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VERTICES BELONGING TO ALL OR TO NO MINIMUM LOCATING DOMINATING SETS OF TREES

Abstract. A set D of vertices in a graph G is a locating-dominating set if for every two vertices u, v of $G \setminus D$ the sets $N(u) \cap D$ and $N(v) \cap D$ are non-empty and different. In this paper, we characterize vertices that are in all or in no minimum locating dominating sets in trees. The characterization guarantees that the γ_L-excellent tree can be recognized in a polynomial time.

Keywords: domination, locating domination.

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1. INTRODUCTION AND PRELIMINARY RESULTS

For a simple graph $G = (V, E)$, the *open neighborhood* of a vertex $v \in V$ is $N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. A set $D \subseteq V$ is a *dominating set* if for each vertex $v \in V - D$, $N(v) \cap D \neq \emptyset$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set in G, see [2].

We are interested in a variation of domination in graphs. A set $D \subseteq V$ is a *locating-dominating set (LDS)* if it is dominating and every two vertices x, y of $V \setminus D$ satisfy $N(x) \cap D \neq N(y) \cap D$. The *locating-domination number* $\gamma_L(G)$ is the minimum cardinality of a locating-dominating set. Locating-domination was introduced by Slater [9, 10]. Moreover, since every *locating-dominating set* is a *dominating set*, then every graph G satisfies the inequality

$$\gamma(G) \leq \gamma_L(G).$$

For any parameter $\mu(G)$ associated with a graph property $\mathcal{P}$, we refer to a set of vertices with Property $\mathcal{P}$ and cardinality $\mu(G)$ as a $\mu(G)$-set. A graph G is called a $\mu(G)$-excellent graph if every vertex of G is contained in a $\mu(G)$-set.

For more details on domination in graphs, see the monographs by Haynes, Hedetniemi and Slater [5, 6] and also [7].

Many researchers have been interested in characterizing the vertices of G that are in all or in no set with the cardinality $\mu(G)$. Indeed, Hammer *et al.* [4] have characterized those vertices in a graph for independent sets with maximum cardinalities, Mynhardt [8] has characterized the vertices in all or in no minimum dominating sets of trees, Cockayne *et al.* [3] have characterized the set of vertices contained in all or in no total dominating sets of trees and Blidia *et al.* [1] have characterized the set of vertices contained in all or in no minimum double dominating sets of trees.

In this paper, we investigate vertices belonging to all or to no minimum locating dominating sets of a tree and we deduce a polynomial algorithm to recognize a γ_L-excellent tree.

For this purpose, we introduce the following notation.

For a tree T we define the sets $\mathcal{A}_L(T)$ and $\mathcal{N}_L(T)$ by

$$\begin{aligned} \mathcal{A}_L(T) &= \{v \in V(T) \mid v \text{ is in every } \gamma_L(T)\text{-set}\} \quad \text{and} \\ \mathcal{N}_L(T) &= \{v \in V(T) \mid v \text{ is in no } \gamma_L(T)\text{-set}\}. \end{aligned}$$

The *degree* of a vertex v, denoted by $\deg_G(v)$, is the number of vertices adjacent to v and the *diameter* of G is $diam(G) = \max\{d(x,y) \mid x, y \in V(G)\}$ where $d(x,y)$ is the length of the shortest path between x and y. Specifically, for a vertex v in a rooted tree T, we let $C(v)$ and $D(v)$ denote the set of children and descendants, respectively, of v, and we define $D[v] = D(v) \cup \{v\}$. The *maximal subtree* at v is the subtree of T induced by $D[v]$; it is denoted by T_v. A *leaf* (or pendent vertex) of T is a vertex of degree one, while a *support vertex* of T is a vertex adjacent to a leaf. We denote the set of leaves and support vertices set of T by $L(T)$ and $S(T)$, respectively. Let T be a rooted tree. We denote by $L(v)$ the set of leaves of T_v distinct from v, that is, $L(v) = D(v) \cap L(T)$. A vertex of degree at least three is called a *branch vertex.* We denote by $B(T)$ the set of all branch vertices of T. We also define the sets

$$L^j(v) = \{u \in L(v) \mid d(u,v) \equiv j \pmod 5\}, \quad \text{where} \quad j = 0,1,2,3,4.$$

A path on n vertices is denoted by P_n.

Below we give some straightforward observations.

Observation 1. *If T is a tree of diameter at least 2 and y a vertex of $L(T)$, then there is a $\gamma_L(T)$-set that does not contain y.*

Observation 2. *In a nontrivial path P_n, $L(P_n) \subseteq \mathcal{N}_L(P_n)$ if and only if $n \equiv 0 \pmod 5$.*

The following lemma will be used in the next section.

