty}^{\infty}|x|e^{\frac{\pi}{2}\left(1-\frac{1}{|\alpha|}\right)|x|}dx\int_{-\infty}^{\infty}|\text{Re } K_{\alpha+i\tau}[f]|\,e^{\frac{\pi}{2|\alpha|}|\tau|}d\tau<\infty.
\end{aligned}$$

Hence the left-hand side of (26) belongs to $L_1(\mathbb{R}; |x|e^{\pi|x|/2}dx)$ and we get (27). Employing again Fubini's theorem, due to the estimate (see inequality (1.100) from [10])

$$\begin{aligned}
&\int_{-\infty}^{\infty}|x|e^{\pi x}|K_{ix}(y)|\int_{-\infty}^{\infty}\frac{|\text{Re } K_{\alpha+i\tau}[f]|}{\cosh\left(\frac{\pi}{2\alpha}(x-\tau)\right)}d\tau dx \le \\
&\le K_0(y\cos\delta)\int_{-\infty}^{\infty}|x|e^{\left(\pi\left(1-\frac{1}{2|\alpha|}\right)-\delta\right)|x|}dx\int_{-\infty}^{\infty}|\text{Re } K_{\alpha+i\tau}[f]|\,e^{\frac{\pi}{2|\alpha|}|\tau|}d\tau<\infty
\end{aligned}$$

when δ is taken from the interval $\left(\pi\left(1-\frac{1}{2|\alpha|}\right),\ \frac{\pi}{2}\right)$, we finally arrive at the inversion formula for the transform (10):

$$f(y) = \int_{-\infty}^{\infty}\mathcal{R}_\alpha(y,\tau)\text{Re } K_{\alpha+i\tau}[f]d\tau, \tag{28}$$

where the kernel is given by

$$\mathcal{R}_\alpha(y,\tau) = \frac{1}{2\pi^2\alpha y}\int_{-\infty}^{\infty}\frac{xe^{\pi x}K_{ix}(y)}{\cosh\left(\frac{\pi}{2\alpha}(x-\tau)\right)}dx, \quad \alpha\in(-1,1)\setminus\{0\}. \tag{29}$$

We will now expand the value of kernel (29) on any real value of the parameter α writing it in a different form. Employing the following representation (cf. [10, p. 125]):

$$\frac{1}{\cosh\left(\frac{\pi}{2\alpha}(x-\tau)\right)} = \frac{2}{\pi}\int_0^{\infty}K_{\frac{i}{\alpha}(\tau-x)}(t)dt, \tag{30}$$

we substitute the integral into (29) and change the order of integration via Fubini's theorem. Thus we obtain

$$\mathcal{R}_\alpha(y,\tau) = \frac{e^{\pi\tau}}{\pi^3\alpha y}\int_0^{\infty}dt\int_{-\infty}^{\infty}xe^{-\pi(\tau-x)}K_{ix}(y)K_{\frac{i}{\alpha}(\tau-x)}(t)dx. \tag{31}$$

The use of Fubini's theorem here can be motivated as above by the estimate

$$\int_0^\infty dt \int_{-\infty}^\infty e^{\pi x}|xK_{ix}(y)K_{\frac{i}{\alpha}(\tau-x)}(t)|dx \le$$
$$\le e^{\frac{\delta}{|\alpha|}|\tau|}K_0(y\cos\delta)\int_0^\infty K_0(t\cos\delta)dt\int_{-\infty}^\infty |x|e^{\left(\pi-\delta\left(1+\frac{1}{|\alpha|}\right)\right)|x|}dx<\infty,$$

for all $y>0,\ \tau\in\mathbb{R}$ and $\delta\in\left(\frac{\pi|\alpha|}{|\alpha|+1},\frac{\pi}{2}\right),\ \alpha\in(-1,1)\setminus\{0\}$. The inner integral in (31) can be treated as a Fourier convolution. Using the formula

$$xK_{ix}(y)=\frac{y}{2i}\int_{-\infty}^\infty e^{-y\cosh u}\sinh u\ e^{ixu}du, \tag{32}$$

as a consequence of (4), the inverse Fourier transform and a differentiation under integral sign, we substitute its right-hand side into (31) and change the order of integration due to the absolute and uniform convergence of the corresponding iterated integral at least for $|\alpha|<\frac{1}{2}$. Calculating the inner integral with respect to x (cf. [7, Vol. 2]), we get

$$\mathcal{R}_\alpha(y,\tau)=\frac{e^{\pi\tau}}{2\pi^2 i}\int_0^\infty dt\int_{-\infty}^\infty e^{-y\cosh u-t\cosh(\alpha(u-i\pi))}\sinh u\ e^{i\tau u}du. \tag{33}$$

Hence an integration with respect to t and simple manipulations finally give the expression of the kernel $\mathcal{R}_\alpha(y,\tau)$

$$\mathcal{R}_\alpha(y,\tau)=\frac{e^{\pi\tau}}{\pi^2}\int_0^\infty \frac{e^{-y\cosh u}\sinh u}{\cos^2\alpha\pi+\sinh^2\alpha u}\times \tag{34}$$
$$\times\left[\cos\alpha\pi\cosh\alpha u\sin u\tau+\sin\alpha\pi\sinh\alpha u\cos u\tau\right]du,$$

where the latter integral in (11)) is absolutely and uniformly convergent on any compact set of real values of α. Since the integral in (29) is a continuous function with respect to $\alpha\in(-1,1)\setminus\{0\}$, by (34) we obtain an extension of the kernel $\mathcal{R}_\alpha(y,\tau)$ on all real values of α.

