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ON A MULTIVALUED SECOND ORDER DIFFERENTIAL PROBLEM WITH HUKUHARA DERIVATIVE

Abstract. Let K be a closed convex cone with the nonempty interior in a real Banach space and let $cc(K)$ denote the family of all nonempty convex compact subsets of K. Assume that continuous linear multifunctions $H, \Psi \colon K \to cc(K)$ are given. We consider the following problem

$$D^2\Phi(t,x) = \Phi(t, H(x)),$$

$$D\Phi(t,x)|_{t=0} = \{0\},$$

$$\Phi(0,x) = \Psi(x)$$

for $t \geq 0$ and $x \in K$, where $D\Phi(t,x)$ denotes the Hukuhara derivative of $\Phi(t,x)$ with respect to t.

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Let X be a real vector space. Throughout this paper, all vector spaces are supposed to be real. We introduce addition and multiplication by scalar as follows:

$$A + B := \{a + b : \ a \in A, \ b \in B\} \quad \text{and} \quad \lambda A := \{\lambda a : \ a \in A\}$$

for $A, B \subset X$ and $\lambda \in \mathbb{R}$.

A subset K of X is called a *cone* if $tK \subset K$ for all $t \in (0, +\infty)$. A cone is said to be *convex* if it is a convex set.

Let X and Y be two vector spaces and let $K \subset X$ be a convex cone. A set-valued function $F \colon K \to n(Y)$, where $n(Y)$ denotes the family of all nonempty subsets of Y, is called *linear* if

$$F(x + y) = F(x) + F(y) \quad \text{and} \quad F(\lambda x) = \lambda F(x)$$

for all $x, y \in K$ and $\lambda \geq 0$.

From now on, we assume that X is a normed vector space, $c(X)$ denotes the family of all compact members of $n(X)$ and $cc(X)$ stands for the family of all convex sets of $c(X)$.

Let A, B, C be in $cc(X)$. The set C is the *Hukuhara difference* of A and B, if $B + C = A$. From Rådström's Cancellation Lemma [18], it follows that if this difference exists, then it is unique.

For a multifunction $F\colon [a, b] \to cc(X)$ such that there exist the Hukuhara differences $F(t) - F(s)$ as $a \le s \le t \le b$, the *Hukuhara derivative* at $t \in (a, b)$ is defined by the formula

$$DF(t) = \lim_{k\to 0^+} \frac{F(t + k) - F(t)}{k} = \lim_{k\to 0^+} \frac{F(t) - F(t - k)}{k},$$

whenever both these limits exist with respect to the Hausdorff distance h (see [13]). Moreover,

$$DF(a) = \lim_{s\to a^+} \frac{F(s) - F(a)}{s - a}, \quad DF(b) = \lim_{s\to b^-} \frac{F(b) - F(s)}{b - s}.$$

The Hukuhara derivative is not the only derivative defined for multifunctions (see for example [5, 12] or [15]). The study of set-valued differentiation started with papers [7,8] and [9] of G. Bouligand and papers [14] of H. Marchaund and [23] of S. C. Zaremba, where Bouligand's definitions have been applied to differential inequalities. To get other information including the rich bibliography, the reader is reffered to [1–4, 19].

Let $(K, +)$ be a semigroup. A one-parameter family $\{F_t :\ t \ge 0\}$ of set-velued functions $F_t : K \to n(K)$ is said to be a *cosine family* if

$$F_0(x) = \{x\} \quad \text{for } x \in K$$

and

$$F_{t+s}(x) + F_{t-s}(x) = 2F_t(F_s(x)) := 2\bigcup\{F_t(y) :\ y \in F_s(x)\} \tag{1}$$

for $x \in K$ and $0 \le s \le t$.