Lemma 1. *Let T' be a tree and v a vertex of $V(T')$. Let u be a vertex of T' such that $u \neq v$. Let T be the tree obtained from T' by adding a path $P_5 = x_1x_2x_3x_4x_5$ and the edge ux_1. Then:*

(1) $\gamma_L(T) = \gamma_L(T') + 2$,
(2) $v \in \mathcal{A}_L(T')$ *if and only if* $v \in \mathcal{A}_L(T)$,
(3) $v \in \mathcal{N}_L(T')$ *if and only if* $v \in \mathcal{N}_L(T)$.

Proof. Let T be the tree obtained from T' by adding a path $P_5 = x_1x_2x_3x_4x_5$ and the edge ux_1 where $u \neq v$.

(1) Every $\gamma_L(T')$-set can be extended to an *LDS* of T by adding the vertices x_2 and x_4, so $\gamma_L(T) \leq \gamma_L(T')+2$. On the other hand, let S be a $\gamma_L(T)$-set. If $u \in S$, then clearly $|S \cap P_5| = 2$ and $S' = S - S \cap P_5$ is an *LDS* of T' with $|S'| = \gamma_L(T) - 2 \geq \gamma_L(T')$. Otherwise ($u \notin S$), if $x_1 \notin S$, then clearly $|S \cap P_5| = 2$ and $S' = S - S \cap P_5$ is an *LDS* of T' with $|S'| = \gamma_L(T) - 2 \geq \gamma_L(T')$. If $x_1 \in S$, in this case $|S \cap P_5| = 3$, let $S' = (S - (S \cap P_5)) \cup \{u\}$ then S' is an *LDS* of T' with $|S'| = \gamma_L(T) - 3 + 1 = \gamma_L(T) - 2$. So in each case we have $\gamma_L(T) \geq \gamma_L(T') + 2$. Therefore, $\gamma_L(T) = \gamma_L(T') + 2$.

(2) Assume that $v \notin \mathcal{A}_L(T')$ and let S' be a $\gamma_L(T')$-set which does not contain v. Then $S = S' \cup \{x_1, x_2\}$ is a $\gamma_L(T)$-set that does not contain v, and so $v \notin \mathcal{A}_L(T)$. Conversely, assume that $v \in \mathcal{A}_L(T')$ and let S be any $\gamma_L(T)$-set with $S' = S \cap V(T')$. If $u \in S$, then S' is an *LDS* of T' with $|S'| = |S| - 2$. Hence, S' is a $\gamma_L(T')$-set with $v \in S' \subset S$. If $u \notin S$, then, as discussed in (1), S' is an *LDS* of T' with $|S'| = |S| - 2$. Hence, S' is a $\gamma_L(T')$-set with $v \in S'$, and $v \in S$ since $v \neq u$. Therefore, $v \in \mathcal{A}_L(T)$.

(3) Suppose that $v \notin \mathcal{N}_L(T')$. Let S' be a $\gamma_L(T')$-set that contains v. Clearly, $S' \cup \{x_2, x_4\}$ is a $\gamma_L(T)$-set containing v so $v \notin \mathcal{N}_L(T)$. Conversely, suppose that $v \in \mathcal{N}_L(T')$ and let S be any $\gamma_L(T)$-set with $S' = S \cap V(T')$. Then, as discussed in (1), S' is an *LDS* of T' with $|S'| = |S| - 2$. Hence, S' is a $\gamma_L(T')$-set with $v \notin S'$, and $v \notin S$ since $v \neq u$. We deduce that $v \in \mathcal{N}_L(T)$. □

2. PRUNING OF A TREE

In order to characterize the sets $\mathcal{A}_L(T)$ and $\mathcal{N}_L(T)$ for any nontrivial tree T, we will use a technique called *tree pruning,* introduced by Mynhart [8] and later used by Cockayne, Henning and Mynhardt [3].

Let v be a vertex of a nontrivial tree T. Using the process described below, with respect to the root v, on every branch vertex (vertex of $B(T)$), the tree T_v is transformed into another tree $\overline{T_v}$, called the pruning of T_v, in which every vertex different from v has degree at most two. As a consequence, if a vertex v is in $\mathcal{A}_L(T)$ or $\mathcal{N}_L(T)$, then it has the same properties with respect to $\overline{T_v}$.

Let $T = T_v$ be a nontrivial tree rooted at a vertex v. If every vertex $u \neq v$ has degree at most two, then $\overline{T_v} = T_v$. Otherwise, let w be a branch vertex (vertex of $B(T)$ with degree at least 3) at maximum distance from v. Then apply the following process:

(a) If $\left|L^1(w)\right| \geq 1$, delete $D(w)$ and attach a P_1 to w.
(b) If $\left|L^1(w)\right| = 0$ and $\left|L^3(w)\right| \geq 1$, delete $D(w)$ and attach a P_3 to w.
(c) If $\left|L^1(w) \cup L^3(w)\right| = 0$ and $\left|L^4(w)\right| \geq 1$, delete $D(w)$ and attach a P_4 to w.
(d) If $\left|L^1(w) \cup L^3(w) \cup L^4(w)\right| = 0$ and $\left|L^2(w)\right| \geq 2$, delete $D(w)$ and attach a P_4 to w.
(e) If $\left|L^1(w) \cup L^3(w) \cup L^4(w)\right| = 0$ and $\left|L^2(w)\right| = 1$, delete $D(w)$ and attach a P_2 to w.
(f) If $\left|L^1(w) \cup L^2(w) \cup L^3(w) \cup L^4(w)\right| = 0$, $\left|L^0(w)\right| \geq 2$, delete $D(w)$ and attach a P_5 to w.