We summarize our results in the following

Theorem 2. *Let $f(y)\in L_1(\mathbb{R}_+;K_\alpha(y\cos\delta)dy)$, where $\alpha\in(-1,1)\setminus\{0\}$ and $\delta\in\left(\pi\left(1-\frac{1}{2|\alpha|}\right),\ \frac{\pi}{2}\right)$. If* Re $K_{\alpha+i\tau}[f]\in L_1(\mathbb{R};e^{\frac{\pi}{2|\alpha|}|\tau|}d\tau)$, *then at each Lebesgue point $y\in\mathbb{R}_+$ of f, inversion formula* (28) *of the Re-transform* (10) *holds with kernel* (29), *which can be calculated by formula* (34) *being valid for all $\alpha\in\mathbb{R}$.*

Some interesting examples of kernel (34) and inversion formula (28) can be obtained directly. For instance, let $\alpha=0$. Then (10) coincides with the Kontorovich-Lebedev operator (3). Meanwhile from (34) we find with (32) that $\mathcal{R}_0(y,\tau)=\frac{\tau e^{\pi\tau}}{y\pi^2}K_{i\tau}(y)$, which leads us to the inversion formula for the Kontorovich-Lebedev transform [4, 9, 10]:

$$f(y)=\frac{1}{y\pi^2}\int_{-\infty}^\infty \tau e^{\pi\tau}K_{i\tau}(y)K_{i\tau}[f]d\tau. \tag{35}$$

It is easily seen that $\mathcal{R}_\alpha(y,\tau)$ is even with respect to α. If $\alpha=\frac{1}{2}$, we get $\mathcal{R}_{\frac{1}{2}}(y,\tau)=\frac{2e^{\pi\tau}}{\pi^2}\mathrm{Re}\ K_{\frac{1}{2}+i\tau}(y)$ and easily again come to inversion formula (23) for the Lebedev -Skalskaya Re-transform. We can derive a new pair of reciprocal formulas putting $\alpha=\frac{1}{4}$. In this case, we recall (14), (15) to deduce

$$\mathcal{R}_{\frac{1}{4}}(y,\tau)=\frac{\sqrt{2}e^{\pi\tau}}{\pi^2}\left[\mathrm{Re}\left(K_{\frac{3}{4}+i\tau}(y)-K_{\frac{1}{4}+i\tau}(y)\right)+\mathrm{Im}\left(K_{\frac{3}{4}+i\tau}(y)+K_{\frac{1}{4}+i\tau}(y)\right)\right],$$

and we find the following pair of direct and inverse integral transforms

$$\mathrm{Re}\ K_{\frac{1}{4}+i\tau}[f]=\int_0^\infty \mathrm{Re}\ K_{\frac{1}{4}+i\tau}(y)f(y)dy,$$

$$f(y)=\frac{2\sqrt{2}}{\pi^2}\int_0^\infty\Big[\cosh\pi\tau\ \mathrm{Re}\left(K_{\frac{3}{4}+i\tau}(y)-K_{\frac{1}{4}+i\tau}(y)\right)+$$
$$+\sinh\pi\tau\ \mathrm{Im}\left(K_{\frac{3}{4}+i\tau}(y)+K_{\frac{1}{4}+i\tau}(y)\right)\Big]\mathrm{Re}\ K_{\frac{1}{4}+i\tau}[f]d\tau.$$

The limit case of $|\alpha|=1$ in (29) can be added to our consideration via the uniform convergence of the integral. In this case, the kernel $\mathrm{Re}\ K_{1+i\tau}(y)$ in (10) is equal to $-\frac{d}{dy}K_{i\tau}(y)$ (see [2]). Hence from (34), (15) we deduce:

$$\mathcal{R}_1(y,\tau)=-\frac{e^{\pi\tau}}{\pi^2}\int_y^\infty \mathrm{Im}\ K_{1+i\tau}(y)dy=-\frac{\tau e^{\pi\tau}}{\pi^2}\int_y^\infty\frac{K_{i\tau}(y)}{y}dy.$$

Consequently, under additional conditions on f, integrating by parts and differentiating under the integral sign, we again come to the Kontorovich-Lebedev reciprocal formulas (3), (35).

Finally, in this section we consider an inversion of the general Im -transform (11). In the same manner, we establish an analog of equation (25), which becomes

$$\frac{2}{\pi}\int_0^\infty \mathrm{Im}\ K_{\alpha+i\tau}[f]\frac{\sin\tau u}{\sinh\alpha u}d\tau=\int_0^\infty e^{-y\cosh u}f(y)dy. \tag{36}$$

Taking the cosine Fourier transform (1) from both sides of (36), changing the order of integration and calculating the inner integrals with (4) and relation (2.5.46.9) in [7, Vol. 1], we end up for $\alpha\neq 0$ with the equation

$$\frac{1}{\alpha}\int_0^\infty \mathrm{Im}\ K_{\alpha+i\tau}[f]\frac{\sinh(\pi\tau/\alpha)}{\cosh(\pi x/\alpha)+\cosh(\pi\tau/\alpha)}d\tau=\int_0^\infty K_{ix}(y)f(y)dy. \tag{37}$$

Reasoning as above for the Re-case, we invert the Kontorovich-Lebedev transform in (37) and we arrive at the following inversion formula for the Im-transform (11)

$$f(y)=\int_0^\infty \mathcal{I}_\alpha(y,\tau)\mathrm{Im}\ K_{\alpha+i\tau}[f]d\tau, \tag{38}$$

where

$$\mathcal{I}_\alpha(y,\tau)=\frac{2\sinh(\pi\tau/\alpha)}{\pi^2\alpha y}\int_0^\infty\frac{x\sinh\pi x\ K_{ix}(y)}{\cosh(\pi x/\alpha)+\cosh(\pi\tau/\alpha)}dx,\ |\alpha|\leq 1,\ \alpha\neq 0. \tag{39}$$

