Opuscula Mathematica • Vol. 29 • No. 1 • 2009

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EXTREMAL TRACEABLE GRAPHS WITH NON-TRACEABLE EDGES

Abstract. By NT(n) we denote the set of graphs of order n which are traceable but have non-traceable edges, i.e. edges which are not contained in any hamiltonian path. The class NT(n) has been considered by Balińska and co-authors in a paper published in 2003, where it was proved that the maximum size $t_{\max}(n)$ of a graph in NT(n) is at least $(n^2-5n+14)/2$ (for $n \geq 12$). The authors also found $t_{\max}(n)$ for $5 \leq n \leq 11$.

We prove that, for $n \geq 5$, $t_{\max}(n) = max\{\binom{n-2}{2}+4, \binom{n-\lfloor\frac{n-1}{2}\rfloor}{2} + \lfloor\frac{n-1}{2}\rfloor^2\}$ and, moreover, we characterize the extremal graphs (in fact we prove that these graphs are exactly those already described in the paper by Balińska *et al.*). We also prove that a traceable graph of order $n \geq 5$ may have at most $\lceil\frac{n-3}{2}\rceil\lfloor\frac{n-3}{2}\rfloor$ non traceable edges (this result was conjectured in the mentioned paper by Balińska and co-authors).

Keywords: traceable graph, non-traceable edge.

Mathematics Subject Classification: 05C35, 05C38.

1. INTRODUCTION

Throughout the paper we use the standard terminology, unless stated otherwise. An edge e of a graph G is called *traceable* if it is contained in a hamiltonian path of G, otherwise it is called *non-traceable* edge of G (NT-*edge*). If G has a hamiltonian path and at least one non-traceable edge, then G is called NT-*graph*. The set of all NT-graphs of order n is denoted by NT(n). NT-graphs were introduced and studied in [1].

It is easy to check that the least number n for which NT(n) is not empty is $n = 5$ and, moreover, NT(n) is not empty for every $n \geq 5$.

By $t_{\max}(n)$ we denote the maximum size of NT-graph of order n. In [1] it was proved that

$$t_{\max}(n) = (3n^2 - 4n + 1)/8 \qquad \text{for} \quad n = 7, 9, \tag{1}$$
$$t_{\max}(n) = (n^2 - 5n + 14)/2 \qquad \text{for} \quad n = 5, 6, 8, 10, 11, \tag{2}$$
$$t_{\max}(n) \geq (n^2 - 5n + 14)/2 \qquad \text{for} \quad n \geq 12. \tag{3}$$

Balińska *et al.* [1] have also indicated that

$$\overline{K}_2 * K_2 * K_{n-4} \qquad \text{for} \quad n = 8 \quad \text{and for} \quad n \geq 10, \tag{4}$$
$$\overline{K}_{\frac{n-1}{2}} * K_{\frac{n-1}{2}} * K_1 \qquad \text{for} \quad n = 5, 7, 9, 11, \tag{5}$$
$$\overline{K}_{\frac{n-2}{2}} * K_{\frac{n-2}{2}} * K_2 \qquad \text{for} \quad n = 6 \quad \text{and for} \quad n = 8, \tag{6}$$

(where $*$ denotes the join of graphs) are NT-graphs with size $(3n^2 - 4n + 1)/8$ for $n = 7, 9$ and $(n^2 - 5n + 14)/2$ for $n \geq 5, n \neq 7, 9$.

We prove that the bounds found in [1] are in fact the best possible and, moreover, the family of graphs described by (4)–(6) is the set of all NT-graphs of order n and maximum size.

Theorem 1. *The maximum size $t_{\max}(n)$ of* NT*-graph of order n is given by the formula*

$$t_{\max}(n) = \max\left\{ \binom{n-2}{2} + 4, \binom{n - \lfloor \frac{n-1}{2} \rfloor}{2} + \left\lfloor \frac{n-1}{2} \right\rfloor^2 \right\}.$$

Moreover, the NT*-graphs of size $t_{\max}(n)$ are the following.*

1. $\overline{K}_2 * K_2 * K_{n-4}$ *for $n = 8$ and for $n \geq 10$,*
2. $\overline{K}_{\frac{n-1}{2}} * K_{\frac{n-1}{2}} * K_1$ *for $n = 5, 7, 9, 11$,*
3. $\overline{K}_{\frac{n-2}{2}} * K_{\frac{n-2}{2}} * K_2$ *for $n = 6, 8$.*

Note that for $n = 8$ and for $n = 11$ there are two extremal graphs. An easy computation shows that $\binom{n-2}{2} + 4 \geq \binom{n-\lfloor \frac{n-1}{2} \rfloor}{2} + \lfloor \frac{n-1}{2} \rfloor^2$ for $n = 5, 6, 8$ and for $n \geq 10$.

For every integer $n \geq 5$ the graph $\overline{K}_2 * K_2 * K_{n-4}$ contains exactly one NT-edge. It is also very easy to check that for $n \geq 5$ the graph $\overline{K}_{\lfloor \frac{n-1}{2} \rfloor} * K_{\lfloor \frac{n-1}{2} \rfloor} * K_{n-2\lfloor \frac{n-1}{2} \rfloor}$ has exactly $\binom{\lfloor \frac{n-1}{2} \rfloor}{2}$ NT-edges. In [1], for every $n \geq 5$, Balińska *et al.* presented a graph of order n with $\lceil \frac{n-3}{2} \rceil \lfloor \frac{n-3}{2} \rfloor$ NT-edges. They also conjectured that the maximum number of NT-edges of a graph in $\mathrm{NT}(n)$ is equal to $\lceil \frac{n-3}{2} \rceil \lfloor \frac{n-3}{2} \rfloor$. We prove this conjecture.