To illustrate this technique, we consider the tree of Figure 1(a) where x, u, s, w, y and z are the branch vertices of T. At this step, z is the branch vertex at maximum distance from v, since $\left|L^1(z)\right| = 2$, so we delete $D(w)$ and attach a path P_1 at z (see Figure 1(b)).

Now there remain five branch vertices: x, u, s, w, y. The vertex y is at distance two from v. Since $\left|L^3(y)\right| = 1$ and $\left|L^1(y)\right| = 0$, we delete $D(y)$ and attach a path P_3 at y (see Figure 1(c)). All the remaining branch vertices s, u, w, x are at distance one from v. Since $(\left|L^1(x)\right| = 0$, $\left|L^3(x)\right| = 0$ and $\left|L^4(x)\right| = 1)$, $(\left|L^1(u)\right| = 0$, $\left|L^3(u)\right| = 0$ and $\left|L^4(u)\right| = 1)$, $(\left|L^1(u)\right| = 0$ and $\left|L^3(u)\right| = 1)$ and $(\left|L^1(w)\right| = 1)$. So, we delete $D(x)$ and attach a path P_4 at x, delete $D(u)$ and attach a path P_4 at u, delete $D(s)$ and attach a path P_3 at s, and finally delete $D(w)$ and attach a path P_1 at w. Now the vertex v is the unique branch vertex, so we have obtained the pruning $\overline{T_v}$ of T_v, where $deg_{\overline{T_v}}(u) \leq 2$ for every $u \in V\left(\overline{T_v}\right) - \{v\}$ (see Figure 1(d)).

By Lemma 1, we may delete the two P_5 attached at v (with x and u) and finally we obtain the pruning $\overline{T_v}^*$ of T_v where $deg_{\overline{T_v}^*}(u) \leq 2$ for every $u \in V\left(\overline{T_v}^*\right) - \{v\}$ and $d(u, v) \leq 4$ (see Figure 1(e)).

Since $\left|L^1(v) \cup L^3(v)\right| = 0$ and $\left|L^2(v) \cup L^4(v)\right| = 3$, then by Lemma 3, $v \in \mathcal{N}_L(T_v)$, and by Corollary 1, T is not a γ_L-excellent tree.

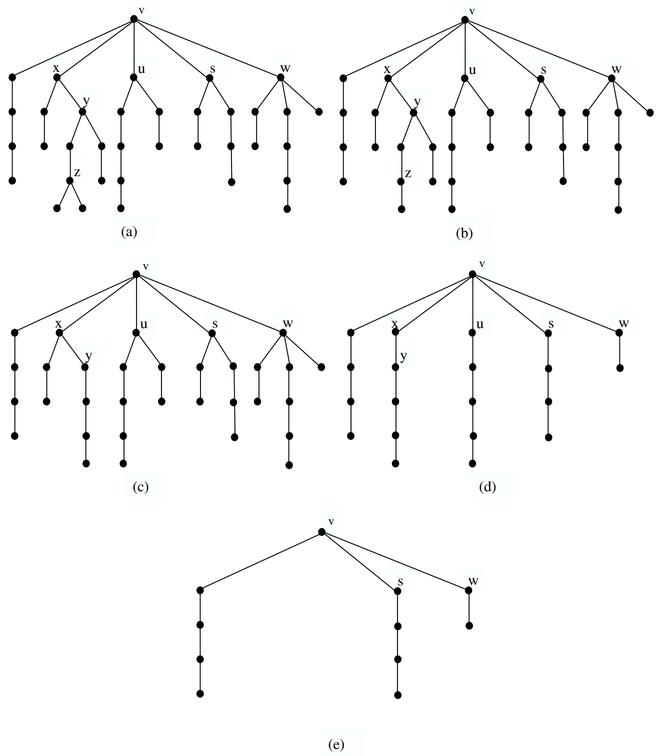

Fig. 1

Lemma 2. *Let T be a tree rooted at a vertex v and w a branch vertex at maximum distance from v ($w \neq v$). Set $k_1 = |L^1(w)|$, $k_2 = |L^2(w)|$, $k_3 = |L^3(w)|$, $k_4 = |L^4(w)|$, and $k_5 = |L^0(w)|$. If:*

