Bożena Piątek

ON THE CONTINUITY
OF THE INTEGRABLE MULTIFUNCTIONS

Abstract. The generalization of the Polovinkin theorem is studied.

Keywords: multifunctions, Riemann integral, Aumann integral, Hausdorff metric.

Mathematics Subject Classification: 26E25, 28B20.

1. INTRODUCTION

In 1975 E.S. Polovinkin showed that the continuity almost everywhere of a bounded multifunction $F\colon [a,b] \to cc(\mathbb{R}^n)$ is a necessary and sufficient condition for the Riemann integrability (see [10]). Our main goal is to give a similar characterization of the Riemann integrability for a more general class of multifunctions. Moreover, we compare the Riemann integral with the Debreu (see [4]) and Aumann ones (e.g. [1]).

Let $(X, \|\cdot\|)$ be a real Banach space. Denote by $cc(X)$ the set of all nonempty convex compact subsets of X. For given $A, B \in cc(X)$, we set $A + B = \{a + b : a \in A, b \in B\}$ and $\lambda A = \{\lambda a : a \in A\}$ for $\lambda \geq 0$. It is easy to see that $(cc(X), +, \cdot)$ satisfies the following properties

$$\lambda(A + B) = \lambda A + \lambda B, \ (\lambda + \mu)A = \lambda A + \mu A, \ \lambda(\mu A) = (\lambda\mu)A, \ 1 \cdot A = A$$

for each $A, B \in cc(X)$ and $\lambda \geq 0$, $\mu \geq 0$. If $A, B, C \in cc(X)$, then the equality $A + C = B + C$ implies $A = B$ (see e.g. [11, Lemma 2]). Thus the cancellation law holds in $cc(X)$ with the additive operation.

The set $cc(X)$ is a metric space with the Hausdorff metric d defined by the relation

$$d(A, B) = \inf \left\{ t > 0 : A \subset B + tS, B \subset A + tS \right\},$$

where S denotes the closed unit ball in X. The metric space $(cc(X), d)$ is complete (see e.g. [3, Theorem II-3, p. 40]). Moreover, the Hausdorff metric d is translation invariant, since

$$d(A + C, B + C) = d(A, B)$$

(cf. [11, Lemma 3]) and positively homogeneous, i.e.,

$$d(\lambda A, \lambda B) = \lambda d(A, B)$$

for all $\lambda \geq 0$ and $A, B, C \in cc(X)$. In the sequel, the continuity is understood with respect to the Hausdorff metric.

Let $F\colon [a, b] \to cc(X)$ be any multifunction. A set $\Delta = \{x_0, x_1, \ldots, x_n\}$, where $a = x_0 < x_1 < \ldots < x_n = b$, is said to be a *partition* of $[a, b]$. For a given partition Δ, we put $\delta(\Delta) := \max_{i \in \{1, \ldots, n\}} |x_i - x_{i-1}|$ and form the approximating sum

$$S(F, \Delta, \tau) = (x_1 - x_0)F(\tau_1) + \ldots + (x_n - x_{n-1})F(\tau_n),$$

where τ is a system $(\tau_1, \ldots, \tau_n)$ of intermediate points corresponding with $\Delta(\tau_i \in [x_{i-1}, x_i])$.

If for every sequence (Δ^n, τ^n), where Δ^n are partitions of $[a, b]$ such that

$$\lim_{n \to \infty} \delta(\Delta^n) = 0, \tag{1}$$

and τ^n, $n \in \mathbb{N}$, are systems of intermediate points corresponding with Δ^n, the sequence of the approximating sums $(S(F, \Delta^n, \tau^n))$ tends to the limit $I \in cc(X)$, then F is said to be *Riemann integrable* over $[a, b]$ and

$$(R)\int_{[a,b]} F(x)dx := I.$$

Obviously, the limit I is independent of the choice of the sequence of partitions and the sequence of systems of intermediate points.

The Riemann integral for multifunctions with nonempty compact convex values in $\mathbb{R}^n$ was introduced by Alexander Dinghas (see [5]) in 1956. Nine years later, Robert Aumann in [2] introduced a different definition of the multivalued integral. This concept was based on the Lebesgue integral for real functions. Next that definition was generalized to the case of an infinite dimension. Suppose for a moment that $F\colon [a, b] \to cl(X)$, where $cl(X)$ means the set of all nonempty closed subsets of a separable Banach space X and let S_F be the set of all Bochner integrable selections of F, i.e., $f(t) \in F(t)$ for almost all $t \in [a, b]$. The *Aumann integral* (see e.g. [1]) is defined as

$$(A)\int_{[a,b]} F dt = \left\{ (B)\int_{[a,b]} f dt \; : \; f \in S_F \right\},$$

where (B) means the Bochner integral. Of course, S_F and in consequence the integral $(A)\int_{[a,b]} F dt$ may be empty sets.