Let X be a normed space. A cosine family $\{F_t :\ t \ge 0\}$ is said to be *regular* if

$$\lim_{t\to 0^+} h(F_t(x), \{x\}) = 0.$$

It was shown in [17] that if K is a closed convex cone with the nonempty interior in a Banach space and $\{F_t :\ t \ge 0\}$ is a regular cosine family of continuous linear set-valued functions $F_t\colon K \to cc(K)$ such that $x \in F_t(x)$ for all $x \in K$, $t \ge 0$ and $F_t \circ F_s = F_s \circ F_t$ for all $s, t \ge 0$, then

$$DF_t(x)|_{t=0} = \{0\} \quad \text{and} \quad D^2F_t(x) = F_t(H(x))$$

for $x \in K$, $t \ge 0$, where $DF_t(x)$ denotes the Hukuhara derivative of $F_t(x)$ with respect to t and $H(x)$ is the second Hukuhara derivative of this multifunction at $t = 0$. It is a reason for studying the existence and uniqueness of a solution $\Phi\colon [0,+\infty)\times K \to cc(K)$ of the following differential problem

$$\begin{aligned} D^2\Phi(t,x) &= \Phi(t,H(x)),\\ D\Phi(t,x)|_{t=0} &= \{0\}, \\ \Phi(0,x) &= \Psi(x), \end{aligned} \tag{2}$$

where $H,\Psi\colon K\to cc(K)$ are given continuous linear set-valued functions and $D\Phi(t,x)$ denotes the Hukuhara derivative of $\Phi(t,x)$ with respect to t, with the condition that this solution is linear with respect to the second variable. The goal of this paper is to study this problem.

A similar first order differential problem was investigated in [20]. The uniqueness and existence theorems for other types of first order differential problem can be found in [10].

Let X be a Banach space and let $[a,b]\subset\mathbb{R}$. If a multifunction $F\colon [a,b]\to cc(X)$ is continuous, then there exists the Riemann integral

$$\int_a^b F(t)dt$$

(see [13]). We need the following properties of the Riemann integral.

Lemma 1 ([13, p. 211])**.** *If $F,G\colon [a,b]\to cc(X)$ are continuous, then*

$$h\left(\int_a^b F(t)dt, \int_a^b G(t)dt\right) \le \int_a^b h(F(t),G(t))dt.$$

Lemma 2 ([13, p. 211])**.** *If $F\colon [a,b]\to cc(X)$ is continuous and $a<c<b$, then*

$$\int_a^b F(t)dt = \int_a^c F(t)dt + \int_c^b F(t)dt.$$

Lemma 3 ([16, Lemma 10])**.** *If $F\colon [a,b]\to cc(X)$ is continuous, then*

$$H(t) = \int_a^t F(u)du \quad \textit{for } a\le t\le b$$

is continuous.

Lemma 4 ([20, Lemma 4])**.** *If $F\colon [a,b]\to cc(X)$ is continuous and $H(t)=\int_a^t F(u)du$, then $DH(t)=F(t)$ for $a\le t\le b$.*

Lemma 5 ([20, Lemma 5]). *If $F, G\colon [a,b] \to cc(X)$ are two differentiable multifunctions such that $DF(t) = DG(t)$ for $t \in [a,b]$ and $F(a) = G(a)$, then*

$$F(t) = G(t) \quad \textit{for } t \in [a,b].$$

Definition 1. *Let K be a convex cone in a Banach space and let $H, \Psi\colon K \to cc(K)$ be two continuous linear multifunctions. A map $\Phi\colon [0,+\infty) \times K \to cc(K)$ is said to be a solution of problem (2) if it is continuous, twice differentiable with respect to t and Φ satisfies (2) everywhere in $[0,+\infty) \times K$ and in K, respectively.*

With problem (2), we associate the following equation

$$\Phi(t,x) = \Psi(x) + \int_0^t \left(\int_0^s \Phi(u, H(x))du \right) ds \tag{3}$$

for $x \in K$, $t \in [0,+\infty)$, where $H, \Psi\colon K \to cc(K)$ are given continuous linear multifunctions.

Definition 2. *Let K be a convex cone in a Banach space and let $H, \Psi\colon K \to cc(K)$ be two continuous linear multifunctions. A map $\Phi\colon [0,+\infty) \times K \to cc(K)$ is said to be a solution of (3) if it is continuous and satisfies (3) everywhere.*

Theorem 1. *Let K be a convex cone in a Banach space X and let $H, \Psi\colon K \to cc(K)$ be continuous linear multifunctions. Let $\Phi\colon [0,+\infty)\times K \to cc(K)$ be a given set-valued function. This Φ is a solution of problem (2) if and only if it is a solution of (3).*