Theorem 2. *Let $n \geq 5$ and let $b_{\max}(n)$ denote the maximum number of* NT*-edges of a graph in* $\mathrm{NT}(n)$*. Then*

$$b_{\max}(n) = \left\lceil \frac{n-3}{2} \right\rceil \left\lfloor \frac{n-3}{2} \right\rfloor.$$

2. PROOF OF THEOREM 1

The proof of Theorem 1 follows from the two below quoted theorems in [2] (these theorems are in fact corollaries from Theorem 3 of [2], which is a much more general result).

Theorem 3 (see [2])**.** *Let G be a graph of order $n \geq 5$ and minimum vertex degree $\delta(G) \geq 2$. If G is a non-hamiltonian graph of maximum size then either*

$$G = \overline{K}_2 * K_2 * K_{n-4}$$

or $n \in \{5, 6, 7, 8, 9, 11\}$ and

$$G = \overline{K}_{\lfloor \frac{n-1}{2} \rfloor} * K_{\lfloor \frac{n-1}{2} \rfloor} * K_{n-2\lfloor \frac{n-1}{2} \rfloor}.$$

Note that the set of extremal graphs in Theorem 3 is exactly the same as that in Theorem 1.

Theorem 4 (see [2])**.** *If G is a non-hamiltonian connected graph of order $n \geq 4$ and of maximum size, then either*

$$G = K_1 * K_2 * K_{n-3}$$

or else $n = 6$ and

$$G = \overline{K}_3 * K_3.$$

Let G be a graph of order $n \geq 5$ and size

$$\|G\| \geq \max\left\{ \binom{n-2}{2} + 4, \binom{n - \lfloor \frac{n-1}{2} \rfloor}{2} + \left\lfloor \frac{n-1}{2} \right\rfloor^2 \right\}.$$

We shall prove that every edge of G is contained in a hamiltonian path of G or G is one of the extremal graphs listed in the assertion of Theorem 1.

Let us first suppose that there is a vertex v in G such that $d_G(v) \leq 1$ (by $d_G(x)$ we denote the degree of the vertex x in G). Note that then $d_G(v) = 1$ since G is traceable. Then $\|G - \{v\}\| = \|G\| - 1$. We now apply Theorem 4 and prove that $G - \{v\}$ is hamiltonian connected and check that then every edge of G is contained in a hamiltonian path of G.

It is very easy to see that if a graph is hamiltonian then every of its edges lies in a hamiltonian path – this fact was also observed in [1].

Suppose now that $\delta(G) \geq 2$. By Theorem 3, either G is hamiltonian, and therefore contains every edge in a hamiltonian path, or it is one of extremal graphs of Theorem 1 and the proof is finished.

3. PROOF OF THEOREM 2

Since it has been shown in [1] that for every $n \geq 5$ there is a graph of order n which is traceable and has $\lfloor \frac{n-3}{2} \rfloor \lceil \frac{n-3}{2} \rceil$ non-traceable edges, it remains to prove that in every traceable graph G of order $n \geq 5$ there are at most $\lfloor \frac{n-3}{2} \rfloor \lceil \frac{n-3}{2} \rceil$ non-traceable edges.

Let $(x_1, \ldots, x_n)$ be a hamiltonian path of a traceable graph G of order n. The following facts are very easy to check.

(i) every edge $x_i x_{i+1}$ (for $i = 1, \ldots, n-1$) is traceable,
(ii) every edge $x_1 x_i$ $(1 < i \leq n)$ is traceable and every edge $x_j x_n$ $(1 \leq j < n)$ is traceable,
(iii) if $i > 1, s \geq 2$ and $i+s < n$, then at most one of the edges $x_i x_{i+s}$ and $x_{i+1} x_{i+s+1}$ is non-traceable (if both edges $x_i x_{i+s}$ and $x_{i+1} x_{i+s+1}$ are present in G, then they are traceable).

By (i)–(ii) only the edges of the form $x_i x_{i+s}$ with $i > 1$, $2 \leq s \leq n-3$ and $i+s \leq n$ may be non traceable. Moreover, by (iii), for any fixed s, $2 \leq s \leq n-2$, we may have at most $\lceil \frac{n-s-2}{2} \rceil$ non-traceable edges of the form $x_i x_{i+s}$. Hence, the maximum number $b(G)$ of non-traceable edges in G verifies

$$b(G) \leq \sum_{s=2}^{n-3} \left\lceil \frac{n-s-2}{2} \right\rceil .$$

One may now prove easily, for instance by induction, the formula

$$\sum_{s=2}^{n-3} \left\lceil \frac{n-s-2}{2} \right\rceil = \left\lfloor \frac{n-3}{2} \right\rfloor \left\lceil \frac{n-3}{2} \right\rceil$$

for $n \geq 5$.

Acknowledgements
The research was partially supported by AGH local grant No 11 420 04.

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Received: July 27, 2008.
Accepted: October 30, 2008.