$A \subset \mathbb{R}$ will be called a null set if its Lebesgue measure is equal to zero.

Further, we will say that a set $K \subset X$ is *totally bounded* if for each $\varepsilon > 0$ one can find a finite set $\{x_1, \ldots, x_m\}$ such that

$$K \subset \bigcup_{k=1}^{m} \bar{B}(x_k, \varepsilon)$$

(see, e.g., [6, Definition I.6.14, p. 22]). Clearly, a subset K of X is totally bounded if and only if for each sequence (x_n), $x_n \in K$, there exists a subsequence $(x_{p(n)})$ which tends to some $x \in X$ (see [6, Theorem I.6.15, p. 22]).

2. MAIN RESULTS

In the sequel, we need the following:

Proposition 1. *Let X be a separable real Banach space. Then there exists a countable set C of continuous linear forms on X such that if K is a non-empty, compact, convex subset of X and B is a closed ball in X disjoint from K, then there is $\xi \in C$ for which*

$$\max \xi(K) = \max_{x \in K} \xi(x) < \inf \xi(B) = \inf_{x \in B} \xi(x).$$

Moreover, if $0 \in B$, then one can find $\xi \in C$ such that

$$\max \xi(K) < -1 < \inf \xi(B). \tag{2}$$

Proof. The first part is due to G. Debreu (see [4, Theorem 5.9]). To prove the next one, it is sufficient to take

$$\bar{C} := \{ q \cdot \xi \ : \ q \in \mathbb{Q}, \ \xi \in C \}.$$

$\square$

Theorem 1. *Let X be a separable real Banach space and let $F \colon [a,b] \to cc(X)$ be a Riemann integrable multifunction. If there is a null set D such that for each $t \in (a,b) \setminus D$ there exist a positive number $\tau \le \min \{t-a, b-t\}$ and a totally bounded set $K \subset X$ for which*

$$\bigcup_{u \in (t-\tau, t+\tau)} F(u) \subset K, \tag{3}$$

then F is continuous a.e. on $[a,b]$.

Proof. First we note that the Riemann integrability of F implies the Riemann integrability of real functions F^ξ, $\xi \in X^*$, defined as

$$F^\xi(t) = \sup \xi(F(t)), \qquad t \in [a,b].$$

Indeed, let (Δ^n) be any sequence of partitions of $[a,b]$ such that (1) holds and suppose that (τ^n) is a sequence of systems of intermediate points corresponding with the respective Δ^n. Let us fix $\varepsilon > 0$. If $m, n \in \mathbb{N}$ are sufficiently large, then we have

$$|S(F^\xi, \Delta^n, \tau^n) - S(F^\xi, \Delta^m, \tau^m)| \le \|\xi\| \ d(S(F, \Delta^n, \tau^n), S(F, \Delta^m, \tau^m)) < \|\xi\|\varepsilon.$$

on account of (2)–(3) from [9], the proof of [9, Lemma 2] and the Riemann integrability of F. Hence we conclude that for each function F^ξ the set A_ξ of points of noncontinuity is null. Let C be defined as in Proposition 1. Obviously,

$$A := \{a, b\} \cup \bigcup_{\xi \in C} A_\xi \cup D$$

is also a null set.

Now we will show that F is ε–u.s.c. (e.g. [8, Section II.2.1, p. 28]) at each point of $[a,b] \setminus A$. Suppose that it is not true. Hence one can find $t_0 \in [a,b] \setminus A$, $\varepsilon > 0$ and a sequence (t_n) such that $t_n \to t_0$ and $F(t_n) \not\subset B(F(t_0), \varepsilon)$, $n \in \mathbb{N}$, where $B(F(t_0), \varepsilon)$ means an ε–neighbourhood of $F(t_0)$.

Let us take any $e_n \in F(t_n) \cap (X \setminus B(F(t_0), \varepsilon))$, $n \in \mathbb{N}$. Of course, according to hypothesis (3) there is $e_n \in K$ for each sufficiently large n. Since the set K is totally bounded, there is a subsequence $(e_{p(n)})$ which tends to some $e \in X$. It is simple to see that

$$\bar{B}\left(e, \frac{\varepsilon}{2}\right) \cap F(t_0) = \emptyset.$$

Let $\xi \in C$ be such that

$$F^\xi(t_0) = \max \xi(F(t_0)) < \inf \xi \left(\bar{B}\left(e, \frac{\varepsilon}{2}\right)\right) =: L.$$

Since $e_{p(n)} \in \bar{B}(e, \frac{\varepsilon}{2})$, note that $F^\xi(t_{p(n)}) \geq L$ for each sufficiently large n. By the continuity of F^ξ at t_0, it follows that $F^\xi(t_0) \geq L$, a contradiction.