Proof. Suppose that a set-valued function $\Phi(t,x)$ is a solution of (3). Then it is continuous. Fix $\varepsilon > 0$, $t > 0$ and $x \in K$. Since the set $[0,t] \times H(x)$ is compact, there exists $\delta > 0$ such that

$$h(\Phi(u,a), \Phi(v,b)) < \varepsilon$$

for $u, v \in [0,t]$, $a, b \in H(x)$, where $|u - v| < \delta$, $\|a - b\| < \delta$. Therefore,

$$\Phi(u,a) \subset \Phi(v,a) + \varepsilon S \subset \Phi(v, H(x)) + \varepsilon S$$

and

$$\Phi(v,a) \subset \Phi(u,a) + \varepsilon S \subset \Phi(u, H(x)) + \varepsilon S$$

for $a \in H(x)$, $u, v \in [0,t]$ such that $|u - v| < \delta$, where S denotes the closed unit ball in X. This implies that

$$\Phi(u, H(x)) \subset \Phi(v, H(x)) + \varepsilon S$$

and

$$\Phi(v, H(x)) \subset \Phi(u, H(x)) + \varepsilon S$$

for $u, v \in [0,t]$ such that $|u - v| < \delta$. Thus for every $x \in K$ the multifunction

$$u \mapsto \Phi(u, H(x))$$

is continuous in $[0, +\infty)$. By Lemmas 3, 4, the set-valued function

$$\Phi(t,x) = \Psi(x) + \int_0^t \left(\int_0^s \Phi(u, H(x))du \right) ds$$

is twice differentiable with respect to t and

$$D^2\Phi(t,x) = D^2\Psi(x) + D^2 \int_0^t \left(\int_0^s \Phi(u, H(x))du \right) ds = \Phi(t, H(x)).$$

Obviously, $\Phi(0,x) = \Psi(x)$ and $D\Phi(t,x) = \int_0^t \Phi(u, H(x))du$ so $D\Phi(t,x)|_{t=0} = \{0\}$. Thus Φ satisfies (2).

Now suppose that $\Phi(t,x)$ is a solution of (2) and let

$$\Pi(t,x) = \Psi(x) + \int_0^t \left(\int_0^s \Phi(u, H(x))du \right) ds, \quad (t,x) \in [0,+\infty) \times K.$$

By Lemma 4, we get

$$D\Pi(t,x) = \int_0^t \Phi(u, H(x))du$$

and

$$D^2\Pi(t,x) = \Phi(t, H(x)).$$

Since $\Pi(0,x) = \Psi(x) = \Phi(0,x)$, $D\Pi(t,x)|_{t=0} = \{0\} = D\Phi(t,x)|_{t=0}$, $D^2\Pi(t,x) = D^2\Phi(t,x)$, then using Lemma 5 we obtain

$$\Pi(t,x) = \Phi(t,x) \qquad \text{for } (t,x) \in [0,+\infty) \times K.$$

□

In the proof of next theorem we use the following two lemmas.

Lemma 6 ([22, Theorem 3]). *Let X and Y be two normed spaces and let K be a convex cone in X. Suppose that $\{F_i : i \in I\}$ is a family of superadditive set-valued functions $F_i \colon K \to n(Y)$ lower semicontinuous in K and $\mathbb{Q}_+$-homogeneous. If K is of the second category in K and $\bigcup_{i\in I} F_i(x) \in b(Y)$ for $x \in K$, then there exists a constant $M \in (0,+\infty)$ such that*

$$\sup_{i\in I} \|F_i(x)\| \le M\|x\| \qquad \text{for } x \in K.$$

Let K be a closed convex cone in X. Applying Lemma 6 we can define the *norm* $\|F\|$ of a continuous linear multifunction $F \colon K \to n(K)$ to be the smallest element of the set

$$\{M > 0 : \ \|F(x)\| \le M\|x\|, \ x \in K\}.$$

Lemma 7 ([21, Lemma 5]). *Let X and Y be two normed spaces and let h be the Hausdorff distance derived from the norm in Y. Assume that K is a convex cone in X such that* $\operatorname{int} K \neq \emptyset$*. Then there exists a constant $M_0 \in (0, +\infty)$ such that the inequality*

$$h(F(x), F(y)) \leq M_0 \|F\| \|x - y\|$$

holds for all continuous additive set-valued functions $F : K \to c(Y)$ and for all $x, y \in K$.