Making use of (32), we substitute it in (39) and change the order of integration by Fubini's theorem. The inner integral can then be calculated for $|\alpha| < 1$ employing relation (2.5.49.3) in [7, Vol. 1]. As a result it becomes

$$\mathcal{I}_\alpha(y,\tau) = -\frac{2}{\pi^2}\int_0^\infty e^{-y\cosh u}\sinh u\ \mathrm{Im}\left[\frac{\sin\tau(u+i\pi)}{\sinh\alpha(u+i\pi)}\right]du.$$

Taking the imaginary part in the latter integrand, we write kernel (39) in the final form, which is valid for all $\alpha \in \mathbb{R}\backslash\{0\}$:

$$\mathcal{I}_\alpha(y,\tau) = \frac{2}{\pi^2}\int_0^\infty \frac{e^{-y\cosh u}\sinh u}{\sin^2\alpha\pi + \sinh^2\alpha u}\,(\sin\alpha\pi\cosh\alpha u\cosh\pi\tau\sin u\tau - \\ -\cos\alpha\pi\sinh\alpha u\sinh\pi\tau\cos u\tau)\,du, \quad \alpha \neq 0. \tag{40}$$

Therefore, the following theorem for the Im-transform (11) holds true.

Theorem 3. *Let* $f(y) \in L_1(\mathbb{R}_+; K_\alpha(y\cos\delta)dy)$, *where* $\alpha \in [-1,1]\backslash\ \{0\}$ *and* $\delta \in \left(\pi\left(1-\frac{1}{|\alpha|}\right),\ \frac{\pi}{2}\right)$. *If* $\mathrm{Im}\ K_{\alpha+i\tau}[f] \in L_1(\mathbb{R}_+; e^{\frac{\pi}{|\alpha|}\tau}d\tau)$, *then at each Lebesgue point* $y \in \mathbb{R}_+$ *of* f *inversion formula* (38) *of the Im-transform* (11) *takes place with the kernel* (39), *which can be calculated by formula* (40) *being valid for all* $\alpha \in \mathbb{R}\backslash\{0\}$.

Concerning examples of the Im-transforms and their kernels, we first note that $\mathcal{I}_\alpha(y,\tau)$ is odd with respect to α. Letting $\alpha = 1$, from (40) we get $\mathcal{I}_1(y,\tau) = \frac{2}{\pi^2}\sinh\pi\tau K_{i\tau}(y)$, which again leads us to Kontorovich-Lebedev operator (3) and its inversion formula (35). If $\alpha = \frac{1}{2}$, by (40) via (15) we easily confirm that $\mathcal{I}_{\frac{1}{2}}(y,\tau) = \frac{4}{\pi^2}\cosh\pi\tau\ \mathrm{Im}\ K_{\frac{1}{2}+i\tau}(y)$, which leads to inversion formula (24) of the Lebedev-Skalskaya Im-transform. And finally putting $\alpha = \frac{1}{4}$ in (40) we calculate the corresponding integral using (14), (15) and end up with a new pair of the Lebedev-Skalskaya type transforms

$$\mathrm{Im}\ K_{\frac{1}{4}+i\tau}[f] = \int_0^\infty \mathrm{Im}\ K_{\frac{1}{4}+i\tau}(y)f(y)dy,$$

$$f(y) = \frac{2\sqrt{2}}{\pi^2}\int_0^\infty \Big[\cosh\pi\tau\ \mathrm{Im}\Big(\ K_{\frac{3}{4}+i\tau}(y) + K_{\frac{1}{4}+i\tau}(y)\Big) - \\ -\sinh\pi\tau\ \mathrm{Re}\Big(\ K_{\frac{3}{4}+i\tau}(y) - K_{\frac{1}{4}+i\tau}(y)\Big)\Big]\ \mathrm{Im}\ K_{\frac{1}{4}+i\tau}[f]d\tau.$$

Remark 1. *The case of* $\alpha = 0$ *in* (40) *is naturally excluded, since the direct kernel* $\mathrm{Im}\ K_{i\tau}(y) \equiv 0$ *and* (11) *yield the null-operator.*

3. OTHER INDEX TRANSFORMS

In this section we will prove analogs of Beurling's theorem for the following index transforms: the Lebedev transform with the square of the modified Bessel function [5, 11]

$$K^2_{ix}[f] = \int_0^\infty K^2_{ix}(y)f(y)dy, \tag{41}$$

index transform with the Nicholson function as the kernel [13]

$$JY_{ix}[f] = \int_0^\infty \left[J^2_{ix/2}(y) + Y^2_{ix/2}(y)\right] f(y)dy, \tag{42}$$

where $J_\nu(y)$, $Y_\nu(y)$ are Bessel functions of the first and second kind [2], and an index transform involving a product of the modified Bessel functions of different arguments [14]

$$KK_{i\tau}[f] = \int_0^\infty \int_0^\infty K_{i\tau}\left(\sqrt{x^2+y^2} - y\right) K_{i\tau}\left(\sqrt{x^2+y^2} + y\right) f(x,y)\frac{dxdy}{x}. \tag{43}$$

The following holds true.

Theorem 4. *Let* $f \in L_1(\mathbb{R}_+; K_0^2(y)dy)$ *and*

$$\int\limits_{\mathbb{R}_+}\int\limits_{\mathbb{R}_+} \left|f(y)K^2_{ix}[f]\right| K^2_x(y)dxdy < \infty. \tag{44}$$

Then $f = 0$.