Next we will show that F is l.s.c. (e.g. [8, Section II.2.2, p. 34]) at each point of $[a,b] \setminus A$. Suppose that $t_0 \in [a,b] \setminus A$ and F is not l.s.c. at t_0. Hence there are an open set $U \subset X$ such that $U \cap F(t_0) \neq \emptyset$ and a sequence (t_n) for which $t_n \to t_0$ and $U \cap F(t_n) = \emptyset$ for all $n \in \mathbb{N}$. The sequence may be chosen in such a way that $|t_n - t_0| < \tau$, where τ is matched to t_0 with respect to the hypothesis. Without loss of generality we may assume that $0 \in U \cap F(t_0)$. Otherwise, let us consider a multifunction $G \colon [a,b] \to cc(X)$ defined as

$$G(x) = F(x) + e,$$

where $-e$ belongs to $U \cap F(t_0)$. Obviously, F and G have the same properties. Then there is a closed ball $\bar{B}(0, R) \subset U$ disjoint with each $F(t_n)$. By Proposition 1, for any $n \in \mathbb{N}$, there is $\xi_n \in C$ such that

$$F^{\xi_n}(t_n) = \max \xi_n(F(t_n)) < -1 < \inf \xi_n \left(\bar{B}(0, R)\right). \tag{4}$$

Let $B^0 := \left\{\xi \in X^* \ : \ \|\xi\| \leq R^{-1}\right\}$. It is not difficult to see that each $\xi_n \in B^0$.

According to Theorem 2.10.1 in [7, p. 37], it follows that B^0 is sequentially compact in weak* topology of X^*, which implies that there is a subsequence $(\xi_{p(n)})$ which tends to some $\xi \in B^0$.

Let τ and a totally bounded set K be matched to t_0. We will show that $F^{\xi_{p(n)}}$ tends uniformly to F^ξ on $(t_0 - \tau, t_0 + \tau)$. Let us fix $\varepsilon > 0$ and take $\delta = \frac{R\varepsilon}{R+2}$. One can choose $\{x_k \in K \ : \ k \in \{1, \ldots, m\}\}$ such that

$$K \subset \bigcup_{k=1}^{m} \bar{B}(x_k, \delta). \tag{5}$$

Moreover, there is $N \in \mathbb{N}$ such that $|\xi_{p(n)}(x_k) - \xi(x_k)| < \delta$ for each $n \geq N$ and $k \in \{1, \ldots, m\}$. Now let us fix $x \in K$. By (5) we find k for which $\|x_k - x\| \leq \delta$ and

$$|\xi_{p(n)}(x) - \xi(x)| \leq |\xi_{p(n)}(x) - \xi_{p(n)}(x_k)| + |\xi_{p(n)}(x_k) - \xi(x_k)| + |\xi(x_k) - \xi(x)| <$$
$$< \frac{\delta}{R} + \delta + \frac{\delta}{R} = \varepsilon$$

when $n \geq N$. Hence for all $t \in (t_0 - \tau, t_0 + \tau)$ and $x \in F(t)$

$$\xi_{p(n)}(x) \leq \sup_{y \in F(t)} \xi(y) + \varepsilon = F^\xi(t) + \varepsilon$$

and finally

$$F^{\xi_{p(n)}}(t) \leq F^\xi(t) + \varepsilon, \quad n \geq N.$$

In the same way we obtain

$$F^\xi(t) \leq F^{\xi_{p(n)}}(t) + \varepsilon$$

for each $t \in (t_0 - \tau, t_0 + \tau)$ and $n \geq N$. Thereby, $F^{\xi_{p(n)}}$ tends uniformly to F^ξ on $(t_0 - \tau, t_0 + \tau)$.

Using this and the continuity of $F^{\xi_{p(n)}}$ at t_0, we obtain $F^{\xi_{p(n)}}(t_{p(n)}) \to F^\xi(t_0)$. On account of (4), it follows that $F^\xi(t_0) \leq -1$, contrary to $0 \in F(t_0)$.