(a) *$k_1 \geq 1$, let T' be the tree obtained from T by deleting $D(w)$ and attaching a P_1 to w.*
(b) *$k_1 = 0$ and $k_3 \geq 1$, let T' be the tree obtained from T by deleting $D(w)$ and attaching a P_3 to w.*
(c) *$k_1 + k_3 = 0$ and $k_4 \geq 1$, let T' be the tree obtained from T by deleting $D(w)$ and attaching a P_4 to w.*
(d) *$k_1 + k_3 + k_4 = 0$ and $k_2 \geq 2$, let T' be the tree obtained from T by deleting $D(w)$ and attaching a P_4 to w.*
(e) *$k_1 + k_3 + k_4 = 0$ and $k_2 = 1$, let T' be the tree obtained from T by deleting $D(w)$ and attaching a P_2 to w.*
(f) *$k_1 + k_2 + k_3 + k_4 = 0$ and $k_5 \geq 2$, let T' be the tree obtained from T by deleting $D(w)$ and attaching a P_5 to w.*

Then in each case:

(a) *$v \in \mathcal{A}_L(T')$ if and only if $v \in \mathcal{A}_L(T)$.*
(b) *$v \in \mathcal{N}_L(T')$ if and only if $v \in \mathcal{N}_L(T)$.*

Proof. For the sake of simplicity with use of Lemma 1, the tree T_v will be simplified by replacing any w-x path with a w-x path of length j, where $j = 1, 2, 3, 4, 5$ if $x \in L^i(w)$ for $i = 1, 2, 3, 4, 0$ respectively.

Let a_i, $b_j c_j$, $d_k e_k f_k$, $g_l h_l p_l q_l$ and $r_m s_m t_m u_m x_m$ be paths of order $1, 2, 3, 4, 5$ respectively, attached to w, where a_i, c_j, f_k, q_l, $x_m \in D(w) \cap L(T)$, for $0 \leq i \leq k_1$, $0 \leq j \leq k_2$, $0 \leq k \leq k_3$, $0 \leq l \leq k_4$ and $0 \leq m \leq k_5$.

Case (a). $k_1 \geq 1$.

Let $T' = T - (D(w) - \{a_1\})$.

Every $\gamma_L(T')$-set can be extended to an *LDS* of T by adding

$$X = \{a_i; i \in \{2, \ldots, k_1\}\} \cup \{b_j, j \in \{1, \ldots, k_2\}\} \cup \{e_k; k \in \{1, \ldots, k_3\}\} \cup \\ \cup \{g_l, p_l;\ l \in \{1, \ldots, k_4\}\} \cup \{s_m, u_m;\ m \in \{1, \ldots, k_5\}\}.$$

Let D' be an arbitrary $\gamma_L(T')$-set. We may assume that $w \in D'$; otherwise, we replace a_1 with w, then $D = D' \cup X$ is an *LDS* of T, so there is $\gamma_L(T) \leq |D'| + |X| = \gamma_L(T') + (k_1 - 1) + k_2 + k_3 + 2k_4 + 2k_5$. On the other hand; let D be an arbitrary $\gamma_L(T)$-set, then $D' = D \cap T'$ is an *LDS of* T'. There follows $|D \cap D(w)| \geq (k_1 - 1) + k_2 + k_3 + 2k_4 + 2k_5 = |X|$ when $a_1 \notin D$ or $|D \cap D(w)| > (k_1 - 1) + k_2 + k_3 + 2k_4 + 2k_5 = |X|$ when $a_1 \in D$ (i.e., all $a_i \in D$), *so* when $a_1 \notin D$, $D' = D - D \cap D(w)$ and when $a_1 \in D, D' = (D - D \cap D(w)) \cup \{a_1\}$. In each case, $\gamma_L(T') \leq |D'| \leq \gamma_L(T) - (k_1 - 1) - k_2 - k_3 - 2k_4 - 2k_5$. Thus $\gamma_L(T) = \gamma_L(T') + (k_1 - 1) + k_2 + k_3 + 2k_4 + 2k_5$.

(1) Suppose that $v \in \mathcal{A}_L(T')$ and let D be an arbitrary $\gamma_L(T)$-set. We have above seen that either $D' = D - (D \cap D(w))$ if $w \in D$ and $a_1 \notin D$ or $D' = (D - (D \cap D(w))) \cup \{a_1\}$ if $w \notin D$ and $a_1 \in D$ (that is $D' = D \cap T'$) is a $\gamma_L(T')$-set. Since $v \in D' \subset D$, then $v \in \mathcal{A}_L(T)$.

Conversely, suppose that $v \in \mathcal{A}_L(T)$ and let S' be a $\gamma_L(T')$-set. We know that S' can be extended to a $\gamma_L(T)$-set S by adding the set X. So, $S = S' \cup X$ is a $\gamma_L(T)$-set. Since $v \in S$ and $v \notin D[w]$, $v \in \mathcal{A}_L(T')$.