Proof. Assuming as in Theorem 1 that $f(y) \neq 0$ on a set of positive measure $K_0^2(y)dy$, we get the estimate

$$\infty > \int\limits_{\mathbb{R}_+}\int\limits_{\mathbb{R}_+} \left|f(y)K^2_{ix}[f]\right| K^2_x(y)dxdy \geq \int\limits_{\mathbb{R}_+} |f(y)|K_0^2(y)dy \int\limits_{\mathbb{R}_+} \left|K^2_{ix}[f]\right| dx.$$

Therefore $K^2_{ix}[f] \in L_1(\mathbb{R}_+; dx)$. Hence using integral representations (see relation (2.16.51.6) in [7, Vol. 2])

$$\int_0^\infty K^2_{ix}(y)\cos\lambda x\ dx = \frac{\pi}{2}K_0\left(2y\cosh\frac{\lambda}{2}\right), \tag{45}$$

$$K^2_{ix}(y) = \int_0^\infty K_0\left(2y\cosh\frac{\lambda}{2}\right)\cos\lambda x\ d\lambda, \tag{46}$$

we calculate the composition of Lebedev operator (41) and the cosine Fourier transform (1) showing that

$$(F_c\ K^2_{ix}[f])(\lambda) = \sqrt{\frac{\pi}{2}}\int_0^\infty K_0\left(2y\cosh\frac{\lambda}{2}\right) f(y)dy. \tag{47}$$

Indeed, the right-hand side of (47) is Lebesgue integrable with respect to $\lambda \in \mathbb{R}_+$, because the function $K_0\left(2y\cosh\frac{\lambda}{2}\right) f(y)$ is Lebesgue integrable as a function in two variables (cf. (46))

$$\int\limits_{\mathbb{R}_+}\int\limits_{\mathbb{R}_+} K_0\left(2y\cosh\frac{\lambda}{2}\right)|f(y)|dyd\lambda = \int\limits_{\mathbb{R}_+} K_0^2\,(y)\,|f(y)|dy < \infty.$$

Thus taking the cosine Fourier transform (1) of the right-hand side of (47) and changing the order of integration via Fubini's theorem, we calculate the inner integral with the use of (46) and we obtain $K^2_{ix}[f]$. Since it belongs to $L_1(\mathbb{R}_+; dx)$, the inversion theorem for the cosine Fourier transform yields (47).

Beurling condition (2) for the pair $K^2_{ix}[f], (F_c K^2_{ix}[f])(\lambda)$ implies

$$\begin{aligned}
&\int\limits_{\mathbb{R}_+}\int\limits_{\mathbb{R}_+} |K^2_{ix}[f](F_c K^2_{ix}[f])(\lambda)| e^{x\lambda} dx d\lambda < \\
&\quad < \sqrt{2\pi} \int\limits_{\mathbb{R}_+}\int\limits_{\mathbb{R}_+} |K^2_{ix}[f]| \cosh x\lambda \int_0^\infty K_0\left(2y \cosh\frac{\lambda}{2}\right) |f(y)| dy dx d\lambda = \\
&\quad = \sqrt{2\pi} \int\limits_{\mathbb{R}_+}\int\limits_{\mathbb{R}_+} |f(y) K^2_{ix}[f]| K^2_x(y) dy dx < \infty.
\end{aligned}$$

Therefore, $K^2_{ix}[f] = 0$ for all $x \in \mathbb{R}_+$ as a continuous function under the condition $f \in L_1(\mathbb{R}_+; K^2_0(y)dy)$. In fact, integral (41) is absolutely and uniformly convergent, since

$$\int_0^\infty K^2_{ix}(y)|f(y)|dy \le \int_0^\infty K^2_0(y)|f(y)|dy < \infty.$$

However, the kernel $K^2_{ix}(y)$ can be represented by the integral (cf. [7, Vol. 2], relation (2.16.9.1))

$$K^2_{ix}(y) = \frac{1}{2}\int_0^\infty e^{-t-\frac{y^2}{2t}} K_{ix}(t)\frac{dt}{t}. \tag{48}$$

Substituting integral (48) into (41) and changing the order of integration by Fubini's theorem, because of the estimate

$$\begin{aligned}
\int_0^\infty \int_0^\infty e^{-t-\frac{y^2}{2t}} |K_{ix}(t) f(y)| \frac{dtdy}{t} &\le \int_0^\infty \int_0^\infty e^{-t-\frac{y^2}{2t}} K_0(t)\, |f(y)| \frac{dtdy}{t} = \\
&= \int_0^\infty K^2_0(y)|f(y)|dy < \infty,
\end{aligned} \tag{49}$$

we find that $K^2_{ix}[f] = K_{ix}[h] = 0$ (see (3)), where

$$h(t) = \frac{e^{-t}}{2t}\int_0^\infty e^{-\frac{y^2}{2t}} f(y) dy. \tag{50}$$

Further, relations (49) guarantee the condition $h(t) \in L_1(\mathbb{R}_+; K_0(t)dt)$ and the existence of the composition

$$\left(F_c \sqrt{\frac{\pi}{2}} (Lf)(\cosh u)\right)(x) = K_{ix}[h] = 0,$$

where Laplace operator (22) is an integrable function, i.e., $(Lf)(\cosh u) \in L_1(\mathbb{R}_+; du)$. Thus $h = 0$ a.e. and we arrive at the equation

$$\int_0^\infty e^{-\frac{y^2}{2t}} f(y) dy = 0 \tag{51}$$

for almost all $t > 0$. To end the proof, we make an elementary substitution in (51) and come out with the equation

$$\int_0^\infty e^{-py} f(\sqrt{y}) \frac{dy}{\sqrt{y}} = 0, \ p = \frac{1}{2t}, \ t > 0,$$

treating its left-hand side as a Laplace transform $(Lg)(p - p_0), \ p > p_0 > 0$ of the integrable function $g(y) = e^{-p_0 y} \frac{f(\sqrt{y})}{\sqrt{y}}$, since (see (6), (8)),

$$\int_0^\infty e^{-p_0 y} |f(\sqrt{y})| \frac{dy}{\sqrt{y}} = 2 \int_0^\infty e^{-p_0 y^2} |f(y)| dy < \infty.$$