Since the multifunction F is l.s.c. and ε–u.s.c. for all $t \in [a, b] \setminus A$ and the values of F are compact, F is continuous at each $t \in [a, b] \setminus A$ (e.g. [8, Section II.2.2, p. 35]). $\square$

The assumption on the existence of a totally bounded set K (3) is an essential condition for the continuity (see Example). Now we discuss the problem of the converse theorem. First we consider the embedding of the metric space $(cc(X), d)$ in a real Banach space, which we will need later. Owing to theorem [11] (see also [4, Theorem 5.5]), the following holds true.

Proposition 2. *The space $(cc(X), d)$ can be embedded as a convex cone in a real normed space $\mathcal{L}$ in such a way that the embedding is isometric and operations of addition and multiplication by nonnegative numbers in $\mathcal{L}$ induce the corresponding operations in $cc(X)$.*

Furthermore, using the Theorem II.5 [6, p. 89], we get

Proposition 3. *Every real normed space is isomorphic to and isometric with a dense linear subspace of a real Banach space.*

Therefore, we obtain

Proposition 4. *Let X be a real Banach space. Then there exist a real Banach space $(\bar{\mathcal{L}}, \| \cdot \|)$ and a mapping $\Phi \colon cc(X) \to \bar{\mathcal{L}}$ such that*

$$\Phi(A + B) = \Phi(A) + \Phi(B), \quad \Phi(\lambda A) = \lambda \Phi(A), \quad \|\Phi(A) - \Phi(B)\| = d(A, B)$$

for all $A, B \in cc(X)$, $\lambda \geq 0$.

Theorem 2. *Let X be a real Banach space and let $F \colon [a, b] \to cc(X)$ be a bounded multifunction. If F is continuous a.e. on $[a, b]$, then F is Riemann integrable.*

Proof. Let (Δ^n) be a sequence of partitions of $[a, b]$ and let (τ^n) be a sequence of systems of intermediate points corresponding to the respective Δ^n. Suppose also that (1) holds. We have to show that the sequence $(S(F, \Delta^n, \tau^n))$ tends to a limit $I \in cc(X)$.

For each $n \in \mathbb{N}$, we define a multifunction $F_n \colon [a,b] \to cc(X)$ as $F_n(t) = F(\tau_k^n)$ if $t \in (t_{k-1}^n, t_k^n]$, $F_n(a) = F(a)$. By the continuity of F and condition (1), a sequence $(F_n(t))$ converges to $F(t)$ for almost each $t \in [a,b]$. According to Proposition 4, there exist a real Banach space $\bar{\mathcal{L}}$ and an isometric embedding $\Phi \colon cc(X) \to \bar{\mathcal{L}}$. The isometry of the mapping implies that

$$\Phi\left(F_n(t)\right) \to \Phi\left(F(t)\right)$$

for almost each $t \in [a,b]$.

Obviously, $\Phi\left(F_n\right) \colon [a,b] \to \bar{\mathcal{L}}$ is a simple vector-valued function, so the Bochner integral

$$(B) \int_{[a,b]} \Phi\left(F_n\right) dt$$

exists for each $n \in \mathbb{N}$. Moreover,

$$(B) \int_{[a,b]} \Phi\left(F_n\right) dt = S\left(\Phi(F), \Delta^n, \tau^n\right) = \Phi\left(S(F, \Delta^n, \tau^n)\right). \tag{6}$$

By the assumptions, one can find a positive number $M > 0$ such that $\|\Phi\left(F_n(t)\right)\| \leq M$ for each $t \in [a,b]$ and $n \in \mathbb{N}$. According to the Lebesgue dominated convergence theorem (see e.g. [6, Corollary III.6.16, p. 151]), it follows that the function $\Phi(F)$ is Bochner integrable and

$$(L) \int_{[a,b]} \|\Phi(F_n) - \Phi(F)\| dt \to 0 \quad \text{if} \quad n \to \infty, \tag{7}$$

where (L) means the Lebesgue integral. Moreover (see [7, Theorem 3.7.6, p. 82]), there is

$$\left\| (B) \int_{[a,b]} \Phi(F_n) dt \; - \; (B) \int_{[a,b]} \Phi(F) dt \right\| \leq (L) \int_{[a,b]} \|\Phi(F_n) - \Phi(F)\| dt,$$

which combined with (6) and (7) yields

$$\Phi\left(S(F, \Delta^n, \tau^n)\right) \to (B) \int_{[a,b]} \Phi(F) dt.$$

On account of the completeness of $(cc(X), d)$ and Proposition 4, we conclude that there exists $I \in cc(X)$ such that

$$\Phi(I) = (B) \int_{[a,b]} \Phi(F) dt$$

which completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Corollary 1. *Let X be a real Banach space. If a bounded multifunction $F : [a,b] \to cc(X)$ is continuous a.e. on $[a,b]$, then the set of the Bochner integrable selections of F is non-empty and*

$$(R) \int_{[a,b]} F dt = (A) \int_{[a,b]} F dt.$$

Proof. From the construction of the sequence (F_n) in the proof of Theorem 2, it follows that F is Debreu integrable (see [4, Section 6]) and

$$(R)\int_{[a,b]} Fdt = (D)\int_{[a,b]} Fdt,$$

where (D) means the integral defined in [4]. This integral is equal to the Aumann one (see [4, Theorem 6.5]). □