Theorem 2. *Let K be a closed convex cone with the nonempty interior in a Banach space and let $H, \Psi : K \to cc(K)$ be two continuous linear multifunctions. Then there exists exactly one solution of problem* (2)*. Moreover, this solution is linear with respect to the second variable.*

Proof. Fix $T > 0$. Let E be the set of all continuous set-valued functions $\Phi : [0, T] \times K \to cc(K)$ such that $x \mapsto \Phi(t, x)$ is linear. We define a functional ρ in $E \times E$ by

$$\rho(\Phi, \Pi) = \sup\{h(\Phi(t, A), \Pi(t, A)) : \ 0 \leq t \leq T, \ A \in cc(K), \ \|A\| \leq 1\}$$

for $\Phi, \Pi \in E$. Since sets

$$\Phi([0, T], x) = \bigcup_{t \in [0,T]} \Phi(t, x)$$

are compact for $\Phi \in E$ and $x \in K$ (see Theorem 3, Chap. IV, p. 110 in [6]), they are bounded. So by Lemma 6, for every $\Phi \in E$ there exists a positive constant M_Φ such that

$$\|\Phi(t, x)\| \leq M_\Phi \|x\|$$

for $t \in [0, T]$ and $x \in K$. Therefore,

$$h(\Phi(t, A), \Pi(t, A)) \leq h(\Phi(t, A), \{0\}) + h(\{0\}, \Pi(t, A)) \leq M_\Phi + M_\Pi$$

for $t \in [0, T]$ and $A \in cc(K)$ such that $\|A\| \leq 1$. Thus

$$\rho(\Phi, \Pi) \leq M_\Phi + M_\Pi < +\infty,$$

hence the functional ρ is finite. It is easy to verify that ρ is a metric in E.

As the space $(cc(K), h)$ is complete (see [11]), (E, ρ) is a complete metric space.

We introduce a map Γ which with every $\Phi \in E$ associates the set-valued function $\Gamma\Phi$ defined by

$$(\Gamma\Phi)(t, x) := \Psi(x) + \int_0^t \left(\int_0^s \Phi(u, H(x)) du \right) ds \tag{4}$$

for $(t, x) \in [0, T] \times K$. We see that every set $(\Gamma\Phi)(t, x)$ belongs to $cc(K)$ and $\Gamma\Phi$ is linear with respect the second variable.

Next we show that $\Gamma\Phi$ is continuous. Let $\Phi \in E$, $x, y \in K$ and $0 \leq t_1 \leq t_2 \leq T$. Similarly as above, by Lemma 6, there exists a positive constant M_Φ such that

$$\|\Phi(u, a)\| \leq M_\Phi \|a\| \tag{5}$$

for $u \in [0,T]$ and $a \in K$. This implies that

$$\|\Phi(u, H(x))\| \le M_\Phi \|H(x)\|$$

for $u \in [0,T]$. Thus

$$\left\| \int_{t_1}^{t_2} \left(\int_0^s \Phi(u, H(x)) du \right) ds \right\| \le \int_{t_1}^{t_2} \left(\int_0^s \|\Phi(u, H(x))\| du \right) ds \le$$
$$\le \int_{t_1}^{t_2} \left(\int_0^s M_\Phi \|H(x)\| du \right) ds = \tag{6}$$
$$= \frac{t_2^2 - t_1^2}{2} M_\Phi \|H(x)\|.$$

By Lemma 7 and (5), there exists a positive constant M_0 such that

$$h(\Phi(u,a), \Phi(u,b)) \le M_0 \|\Phi(u,\cdot)\| \|a-b\| \le M_0 M_\Phi \|a-b\|$$

for $u \in [0,T]$ and $a, b \in K$. Therefore,

$$\Phi(u,a) \subset \Phi(u,b) + M_0 M_\Phi \|a-b\| S$$

for $u \in [0,T]$ and $a, b \in K$.