By (51) $(Lg)(p - p_0)$ is zero at at least the countable set of points $p_n = p_0 + jn$, $j > 0, n = 1, 2, \ldots$. Hence, as in the proof of Theorem 1, we conclude that $f = 0$ a.e. Theorem 4 is proved. □

Index transform (42) is based on the following Nicholson formula for the sum of squares of Bessel functions [2, p. 54]

$$J_{x/2}^2(y) + Y_{x/2}^2(y) = \frac{8}{\pi^2} \int_0^\infty K_0(2y \sinh \lambda) \cosh x\lambda \ d\lambda, \ y > 0. \tag{52}$$

This transform was introduced for the first time in [13] as an adjoint operator to (42), where the integration was performed with respect to the pure imaginary index of Nicholson function (52). Here we will prove an analog of Beurling's theorem for operator (42) and will find its inversion by the Sneddon operational method [8].

Theorem 5. *Let* $f \in L_1 \left(\mathbb{R}_+; \ \left[J_0^2(y) + Y_0^2(y)\right] dy\right)$ *and*

$$\int\limits_{\mathbb{R}_+} \int\limits_{\mathbb{R}_+} |f(y) J Y_{ix}[f]| \left[J_{x/2}^2(y) + Y_{x/2}^2(y)\right] dxdy < \infty. \tag{53}$$

Then $f = 0$.

Proof. Assuming again that $f(y) \neq 0$ on a set of positive measure $K_0^2(y)dy$, from (52) we deduce:

$$\infty > \int\limits_{\mathbb{R}_+} \int\limits_{\mathbb{R}_+} |f(y) J Y_{ix}[f]| \left[J_{x/2}^2(y) + Y_{x/2}^2(y)\right] dxdy \geq$$
$$\geq \int\limits_{\mathbb{R}_+} |f(y)| \left[J_0^2(y) + Y_0^2(y)\right] dy \int\limits_{\mathbb{R}_+} |J Y_{ix}[f]| \, dx.$$

Therefore, $JY_{ix}[f] \in L_1(\mathbb{R}_+; dx)$. Reasoning as in the proofs of Theorems 1, 4 we establish the equality

$$(F_c \ JY_{ix}[f])(\lambda) = \frac{4}{\pi} \sqrt{\frac{2}{\pi}} \int_0^\infty K_0 \left(2y \sinh \lambda\right) f(y) dy, \tag{54}$$

where the right-hand side is integrable with respect to λ due to the estimate (see (52)):

$$\int_0^\infty d\lambda \left| \int_0^\infty K_0\left(2y\sinh\lambda\right) f(y)dy \right| \le \int_0^\infty d\lambda \int_0^\infty K_0\left(2y\sinh\lambda\right)|f(y)|dy = \\ = \frac{\pi^2}{8}\int_0^\infty \left[J_0^2(y)+Y_0^2(y)\right]|f(y)|dy < \infty.$$

Further, Beurling condition (2) for the pair $JY_{ix}[f], (F_cJY_{ix}[f])(\lambda)$ gives

$$\int\limits_{\mathbb{R}_+}\int\limits_{\mathbb{R}_+} |JY_{ix}[f](F_cJY_{ix}[f])(\lambda)|e^{x\lambda}dxd\lambda < \\ < \frac{8}{\pi}\sqrt{\frac{2}{\pi}}\int\limits_{\mathbb{R}_+}\int\limits_{\mathbb{R}_+} |JY_{ix}[f]|\cosh x\lambda \int_0^\infty K_0\left(2y\sinh\lambda\right)|f(y)|dydxd\lambda = \\ = \sqrt{2\pi}\int\limits_{\mathbb{R}_+}\int\limits_{\mathbb{R}_+} |f(y)JY_{ix}[f]|\left[J_{x/2}^2(y)+Y_{x/2}^2(y)\right]dydx < \infty.$$

Therefore, $JY_{ix}[f]=0$. Consequently, equation (54) yields

$$\int_0^\infty K_0\left(2y\sinh\lambda\right) f(y)dy = 0 \tag{55}$$

for all $\lambda > 0$. Taking $p = 2\sinh\lambda$ and the representation (see (4)) of the modified Bessel function:

$$K_0(py) = \int_0^\infty e^{-py\cosh u}du = \int_y^\infty e^{-pt}\frac{dt}{\sqrt{t^2-y^2}},$$

we substitute it into (55) and inverting the order of integration by Fubini's theorem, it becomes a composition of Laplace transform (22) and a simple Erdélyi-Kober fractional integration operator [9]

$$\int_0^\infty e^{-pt}dt\int_0^t \frac{f(y)}{\sqrt{t^2-y^2}}dy = 0,\ p>0.$$

It is convergent and equal to 0 at at least a countable set of points. Thus, for all $t > 0$:

$$\int_0^t \frac{f(y)}{\sqrt{t^2-y^2}}dy = 0. \tag{56}$$

Via asymptotic of Bessel functions [2], we observe that the condition

$$f \in L_1\left(\mathbb{R}_+;\ \left[J_0^2(y)+Y_0^2(y)\right]dy\right)$$

means $f \in L_1((0,1);(1+\log^2 y)dy)\ \cap L_1((1,\infty);y^{-1}dy)$. Hence it follows that f is locally integrable on $\mathbb{R}_+$. Making elementary change of variables, we write (56) in the form of Abel's homogeneous equation

$$\int_0^t \frac{f(\sqrt{y})}{\sqrt{t-y}}\frac{dy}{\sqrt{y}} = 0. \tag{57}$$

It has a trivial solution, which can easily be checked by taking Laplace's transform of the both sides of (57) and treating its left-hand side as a Laplace convolution [9]. So $f = 0$ a.e. and we conclude the proof of Theorem 5. □