If we assume that F is a bounded multifunction with values in $\mathbb{R}^n$, the condition from Theorem 1 is obviously satisfied. Hence we get the following Polovinkin theorem as an immediate consequence of Theorems 1 and 2.

Corollary 2 (see [10, Theorem 1]). *Let $F\colon [a,b] \to cc(\mathbb{R}^n)$ be a bounded multifunction. Then F is Riemann integrable on $[a,b]$ if and only if F is continuous a.e. on $[a,b]$. Moreover,*

$$(R)\int_{[a,b]} Fdt = (A)\int_{[a,b]} Fdt.$$

As it has already been mentioned, the existence of a totally bounded set K from Theorem 1 is an essential condition for the continuity. To illustrate it, we present the following example.

Example

Let $Q = \{q_1, q_2, \ldots\}$ be the set of all rational numbers between 0 and 1. We define a multifunction $F\colon [0,1] \to cc(c_0)$ as

$$F(t) := \begin{cases} \{(u_k) \ : \ u_k = 0, \ k \in \mathbb{N}\} & \text{if } t \notin Q, \\ \{(u_k) \ : \ u_n \in [0,1], \ u_k = 0 \text{ for } k \neq n\} & \text{if } t = q_n. \end{cases}$$

Obviously, the multifunction F is discontinuous at each point of $[0,1]$.

On the other hand, it is not difficult to see that F is integrable on $[0,1]$ and

$$(R)\int_{[a,b]} Fdt = \{0\}.$$

Indeed, let $\Delta = \{t_0, t_1, \ldots, t_n\}$ be any partition of $[0,1]$ and let $\tau = (\tau_1, \ldots, \tau_n)$ be any system of intermediate points corresponding with Δ. If $x \in S(F, \Delta, \tau)$, then for each $k \in \mathbb{N}$ there is:

— $x_k = 0$ if $q_k \notin \{\tau_1, \ldots, \tau_n\}$;
— $x_k \in [0, t_i - t_{i-1}]$ if there is only one $i \in \{1, \ldots, n\}$ such that $\tau_i = q_k$;
— $x_k \in [0, t_{i+1} - t_{i-1}]$ if there is $i \in \{1, \ldots, n\}$ for which $\tau_i = \tau_{i+1} = q_k$.

Thus $\|x\| \leq 2\delta(\Delta)$.

REFERENCES

[1] J.P. Aubin, H. Frankowska, *Set-valued analysis,* Birkhäuser, Boston, Basel, Berlin, 1990.

[2] R.J. Aumann, *Integrals of set-valued functions,* J. Math. Anal. Appl. **12** (1965), 1–12.

[3] C. Castaing, M. Valadier, *Convex analysis and mesurable multifunctions,* Springer-Verlag, Berlin, Heidelberg, New York, 1977.

[4] G. Debreu, *Integration of correspondences*, Proc. Fifth Berkeley Sympos. Math. Statist. and Probability 2 (1) (1965/66), 351–372, Univ. California Press, Berkeley, Calif.

[5] A. Dinghas, *Zum Minkowskischen Integralbegriff abgeschlossener Mengen*, Math. Z. **66** (1956), 173–188.

[6] N. Dunford, F.T. Schwartz, *Linear operators I*, Wiley, New York, 1958.

[7] E. Hille, R. Phillips, *Functional analysis and semi-groups*, Amer. Math. Soc. Colloquium Publ. 31, Providence, 1957.

[8] M. Kisielewicz, *Differential inclusions and optimal control*, Kluwer Academic Pres, Dordrecht, 1990.

[9] B. Piątek, *On convex and ∗-concave multifunctions*, Ann. Polon. Math. **86** (2005), 165–170.

[10] E.S. Polovinkin, *Riemannian integral of set-valued functions*, Lecture Notes in Comput. Sci. **27** (1975), 405–410.

[11] H. Rådström, *An embedding theorem for spaces of convex sets*, Proc. Amer. Math. Soc. **3** (1952), 165–169.

Bożena Piątek
b.piatek@polsl.pl

Silesian University of Technology
Institute of Mathematics
Kaszubska 23, 44–100 Gliwice, Poland