(2) Suppose now that $v \in \mathcal{N}_L(T')$ and let D be an arbitrary $\gamma_L(T)$-set. As seen above, $D' = D \cap T'$ is a $\gamma_L(T')$-set. Since $v \notin D'$ and $v \notin D[w]$, then $v \notin D$ and thus $v \in \mathcal{N}_L(T)$. Conversely, assume that $v \in \mathcal{N}_L(T)$ and let S' be a $\gamma_L(T')$-set. Then S' can be extended to a $\gamma_L(T)$-set S by adding the set X, and since $v \notin S$ and $v \notin D[w]$, then $v \notin S'$ and so $v \in \mathcal{N}_L(T')$.

Case (b). $k_1 = 0$ *and* $k_3 \geq 1$.

Let $T' = T - (D(w) - \{d_1, e_1, f_1\})$.

Every $\gamma_L(T')$-set can be extended to an *LDS* of T by adding

$$X = \{b_j, j \in \{1, \dots, k_2\}\} \cup \{e_k; k \in \{2, \dots, k_3\}\} \cup \\ \cup \{g_l, p_l;\ l \in \{1, \dots, k_4\}\} \cup \{s_m, u_m;\ m \in \{1, \dots, k_5\}\}.$$

Let D' be an arbitrary $\gamma_L(T')$-set. Without loss of generality, we may assume that D' contains e_1 and w; otherwise, we replace f_1 with e_1 in the first case and d_1 (or f_1) with w in the second case. So $\gamma_L(T) \leq \gamma_L(T') + k_2 + (k_3 - 1) + 2k_4 + 2k_5$. On the other hand, let D be an arbitrary $\gamma_L(T)$-set. We may assume that $w \in D$; otherwise, we replace d_1 or f_1 with w and take e_1 in D ; then $D' = D \cap T'$ is an *LDS of* T' and clearly $|D \cap (D(w) - \{d_1, e_1, f_1\})| \geq k_2 + (k_3 - 1) + 2k_4 + 2k_5 = |X|$, then $\gamma_L(T') \leq |D'| = |D| - |D \cap (D(w) - \{d_1, e_1, f_1\})| \leq \gamma_L(T) - k_2 - (k_3 - 1) - 2k_4 - 2k_5$. Thus $\gamma_L(T) = \gamma_L(T') + k_2 + (k_3 - 1) + 2k_4 + 2k_5$.

In this case and also in cases (c), (d), (e) and (f), the proofs of part (1) and (2) are similar to the proof of "case (a) (part (1) and part (2))". So the similar proofs are omitted.

Case (c). $k_1 + k_3 = 0$ and $k_4 \geq 1$.

Let $T' = T - (D(w) - \{g_1, h_1, p_1, q_1\})$.

Every $\gamma_L(T')$-set can be extended to an *LDS* of T by adding

$$X = \{b_j, j \in \{1, \dots, k_2\}\} \cup \{g_l, p_l;\ l \in \{2, \dots, k_4\}\} \cup \{s_m, u_m;\ m \in \{1, \dots, k_5\}\}.$$

Let D' be an arbitrary $\gamma_L(T')$-set. Then $\gamma_L(T) \leq \gamma_L(T') + k_2 + 2(k_4 - 1) + 2k_5$. On the other hand, let D be an arbitrary $\gamma_L(T)$-set. Without loss of generality, we may assume that D contains g_1 and p_1; otherwise, we replace h_1 with g_1 and q_1 with p_1, then $D' = D \cap T'$ is an *LDS of* T' and clearly $|D \cap (D(w) - \{g_1, h_1, p_1, q_1\})| \geq k_2 + 2(k_4 - 1) + 2k_5 = |X|$, so $\gamma_L(T') \leq |D'| = |D| - |D \cap (D(w) - \{g_1, h_1, p_1, q_1\})| \leq \gamma_L(T) - k_2 - 2(k_4 - 1) - 2k_5$. Thus $\gamma_L(T) = \gamma_L(T') + k_2 + 2(k_4 - 1) + 2k_5$.

Case (d). $k_1 + k_3 + k_4 = 0$ and $k_2 \geq 2$.

Let $T' = T - (D(w) - \{b_1, c_1, b_2, c_2\})$ and let $T'' = T' - \{b_1, c_1, b_2, c_2\} + P_4$, that is we replace $\{b_1, c_1, b_2, c_2\}$ by attaching P_4 to w, where $P_4 = b_1 c_1 b_2 c_2$.