An inversion formula for Nicholson kernel transform (42) can be proved with

Theorem 6. *Let $f(y) \in L_1((0,1); (1+\log^2 y)dy) \cap L_1((1,\infty); y^{1-\gamma}dy)$, $1/2 < \gamma < 1$ and $JY_{ix}[f] \in L_1(\mathbb{R}_+; xe^{\pi x}dx)$. Then for almost all $y > 0$ the following inversion formula holds for operator* (42)

$$f(y) = -\frac{\pi}{4}\frac{d}{dy}\int_0^\infty x\ \mathrm{Im} J^2_{ix/2}(y)\ JY_{ix}[f]\ dx. \tag{58}$$

Proof. In fact, it is easily seen that

$$L_1((0,1); (1+\log^2 y)dy) \subset L_1\left((0,1);\ y^{1-\gamma}dy\right), \qquad \frac{1}{2} < \gamma < 1,$$
$$L_1((1,\infty); y^{1-\gamma}dy) \subset L_1\left((1,\infty);\ \left[J_0^2(y) + Y_0^2(y)\right]dy\right), \quad \frac{1}{2} < \gamma < 1,$$

and $JY_{ix}[f]$ is continuous on $\mathbb{R}_+$ via the estimate

$$|JY_{ix}[f]| \le \int_0^\infty \left[J_0^2(y) + Y_0^2(y)\right]|f(y)|dy < \infty.$$

Differentiating (54) with respect to λ, taking into account the condition $JY_{ix}[f] \in L_1(\mathbb{R}_+; xe^{\pi x}dx)$ and the formula (see [2]) $K_0'(z) = -K_1(z)$, we obtain

$$\int_0^\infty JY_{ix}[f] x \sin x\lambda dx = \frac{8}{\pi}\cosh\lambda \int_0^\infty yK_1(2y\sinh\lambda)\, f(y)dy.$$

Hence, the simple change of variable $p = \sinh\lambda$ gives

$$\int_0^\infty xJY_{ix}[f]\frac{\sin\left(x\log(p+\sqrt{p^2+1})\right)}{\sqrt{p^2+1}}dx = \frac{8}{\pi}\int_0^\infty yK_1(2yp)\, f(y)dy. \tag{59}$$

The right-hand side of (59) can be treated as the Mellin convolution transform [9]. Therefore, invoking the generalized Parseval equality for the Mellin transform and relation (2.16.2.2) in [7, Vol. 2], it can be written in the form

$$\frac{8}{\pi}\int_0^\infty yK_1(2yp)\, f(y)dy = -\frac{4}{\pi}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma^2\left(\frac{1+s}{2}\right)\frac{(\mathcal{M}f)(2-s)}{1-s}p^{-s}ds, \tag{60}$$

where we choose $\gamma \in (1/2, 1)$, $\Gamma(z)$ is the Euler gamma-function [7] and $\frac{(\mathcal{M}f)(2-s)}{1-s}$ is the Mellin transform of the integration operator

$$\frac{(\mathcal{M}f)(2-s)}{1-s} = \frac{1}{1-s}\int_0^\infty f(y)y^{1-s}dy = \int_0^\infty v^{-s}\int_v^\infty f(y)dy\, dv. \tag{61}$$

On the other hand, via relation (2.5.46.15) in [7, Vol. 1], we find

$$\begin{aligned}&\frac{\sin\left(x\log(p+\sqrt{p^2+1})\right)}{\sqrt{p^2+1}}=\\&=\frac{1}{2\sqrt{\pi}}\frac{\sinh(\pi x/2)}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma\left(\frac{1+s}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)}\Gamma\left(\frac{1-s}{2}+\frac{ix}{2}\right)\Gamma\left(\frac{1-s}{2}-\frac{ix}{2}\right)p^{-s}ds,\end{aligned}\tag{62}$$

with the same $\gamma=\operatorname{Re} s\in(1/2,1)$. Substituting (62) into (59), we change the order of integration by Fubini's theorem, which is motivated by the following estimate (see the asymptotic behavior of the gamma-function on the vertical line in a complex plane and elementary inequality for the Euler beta-function $|B(a,b)|\le B(\operatorname{Re} a,\operatorname{Re} b)$ [9]):

$$\int_0^\infty x\sinh\left(\frac{\pi x}{2}\right)|JY_{ix}[f]|\int_{\gamma-i\infty}^{\gamma+i\infty}\left|\frac{\Gamma\left(\frac{1+s}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)}\Gamma\left(\frac{1-s}{2}+\frac{ix}{2}\right)\Gamma\left(\frac{1-s}{2}-\frac{ix}{2}\right)p^{-s}ds\right|dx\le$$

$$\le\frac{(2p)^{-\gamma}B((1-\gamma)/2,(1-\gamma)/2)}{\sqrt{\pi}}\int_0^\infty x\sinh(\pi x/2)|JY_{ix}[f]|dx\times$$

$$\times\int_{\gamma-i\infty}^{\gamma+i\infty}\left|\Gamma\left(\frac{1+s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)ds\right|<\infty.$$

