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A FIRST-ORDER SPECTRAL PHASE TRANSITION IN A CLASS OF PERIODICALLY MODULATED HERMITIAN JACOBI MATRICES

Abstract. We consider self-adjoint unbounded Jacobi matrices with diagonal $q_n = b_n n$ and off-diagonal entries $\lambda_n = n$, where b_n is a 2-periodical sequence of real numbers. The parameter space is decomposed into several separate regions, where the spectrum of the operator is either purely absolutely continuous or discrete. We study the situation where the spectral phase transition occurs, namely the case of $b_1 b_2 = 4$. The main motive of the paper is the investigation of asymptotics of generalized eigenvectors of the Jacobi matrix. The pure point part of the spectrum is analyzed in detail.

Keywords: Jacobi matrices, spectral phase transition, absolutely continuous spectrum, pure point spectrum, discrete spectrum, subordinacy theory, asymptotics of generalized eigenvectors.

Mathematics Subject Classification: 47A10, 47B36.

1. INTRODUCTION

In the present paper, we study a class of Jacobi matrices with unbounded entries: a periodically modulated linearly growing diagonal and linearly growing weights.

First define the operator J on the linear set of finite (having a finite number of non-zero elements) vectors $l_{fin}(\mathbb{N})$:

$$(Ju)_n = \lambda_{n-1} u_{n-1} + q_n u_n + \lambda_n u_{n+1},\ n \geq 2 \tag{1}$$

with the initial condition $(Ju)_1 = q_1 u_1 + \lambda_1 u_2$, where $q_n = b_n n,\ \lambda_n = n$, and b_n is a real 2-periodic sequence, generated by the parameters b_1 and b_2.

Let $\{e_n\}_{n\in\mathbb{N}}$ be the canonical basis in $l^2(\mathbb{N})$. With respect to this basis, the operator J has the following matrix representation:

$$J = \begin{pmatrix} q_1 & \lambda_1 & 0 & \cdots \\ \lambda_1 & q_2 & \lambda_2 & \cdots \\ 0 & \lambda_2 & q_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Due to the Carleman condition (cf. [2]) $\sum\limits_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty$, the operator J is essentially self-adjoint. So we will in what follows assume that J is a closed self-adjoint operator in $l^2(\mathbb{N})$, defined on its natural domain $D(J) = \{u \in l^2(\mathbb{N}) : Ju \in l^2(\mathbb{N})\}$.

The main tool in our investigation is Gilbert and Pearson's subordinacy theory (cf. [5]), which was generalized to the case of Jacobi matrices by Khan and Pearson (cf. [9]).

Our example is a special case of the situation investigated by S. Naboko and J. Janas in [7] and [8]. In these articles, there was shown that the space of parameters $(b_1; b_2) \in \mathbb{R}^2$ can be divided into a set of regions of two types. In the regions of the first type, the spectrum of the operator J is purely absolutely continuous and covers the real line $\mathbb{R}$, and in the regions of the second type, the spectrum is discrete.

According to the results of [7] and [8], spectral properties of Jacobi matrices of the class considered are determined by the location of the point zero relatively to the absolutely continuous spectrum of some periodic matrix J_{per}, constructed out of the modulation parameters b_1 and b_2. In our case this leads to:

$$J_{per} = \begin{pmatrix} b_1 & 1 & 0 & 0 & \dots \\ 1 & b_2 & 1 & 0 & \dots \\ 0 & 1 & b_1 & 1 & \dots \\ 0 & 0 & 1 & b_2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Considering the characteristic polynomial

$$d_{J_{per}}(\lambda) = Tr\left(\begin{pmatrix} 0 & 1 \\ -1 & \lambda - b_1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & \lambda - b_2 \end{pmatrix}\right) = \lambda^2 - (b_1 + b_2)\lambda + b_1 b_2 - 2,$$

we can find the absolutely continuous spectrum $\sigma_{ac}(J_{per})$ of J_{per} from the following condition (cf. [2]):

$$\lambda \in \sigma_{ac}(J_{per}) \Leftrightarrow \left|d_{J_{per}}(\lambda)\right| \le 2. \tag{2}$$

This leads to the following result (cf. [8]) about the spectral structure of the operator J.