Clearly, every $\gamma_L(T')$-set of $T'$is a $\gamma_L(T'')$-set of T'' and can be extended to an *LDS* of T by adding

$$X = \{b_j, j \in \{3, \dots, k_2\}\} \cup \{s_m, u_m;\ m \in \{1, \dots, k_5\}\}.$$

Let D' be an arbitrary $\gamma_L(T')$-set. So $\gamma_L(T) \leq \gamma_L(T') + (k_2 - 2) + 2k_5$. On the other hand, let D be an arbitrary $\gamma_L(T)$-set. Without loss of generality, we may assume that D contains s_1 and u_1; otherwise, we replace r_1 with w and take s_1 and u_1 in D. Then $D' = D \cap T'$ is an *LDS of* T' and clearly $|D \cap (D(w) - \{b_1, c_1, b_2, c_2\})| \geq (k_2 - 2) + 2k_5 = |X|$. *So* $\gamma_L(T') \leq |D'| = |D| - |D \cap (D(w) - \{b_1, c_1, b_2, c_2\})| \leq \gamma_L(T) - (k_2 - 2) - 2k_5$. Thus $\gamma_L(T) = \gamma_L(T') + (k_2 - 2) + 2k_5$.

Case (e). $k_1 + k_3 + k_4 = 0$ and $k_2 = 1$.

Let $T' = T - (D(w) - \{b_1, c_1\})$.

Every $\gamma_L(T')$-set can be extended to an *LDS* of T by adding

$$X = \{s_m, u_m;\ m \in \{1, \ldots, k_5\}\}.$$

Let D' be an arbitrary $\gamma_L(T')$-set. So $\gamma_L(T) \leq \gamma_L(T') + |X| = \gamma_L(T') + 2k_5$. On the other hand, let D be an arbitrary $\gamma_L(T)$-set. Without loss of generality, we may replace r_j with w and take s_j, u_j in D if $w \notin D$ and $r_j \in D$, then $D' = D \cap T'$ is an *LDS of* T' and clearly $|D \cap (D(w) - \{b_1, c_1\})| \geq |X| = 2k_5$. So $\gamma_L(T') \leq |D'| = |D| - |D \cap (D(w) - \{b_1, c_1\})| \leq \gamma_L(T) - |X| = \gamma_L(T) - 2k_5$. Thus $\gamma_L(T) = \gamma_L(T') + |X| = \gamma_L(T') + 2k_5$.

Case (f). $k_1 + k_2 + k_3 + k_4 = 0$ and $k_5 \geq 2$.

Let $T' = T - (D(w) - \{r_1, s_1, t_1, u_1, x_1, \})$.

Every $\gamma_L(T')$-set can be extended to an *LDS* of T by adding

$$X = \{s_m, u_m;\ m \in \{2, \ldots, k_5\}\}.$$

Let D' be an arbitrary $\gamma_L(T')$-set. So we have $\gamma_L(T) \leq \gamma_L(T') + |X| = \gamma_L(T') + 2(k_5 - 1)$. On the other hand, let D be an arbitrary $\gamma_L(T)$-set. Without loss of generality, we may replace r_j with w and take s_j, u_j in D if $w \notin D$ and $r_j \in D$, then $D' = D \cap T'$ is an *LDS of*T' and clearly $|D \cap (D(w) - \{r_1, s_1, t_1, u_1, x_1, \})| \geq |X| = 2(k_5 - 1)$, then $\gamma_L(T') \leq |D'| = |D| - |D \cap (D(w) - \{r_1, s_1, t_1, u_1, x_1, \})| \leq \gamma_L(T) - 2(k_5 - 1)$. Thus $\gamma_L(T) = \gamma_L(T') + |X| = \gamma_L(T') + 2(k_5 - 1)$. □

3. CHARACTERIZATIONS

The following lemma gives a necessary and sufficient condition for the special vertex v of a nontrivial tree T_v to be in $\mathcal{A}_L(T)$ (resp. in $\mathcal{N}_L(T)$).

Lemma 3. *Let T be a nontrivial tree rooted at a vertex v such that* $\deg_T(u) \leq 2$ *for every vertex $u \in V(T) - \{v\}$. Then:*

1) $v \in \mathcal{A}_L(T)$ *if and only if either* $(|L^3(v)| \geq 2)$, *or* $(|L^3(v)| = 1$ *and* $|L^1(v)| \geq 1)$.
2) $v \in \mathcal{N}_L(T)$ *if and only if* $(|L^3(v) \cup L^1(v)| = 0$ *and* $|L^2(v) \cup L^4(v)| \geq 2)$ *or* $(|L^3(v) \cup L^2(v) \cup L^1(v)| = 0$ *and* $|L^4(v)| = 1))$.

Proof. Clearly if $L(v) = L^0(v)$, that is all the vertices of T are at distance j from v, then $j \equiv 0 \pmod 5$. In this case we may obtain T from $\overline{T_v}^* = P_6$ by applying Lemma 1 and then $v \notin \mathcal{A}_L(\overline{T_v}^*) \cup \mathcal{N}_L(\overline{T_v}^*)$, therefore $v \notin \mathcal{A}_L(T) \cup \mathcal{N}_L(T)$.