Thus equating (59) with (60), taking into account (61) and cancelling the inverse Mellin transform [9] by the uniqueness property via the summability of the integrands, we come out with the equality

$$\begin{aligned}&\frac{1}{2\sqrt{\pi}}\int_0^\infty x\sinh(\pi x/2)\ JY_{ix}[f]\ \frac{\Gamma\left(\frac{1-s}{2}+\frac{ix}{2}\right)\Gamma\left(\frac{1-s}{2}-\frac{ix}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)\Gamma\left(\frac{1+s}{2}\right)}dx=\\&=-\frac{4}{\pi}\int_0^\infty v^{-s}\int_v^\infty f(y)dy\ dv.\end{aligned}\tag{63}$$

Now making use of the representation of the gamma-ratio in (63) as a reciprocal Mellin transform of relation (1.11) in [13]:

$$\frac{\Gamma\left(\frac{1-s}{2}+\frac{ix}{2}\right)\Gamma\left(\frac{1-s}{2}-\frac{ix}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)\Gamma\left(\frac{1+s}{2}\right)}=-\frac{2\sqrt{\pi}}{\sinh(\pi x/2)}\int_0^\infty\operatorname{Im}\ J_{ix/2}^2(y)\,y^{-s}dy,$$

we substitute it in the left-hand side of (63) and it becomes

$$\begin{aligned}&\int_0^\infty x\ JY_{ix}[f]\int_0^\infty\operatorname{Im}J_{ix/2}^2(y)\ y^{-s}dy\,dx=\\&=\frac{4}{\pi}\int_0^\infty v^{-s}\int_v^\infty f(y)dy\,dv,\ \operatorname{Re} s\in\left(\frac{1}{2},1\right).\end{aligned}\tag{64}$$

It is known [2] that $|\mathrm{Im}J^2_{ix/2}(y)|$ is bounded for all $x, y > 0$ and satisfies the inequality

$$|\mathrm{Im}J^2_{ix/2}(y)| < C\ \frac{e^{\pi x}}{y},$$

where $C > 0$ is an absolute constant. Consequently, under conditions of the theorem,

$$x\ JY_{ix}[f]\mathrm{Im}J^2_{ix/2}(y)\ y^{-s} \in L_1(\mathbb{R}_+ \times \mathbb{R}_+; e^{\pi x} y^{-\gamma} dydx),\ \gamma \in \left(\frac{1}{2}, 1\right)$$

and the change of the order of integration is indeed possible in the left-hand side (64) by Fubini's theorem. Furthermore,

$$\int_0^\infty \left| v^{-s} \int_v^\infty f(y)dy \right| dv \le \int_0^\infty v^{-\gamma} \int_v^\infty |f(y)|dydv =$$
$$= \frac{1}{1-\gamma} \int_0^\infty |f(y)| y^{1-\gamma} dy < \infty.$$

Therefore, changing the order of integration in the left-hand side of (64) and then omitting the Mellin transform on the both sides via the uniqueness theorem for functions integrable with respect to the measure $y^{-\gamma}dy$, we find

$$\int_0^\infty x\ \mathrm{Im}J^2_{ix/2}(y)\ JY_{ix}[f]\ dx = \frac{4}{\pi} \int_y^\infty f(v)\ dv.$$

Differentiating this equality with respect to y, we get for almost all $y > 0$ inversion formula (58). Theorem 6 is proved. □

Finally, we prove an analog of Beurling's theorem for index transform (43). We note that the corresponding Plancherel theory and adjoint operator have been considered in [14].

Theorem 7. *Let*

$$f(x,y) \in L_1\left(\mathbb{R}_+ \times \mathbb{R}_+; K_0^2\left(\sqrt{x^2+y^2} - y\right) x^{-1} dxdy\right) \cap L_2\left(\mathbb{R}_+ \times \mathbb{R}_+; x^{-1}dxdy\right)$$

and

$$\int\limits_{\mathbb{R}_+}\int\limits_{\mathbb{R}_+}\int\limits_{\mathbb{R}_+} |f(x,y)KK_{i\tau}[f]|\, K_\tau^2\left(\sqrt{x^2+y^2} - y\right) x^{-1} dxdyd\tau < \infty.$$

Then $f(x,y) = 0$.

Proof. Suppose that $f(x,y) \neq 0$ on a set of the positive measure in $\mathbb{R}_+ \times \mathbb{R}_+$. We have

$$\infty > \int\limits_{\mathbb{R}_+}\int\limits_{\mathbb{R}_+}\int\limits_{\mathbb{R}_+} |f(x,y)KK_{i\tau}[f]|\, K_\tau^2\left(\sqrt{x^2+y^2} - y\right) x^{-1} dxdyd\tau >$$
$$> \int\limits_{\mathbb{R}_+}\int\limits_{\mathbb{R}_+} |f(x,y)|\, K_0^2\left(\sqrt{x^2+y^2} - y\right) x^{-1} dxdy \int\limits_{\mathbb{R}_+} |KK_{i\tau}[f]| d\tau.$$

Therefore, $KK_{i\tau}[f] \in L_1(\mathbb{R}_+; d\tau)$. On the other hand, $KK_{i\tau}[f]$ is a continuous function on $\mathbb{R}_+$ via the absolute and uniform convergence of integral (43), which is guaranteed by the estimate

$$|KK_{i\tau}[f]| \le \int_0^\infty \int_0^\infty K_0\left(\sqrt{x^2+y^2}-y\right) K_0\left(\sqrt{x^2+y^2}+y\right) |f(x,y)| \frac{dxdy}{x} \le$$
$$\le \int_0^\infty \int_0^\infty K_0^2\left(\sqrt{x^2+y^2}-y\right) |f(x,y)| \frac{dxdy}{x} < \infty.$$