If $\left|d_{J_{per}}(0)\right| < 2$, *then the spectrum of the operator* J *is purely absolutely continuous of local multiplicity* 1 *a.e. on the whole real line.*

If $\left|d_{J_{per}}(0)\right| > 2$, *then the spectrum of the operator* J *is discrete.*

So the condition $|d_{J_{per}}(0)| = |b_1 b_2 - 2| = 2$, equivalent to $\{b_1 b_2 = 0 \text{ or } b_1 b_2 = 4\}$, determines the location of the regions on the plane $(b_1; b_2)$ where the spectrum of the operator J is either purely absolutely continuous or discrete. Namely:

If $0 < b_1 b_2 < 4$ *then the spectrum of the operator* J *is purely absolutely continuous of local multiplicity* 1 *a.e. on the whole real line.*

If $b_1 b_2 < 0$ *or* $b_1 b_2 > 4$ *then the spectrum of the operator* J *is discrete.*

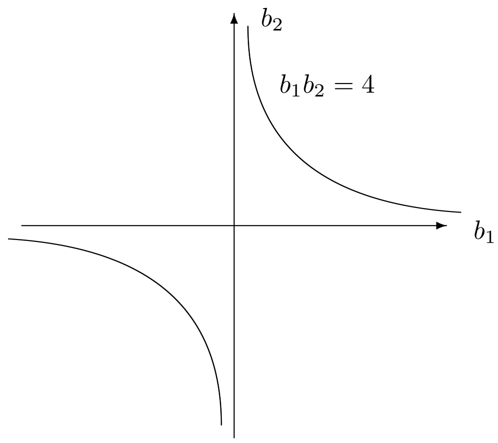

Fig. 1

The boundary situations, i.e., $b_1 b_2 = 0$ or $b_1 b_2 = 4$ (Fig. 1), are of most interest to us, because they show the spectral behavior of the operator J when the parameters cross the line where the spectral phase transition occurs. The first case ($b_2 = 0$) was studied in [3] and the following results were obtained:

— The interval $(-\infty; 0]$ is covered by the purely absolutely continuous spectrum (of local multiplicity 1 a.e.) of the operator J.
— In the interval $(0; +\infty)$, the spectrum is discrete and there is no accumulation of eigenvalues at the point zero.
— The n-th eigenvalue grows linearly with n: $\lambda_n \asymp n$, which means that the operator J is not semi-bounded.

In the present paper, we study the spectral properties of the operator J in the second boundary case, i.e., $b_1 b_2 = 4$, where the corresponding Jacobi matrix is semi-bounded, as will be shown below.

A similar example of the Jacobi matrix with modulated weights was considered in paper [10].

Remark 1. *We note that a Jacobi matrix* J_{b_1,b_2} *with modulation parameters equal to* b_1 *and* b_2*, is unitarily equivalent to the matrix* $-J_{-b_1,-b_2}$*. Therefore, a study of the situation considered can be reduced to the case of* $b_1,\ b_2 > 0$*.*

Now let us formulate the main result of this paper.

Theorem. 1. *Let $b_1 b_2 = 4$, $b_1, b_2 > 0$, $b_1 \neq b_2$. Then:*

— *The interval* $(\frac{4}{b_1+b_2}; +\infty)$ *is covered by the purely absolutely continuous spectrum (of local multiplicity* 1 *a.e.) of the operator* J.
— *The operator is semi-bounded from below by the constant* $\min\{|b_1|, |b_2|\}$.
— *On the interval* $[\min\{|b_1|, |b_2|\}; \frac{4}{b_1+b_2})$, *the spectrum is discrete and the following estimate on the number of eigenvalues in the interval* $[\min\{|b_1|, |b_2|\}; \frac{4}{b_1+b_2} - \varepsilon)$ *for positive* ε *holds:*

$$\#\left\{\lambda_n : \min\{|b_1|, |b_2|\} < \lambda_n < \frac{4}{b_1 + b_2} - \varepsilon\right\} < \frac{(b_2 - b_1)^2(\sqrt{b_1} + \sqrt{b_2})^2}{16\varepsilon(b_1 + b_2)^2} + 1.$$

2. *Let $b_1 = b_2 = 2$. Then the spectrum of the operator J is $\sigma(J) = [1; +\infty)$ and in the interval $(1; +\infty)$, the spectrum is purely absolutely continuous.*

The paper is organized as follows.