So now we suppose that $L(v) \neq L^0(v)$. According to Lemma 1, it will be sufficient to prove the lemma by considering the tree $\overline{T_v}^*$ in which every vertex distinct from v has degree at most 2 and every leaf of $\overline{T_v}^*$ is at distance at most 4 from v (that is if $\overline{T_v}^*$ contains leaves at distance 5 from v, we just consider the remaining tree obtained by removing the paths P_5 attached to v). So we may assume that no path P_5 is attached to v in $\overline{T_v}^*$.

Let k_i denote the number of leaves in $\overline{T_v}^*$ at distance i from v, where $i = 1, 2, 3, 4$. So $v \in \mathcal{A}_L(T)$ (resp. $\mathcal{N}_L(T)$) if and only if $v \in \mathcal{A}_L(\overline{T_v}^*)$ (resp. $\mathcal{N}_L(\overline{T_v}^*)$).

If v is a pendent vertex, then $\overline{T_v}^*$ is a path P_n with $5 \geq n \geq 2$ and $\left|L^1(v) \cup L^2(v) \cup L^3(v) \cup L^4(v)\right| = 1$. Then, by Observations 1 and 2, $v \notin \mathcal{A}_L(\overline{T_v}^*) \cup \mathcal{N}_L(\overline{T_v}^*)$ if and only if $n = 2, 3, 4$, and $v \in \mathcal{N}_L(\overline{T_v}^*)$ if and only if $n = 5$, which yields the result. We will from now assume that v is not a pendent vertex. Thus v has degree at least 2.

Let D be a $\gamma_L(\overline{T_v}^*)$-set. For a leaf t_i at distance i from v, we denote the v-t_i path by $v, t_1, \ldots, t_i$. It remains now to examine the following cases:

Case 1. $k_3 \geq 2$.

Let x_3 and y_3 be two leaves at distance 3 from v. Assume that $v \notin D$. Then D must contain two vertices from each of $\{x_1, \ldots, x_3\}$ and $\{y_1, \ldots, y_3\}$. Without loss of generality, suppose that $x_i, y_i \in D$ for $i = 1, 2$. In this case, $D' = \{v\} \cup (D - \{x_1, y_1\})$ is an *LDS* of $\overline{T_v}^*$ of size $\gamma_L(\overline{T_v}^*) - 1$, a contradiction. Then $v \in D$ and $v \in \mathcal{A}_L(T)$.

Case 2. $k_3 = 1$ and $k_1 \geq 1$.

Let x_3 and y_1 be two leaves at distances 3 and 1 from v, respectively, and assume that $v \notin D$. Then D must contain y_1 and two vertices from the x_1-x_3 path. Without loss of generality, we assume that $x_i \in D$ for $1 \leq i \leq 2$. Then $D' = \{v\} \cup D - \{y_1, x_1\}$ is an *LDS* of $\overline{T_v}^*$ of size less than D, a contradiction. Then $v \in D$ and $v \in \mathcal{A}_L(\overline{T_v}^*)$.

Case 3. $k_1 + k_3 = 0$ and $k_2 + k_4 \geq 2$.

Subcase 3.1. $k_2 \geq 2$ and $k_1 + k_3 = 0$.

Let x_2, y_2 be two leaves at distance 2 from v and suppose that $v \in D$. Then D must contain one vertex from each of $\{x_1, x_2\}$ and $\{y_1, y_2\}$. Without loss of generality, we assume that $x_2, y_2 \in D$. In this case, $D' = (D - \{v, x_2, y_2\}) \cup \{x_1, y_1\}$ is an *LDS* of $\overline{T_v}^*$ of size $\gamma_L(\overline{T_v}^*) - 1$, because $k_1 + k_3 = 0$; a contradiction. Hence $v \notin D$ and $v \in \mathcal{N}_L(\overline{T_v}^*)$.

Subcase 3.2. $k_4 \geq 2$ and $k_1 + k_3 = 0$.

Let x_4, y_4 be two leaves at distance 4 from v and suppose that $v \in D$. Then D must contain two vertices from each of $\{x_1, \ldots, x_4\}$ and $\{y_1, \ldots, y_4\}$. Without loss of generality, we assume that $x_i, y_i \in D$ for $i = 2, 3$. In this case, $D' = (D - \{v, x_2, x_3, y_2, y_3\}) \cup \{x_1, x_3, y_1, y_3\}$ is an *LDS* of $\overline{T_v}^*$ of size $\gamma_L(\overline{T_v}^*) - 1$, because $k_1 + k_3 = 0$; a contradiction. Hence $v \notin D$ and $v \in \mathcal{N}_L(\overline{T_v}^*)$.

Subcase 3.3. $k_4 = 1$, $k_2 = 1$ and $k_1 + k_3 = 0$.