Therefore, using relation (2.16.51.6) in [7, Vol. 2]), we calculate the composition of this operator and the cosine Fourier transform (1) to obtain

$$(F_c\ KK_{i\tau}[f])(\lambda) = \sqrt{\frac{\pi}{2}} \int_0^\infty \int_0^\infty K_0\left(2\sqrt{y^2+x^2\cosh^2\frac{\lambda}{2}}\right) f(x,y) \frac{dxdy}{x}.$$

Further, Beurling condition (2) yields

$$\int\limits_{\mathbb{R}_+}\int\limits_{\mathbb{R}_+} |KK_{i\tau}[f](F_c KK_{i\tau}[f])(\lambda)| e^{\tau\lambda} d\tau d\lambda <$$

$$< \sqrt{2\pi} \int\limits_{\mathbb{R}_+}\int\limits_{\mathbb{R}_+} |KK_{i\tau}[f]| \cosh\tau\lambda \int\limits_0^\infty\int\limits_0^\infty K_0\left(2\sqrt{y^2+x^2\cosh^2\frac{\lambda}{2}}\right) |f(x,y)| x^{-1} dydxd\lambda d\tau =$$

$$= \sqrt{2\pi} \int\limits_{\mathbb{R}_+}\int\limits_{\mathbb{R}_+}\int\limits_{\mathbb{R}_+} |f(x,y) KK_{i\tau}[f]| K_\tau\left(\sqrt{x^2+y^2}-y\right) K_\tau\left(\sqrt{x^2+y^2}+y\right) x^{-1} dydxd\tau <$$

$$< \sqrt{2\pi} \int\int\int |f(x,y) KK_{i\tau}[f]| K_\tau^2\left(\sqrt{x^2+y^2}-y\right) x^{-1} dydxd\tau < \infty.$$

So $KK_{i\tau}[f] \equiv 0$. But $f(x,y) \in L_2\left(\mathbb{R}_+ \times \mathbb{R}_+; x^{-1}dxdy\right)$. Hence, recalling the Plancherel theorem for transformation (43) (see [14]), it satisfies the following Parseval identity:

$$\int\limits_{\mathbb{R}_+}\int\limits_{\mathbb{R}_+} |f(x,y)|^2 \frac{dydx}{x} = \frac{16}{\pi^4} \int\limits_{\mathbb{R}_+} \tau \sinh 2\pi\tau |KK_{i\tau}[f]|^2 d\tau.$$

Thus $f(x,y) = 0$ a.e. Theorem 7 is proved. □

Acknowledgements

The present investigation was in part supported by the "Centro de Matemática" of the University of Porto.

REFERENCES

[1] V.A. Ditkin, A.P. Prudnikov, *Operational Calculus*, Vysh. Shkola, Moscow (1975) [in Russian].

[2] A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, *Higher Transcendental Functions*, Vol. II, McGraw-Hill, New York, London and Toronto, 1953.

[3] L. Hörmander, *A uniqueness theorem of Beurling for Fourier transform pairs,* Ark. Mat., **29** (1991), 237–240.

[4] N.N. Lebedev, *Sur une formule d'inversion,* C.R.Acad. Sci. URSS **52** (1946), 655–658.

[5] N.N. Lebedev, *On a representation of an arbitrary function by integrals involving a square of the Macdonald function with an imaginary index.* Siberian Mathem. Journ. **3** (1962) 2, 213–222 [in Russian].

[6] N.N. Lebedev, I.P.Skalskaya, *Certain integral transformations related to the Kontorovich-Lebedev transform.* [in:] Problems of Math. Physics. Leningrad (1976), 68–79 [in Russian].

[7] A.P. Prudnikov, Yu.A. Brychkov, O.I. Marichev, *Integrals and Series: Vol. I: Elementary Functions; Vol. 2: Special Functions*, Gordon and Breach, New York, 1986.

[8] I.N. Sneddon, *The Use of Integral Transforms,* McGraw-Hill, New York, 1972.

[9] S.B. Yakubovich, Yu.F. Luchko, *The Hypergeometric Approach to Integral Transforms and Convolutions* (Kluwers Ser. Math. and Appl.: Vol. 287), Dordrecht, Boston, London, 1994.

[10] S.B. Yakubovich, *Index Transforms,* World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, 1996.

[11] S.B. Yakubovich, *On Lebedev type integral transformations associated with modified Bessel functions,* Nieuw Arc. Wiskd. **17** (1999) 2, 219–227.

[12] S.B. Yakubovich, *The Lebedev-Skalskaya transform.* Kluwer Encycl. of Math. Suppl. II (2000), 304–305.

[13] S.B. Yakubovich, *On the index integral transformation with Nicholson's function as the kernel,* J. Math. Anal. Appl. **269** (2002), 689–701.

[14] S.B. Yakubovich, *The Plancherel and Hausdorff-Young type theorems for an index transformation,* Journal for Analysis and its Applications **25** (2006), 193–204.

[15] S.B. Yakubovich, *Uncertainty principles for the Kontorovich-Lebedev transform,* Math. Model. Anal. **13** (2008) 2, 289–302.

[16] S.B. Yakubovich, *A class of integral equations and index transformations related to the modified and incomplete Bessel functions.* Preprint. CMUP. 2008, 19 pp.

S.B. Yakubovich
syakubov@fc.up.pt

University of Porto
Faculty of Sciences
Department of Pure Mathematics
Campo Alegre st., 687, 4169-007 Porto, Portugal

Received: February 26, 2008.
Accepted: May 21, 2008.