Let $\varepsilon > 0$ and $a \in H(x)$. There exists $b \in H(y)$ for which

$$\|a-b\| < d(a, H(y)) + \frac{\varepsilon}{M_0 M_\Phi}.$$

This shows that for every $a \in H(x)$ there exists $b \in H(y)$ such that

$$\Phi(u,a) \subset \Phi(u,b) + M_0 M_\Phi d(a, H(y)) S + \varepsilon S \subset$$
$$\subset \Phi(u, H(y)) + M_0 M_\Phi h(H(x), H(y)) S + \varepsilon S,$$

thus

$$\Phi(u, H(x)) \subset \Phi(u, H(y)) + M_0 M_\Phi h(H(x), H(y)) S + \varepsilon S$$

for $u \in [0,T]$. Since $\varepsilon > 0$ and $x, y \in K$ are arbitrary, we obtain

$$h(\Phi(u, H(x)), \Phi(u, H(y))) \le M_0 M_\Phi h(H(x), H(y)).$$

Hence by Lemma 1,

$$h\left(\int_0^t \left(\int_0^s \Phi(u, H(x)) du \right) ds, \int_0^t \left(\int_0^s \Phi(u, H(y)) du \right) ds \right) \le$$
$$\le \int_0^t \left(\int_0^s h(\Phi(u, H(x)), \Phi(u, H(y))) du \right) ds \le$$
$$\le \int_0^t \left(\int_0^s M_0 M_\Phi h(H(x), H(y)) du \right) ds = \tag{7}$$
$$= \frac{t^2}{2} M_0 M_\Phi h(H(x), H(y)).$$

By (4), (6) and (7), we get

$$
\begin{aligned}
&h((\Gamma\Phi)(t_1,x),(\Gamma\Phi)(t_2,y)) \le \\
&\quad \le h(\Psi(x),\Psi(y)) + h\left(\int_0^{t_1}\left(\int_0^s \Phi(u,H(x))du\right)ds, \int_0^{t_2}\left(\int_0^s \Phi(u,H(y))du\right)ds\right) \le \\
&\quad \le h(\Psi(x),\Psi(y)) + h\left(\int_0^{t_1}\left(\int_0^s \Phi(u,H(x))du\right)ds, \int_0^{t_1}\left(\int_0^s \Phi(u,H(y))du\right)ds\right) + \\
&\qquad + h\left(\{0\}, \int_{t_1}^{t_2}\left(\int_0^s \Phi(u,H(y))du\right)ds\right) \le \\
&\quad \le h(\Psi(x),\Psi(y)) + \frac{t_1^2}{2}M_0M_\Phi h(H(x),H(y)) + \frac{t_2^2-t_1^2}{2}M_\Phi\|H(y)\|.
\end{aligned}
$$

This shows that $\Gamma\Phi$ is a continuous set-valued function, because Ψ and H are continuous. It is obvious that $x \mapsto (\Gamma\Phi)(t,x)$, $t \in [0,T]$, are linear. Therefore,

$$\Gamma\colon E \to E.$$

Now, we prove that Γ has exactly one fixed point.

From Lemma 1 and properties of the Hausdorff metric there follows

$$
\begin{aligned}
&h((\Gamma\Phi)(t,x),(\Gamma\Pi)(t,x)) = \\
&\quad = h\Bigg(\Psi(x) + \int_0^t\left(\int_0^s \Phi(u,H(x))du\right)ds, \\
&\qquad\qquad \Psi(x) + \int_0^t\left(\int_0^s \Pi(u,H(x))du\right)ds\Bigg) = \\
&\quad = h\left(\int_0^t\left(\int_0^s \Phi(u,H(x))du\right)ds, \int_0^t\left(\int_0^s \Pi(u,H(x))du\right)ds\right) \le \\
&\quad \le \int_0^t\left(\int_0^s h(\Phi(u,H(x)),\Pi(u,H(x)))du\right)ds
\end{aligned}
\tag{8}
$$

for $t \in [0,T]$ and $x \in S \cap K$. Thus

$$h((\Gamma\Phi)(t,x),(\Gamma\Pi)(t,x)) \le \frac{t^2}{2}\rho(\Phi,\Pi)\|H(x)\| \tag{9}$$

for $t \in [0,T]$ and $x \in S \cap K$. This implies that

$$\rho(\Gamma\Phi,\Gamma\Pi) \le \frac{T^2}{2}\|H\|\rho(\Phi,\Pi).$$

Let

$$\Phi_1(t,x) := (\Gamma\Phi)(t,x) \quad \text{and} \quad \Pi_1(t,x) := (\Gamma\Pi)(t,x).$$