Let x_4, y_2 be two leaves at distances 4 and 2 from v and suppose that $v \in D$. Then D must contain two vertices from $\{x_1, \ldots, x_4\}$ and one vertex from $\{y_1, y_2\}$. Without loss of generality, we assume that $x_2, x_3, y_2 \in D$. In this case, $D' = (D -$

$\{v, x_2, x_3, y_2\}) \cup \{x_1, x_3, y_1\}$ is an *LDS* of $\overline{T_v}^*$ of size $\gamma_L(\overline{T_v}^*) - 1$, because $k_1 + k_3 = 0$; a contradiction. Hence $v \notin D$ and $v \in \mathcal{N}_L(\overline{T_v}^*)$.

Case 4. $k_1 + k_2 + k_3 = 0$ and $k_4 = 1$.

Let x_4 be a leaf at distance 4 from v and suppose that $v \in D$. Then D must contain two vertices from $\{x_1, \ldots, x_4\}$. Without loss of generality, we assume that $x_2, x_3 \in D$. In this case, $D' = (D - \{v, x_2, x_3\}) \cup \{x_1, x_3\}$ is an *LDS* of $\overline{T_v}^*$ of size $\gamma_L(\overline{T_v}^*) - 1$, because $k_1 + k_2 + k_3 = 0$; a contradiction. Hence $v \notin D$ and $v \in \mathcal{N}_L(\overline{T_v}^*)$.

Conversely, according to cases $1, 2, 3$ above and the fact that $\mathcal{A}_L(\overline{T_v}^*) \cap \mathcal{N}_L(\overline{T_v}^*) = \varnothing$, it remains to examine the following cases to complete the proof:

Case 5. $k_3 = 0$ and $k_1 \geq 1$.

All the leaves are at distances 1, 2 or 4 from v. Since $k_1 \geq 1$, let x_1 be a leaf adjacent to v; by Observation 1, there exists D such that $v \in D$. Clearly we may deduce a $\gamma_L(\overline{T_v}^*)$-set D' which contains x_1 and not v, implying that $v \notin \mathcal{A}_L(\overline{T_v}^*) \cap \mathcal{N}_L(\overline{T_v}^*)$.

Case 6. $k_3 = 1$ and $k_1 = 0$.

Let x_3 be a leaf at distance 3 from v. If $v \in D$, then D must contain x_2 or x_3. In this case, we can deduce a $\gamma_L(\overline{T_v}^*)$-set D' which contains x_1 and not v. If $v \notin D$, then D must contain two vertices from $\{x_1, x_2, x_3\}$. Without loss of generality, we assume that $x_2, x_3 \in D$. In this case, $D' = (D - \{x_3\}) \cup \{v\}$ is a $\gamma_L(\overline{T_v}^*)$-set which contains v, implying in all cases that $v \notin \mathcal{A}_L(\overline{T_v}^*) \cap \mathcal{N}_L(\overline{T_v}^*)$.

Case 7. $k_3 + k_1 = 0$ and $k_2 + k_4 = 1$.

Subcase 7.1. $k_4 + k_3 + k_1 = 0$ and $k_2 = 1$.Thus $\overline{T_v}^*$ is P_3. Therefore, this case has been considered at the beginning, implying that $v \notin \mathcal{A}_L(\overline{T_v}^*) \cap \mathcal{N}_L(\overline{T_v}^*)$.

Subcase 7.2. $k_3 + k_2 + k_1 = 0$ and $k_4 = 1$.Thus $\overline{T_v}^*$ is P_5. This case too has already been considered at the beginning, implying that $v \in \mathcal{N}_L(\overline{T_v}^*)$.

Case 8. $k_4 + k_3 + k_2 + k_1 = 0$.

Finally, this case has also been considered at the beginning, implying that $v \notin \mathcal{A}_L(\overline{T_v}^*) \cap \mathcal{N}_L(\overline{T_v}^*)$. □

From Lemmas 2 and 3, our main result follows:

Theorem 3. *Let v be a vertex of the tree T, then:*

— $v \in \mathcal{A}_L(T)$ *if and only if* $v \in \mathcal{A}_L(\overline{T_v}^*)$.
— $v \in \mathcal{N}_L(T)$ *if and only if* $v \in \mathcal{N}_L(\overline{T_v}^*)$.

Therefore, the following corollary holds true.

Corollary 1. *T is a γ_L-excellent tree if and only if $\mathcal{N}_L(\overline{T_v}^*) = \varnothing$ for every vertex v of T. (That is in a pruning tree $\overline{T_v}^*$ there is: $(\left|L^3(v) \cup L^1(v)\right| \neq 0$ or $\left|L^2(v) \cup L^4(v)\right| \leq 1)$ and $(\left|L^3(v) \cup L^2(v) \cup L^1(v)\right| \neq 0$ or $\left|L^4(v)\right| \neq 1))$.*

It is easy to verify that a pruning tree can be found in a polynomial time with the process defined above. So, if $\mathcal{N}_L(\overline{T_v}^*) = \varnothing$ for every vertex v of T, then the γ_L-excellence property of a tree can be verified in a polynomial time.

Corollary 2. *γ_L-excellent trees can be recognized in a polynomial time.*

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