By (8), there is

$$h\left(\left(\Gamma^2\Phi\right)(t,x), \left(\Gamma^2\Pi\right)(t,x)\right) = h\left((\Gamma\Phi_1)(t,x), (\Gamma\Pi_1)(t,x)\right) \leq$$
$$\leq \int_0^t \left(\int_0^s h\left(\Phi_1(u,H(x)), \Pi_1(u,H(x))\right) du \right) ds.$$

According to (9), we get

$$h\left(\Phi_1(u,y), \Pi_1(u,y)\right) \leq \frac{u^2}{2}\rho(\Phi,\Pi)\|H(y)\|$$

for $y \in H(x)$. Thus

$$h\left(\Phi_1(u,y), \Pi_1(u,y)\right) \leq \frac{u^2}{2}\rho(\Phi,\Pi)\|H(H(x))\|$$

so

$$\Phi_1(u,y) \subset \Pi_1(u,y) + \frac{u^2}{2}\rho(\Phi,\Pi)\|H^2(x)\|S$$

and

$$\Pi_1(u,y) \subset \Phi_1(u,y) + \frac{u^2}{2}\rho(\Phi,\Pi)\|H^2(x)\|S.$$

Hence

$$\Phi_1(u,H(x)) \subset \Pi_1(u,H(x)) + \frac{u^2}{2}\rho(\Phi,\Pi)\|H^2(x)\|S$$

and

$$\Pi_1(u,H(x)) \subset \Phi_1(u,H(x)) + \frac{u^2}{2}\rho(\Phi,\Pi)\|H^2(x)\|S,$$

i.e.,

$$h\left(\Phi_1(u,H(x)), \Pi_1(u,H(x))\right) \leq \frac{u^2}{2}\rho(\Phi,\Pi)\|H^2(x)\|.$$

Therefore,

$$h\left(\left(\Gamma^2\Phi\right)(t,x), \left(\Gamma^2\Pi\right)(t,x)\right) \leq \int_0^t \left(\int_0^s \frac{u^2}{2}\rho(\Phi,\Pi)\left\|H^2(x)\right\| du \right) ds$$
$$= \frac{t^4}{4!}\rho(\Phi,\Pi)\|H^2(x)\|$$

for $t \in [0,T]$ and $x \in S \cap K$. Thus

$$\rho\left(\Gamma^2\Phi, \Gamma^2\Pi\right) \leq \frac{T^4}{4!}\rho(\Phi,\Pi)\|H\|^2.$$

By induction we obtain

$$\rho\left(\Gamma^n\Phi, \Gamma^n\Pi\right) \le \frac{T^{2n}\|H\|^n}{(2n)!}\rho(\Phi,\Pi)$$

for $n \in \mathbb{N}$.

We observe that for every $T > 0$ there exists $n \in \mathbb{N}$ such that $\frac{T^{2n}\|H\|^n}{(2n)!} < 1$.

From the Banach fixed point theorem we conclude that Γ^n has exactly one fixed point, whence it follows that Γ has exactly one fixed point. This means that there exists exactly one solution of problem (2) for $(t,x) \in [0,T] \times K$. □

Now we describe an application. Let K be a closed convex cone with the nonempty interior in a Banach space. Suppose that $\{F_t:\ t \ge 0\}$ and $\{G_t:\ t \ge 0\}$ are regular cosine families of continuous linear multifunctions $F_t\colon K \to cc(K)$, $G_t\colon K \to cc(K)$ such that $x \in F_t(x)$, $x \in G_t(x)$, $F_t \circ F_s = F_s \circ F_t$, $G_t \circ G_s = G_s \circ G_t$ for $x \in K$, $s,t \ge 0$ and

$$H(x) := D^2F_t(x)|_{t=0} = D^2G_t(x)|_{t=0}.$$

Then multifunctions $(t,x) \mapsto F_t(x)$ and $(t,x) \mapsto G_t(x)$ are linear with respect to x and satisfy (2) with $\Psi(x) = \{x\}$. By virtue of Theorem 2, $F_t(x) = G_t(x)$ for $(t,x) \in [0,+\infty) \times K$. This means that if two regular cosine families as those above have the same second order infinitesimal generator, then there are equal.

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