In Section 2, we establish the asymptotics of the generalized eigenvectors and determine the spectral structure of the operator J via the Gilbert-Khan-Pearson subordinacy theory, dividing the real axis into two intervals with pure point and purely absolutely continuous spectrum.

In Section 3, we study the pure point part of the spectrum, proving its non-emptiness and then discreteness and obtaining the estimate for the accumulation of eigenvalues.

2. GENERALIZED EIGENVECTORS AND THE SPECTRUM IN THE BOUNDARY CASE $b_1 b_2 = 4$

In this section, we find the asymptotics of the generalized eigenvectors using the Birkhoff-Adams Theorem and draw first conclusions about the spectrum.

Consider the spectral recurrence relation

$$(n-1)u_{n-1} + b_n n u_n + n u_{n+1} = \lambda u_n, \ n \geq 2. \tag{3}$$

Since the coefficients of this recurrence relation are linear with resect to n, we attempt to use the Birkhoff-Adams Theorem. It deals with the recurrence relation of the form

$$x_{n+2} + F_1(n)x_{n+1} + F_2(n)x_n = 0, \ n \geq 1, \tag{4}$$

where the coefficients $F_1(n)$ and $F_2(n)$ have asymptotical expansions as $n \to \infty$ of the form:

$$F_1(n) \sim \sum_{k=0}^{\infty} \frac{\mathbf{a}_k}{n^k}, \quad F_2(n) \sim \sum_{k=0}^{\infty} \frac{\mathbf{b}_k}{n^k}$$

with $\mathbf{b}_0 \neq 0$. Consider the characteristic equation $\alpha^2 + \mathbf{a}_0\alpha + \mathbf{b}_0 = 0$ and denote its roots α_1 and α_2. The following result takes place (cf. [4]):

Theorem (Birkhoff-Adams). *If the roots α_1 and α_2 coincide, $\alpha := \alpha_1 = \alpha_2$, and an additional condition $\boldsymbol{a}_1\alpha + \boldsymbol{b}_1 \neq 0$ holds, where $\beta = \frac{1}{4} + \frac{\boldsymbol{b}_1}{2\boldsymbol{b}_0}$, $\delta_1 = 2\sqrt{\frac{\boldsymbol{a}_0\boldsymbol{a}_1 - 2\boldsymbol{b}_1}{2\boldsymbol{b}_0}} = -\delta_2$, then there exist two linearly independent solutions $x_n^{(1)}$ and $x_n^{(2)}$ of recurrence relation* (4) *with the following asymptotics, as $n \to \infty$:*

$$x_n^{(i)} = \alpha^n e^{\delta_i \sqrt{n}} n^\beta \left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right), \; i = 1, 2.$$

Obviously, we cannot apply this Theorem directly, because we do not have the required asymptotic expansions. In order to negotiate this problem, we derive equations for even and odd components of the sequence $\{u_n\}$. Taking recurrence relations (5) for three consecutive values of n (for n, $n+1$ and $n+2$),

$$\begin{cases} \lambda_{n-1}u_{n-1} + q_n u_n + \lambda_n u_{n+1} = \lambda u_n, \\ \lambda_n u_n + q_{n+1}u_{n+1} + \lambda_{n+1}u_{n+2} = \lambda u_{n+1}, \\ \lambda_{n+1}u_{n+1} + q_{n+2}u_{n+2} + \lambda_{n+2}u_{n+3} = \lambda u_{n+2}, \end{cases}$$

and having eliminated the values u_n and u_{n+2} from them, we obtain one equation containing the values u_{n-1}, u_{n+1} and u_{n+3} only:

$$u_{n+3} + p_1(n)u_{n+1} + p_2(n)u_{n-1} = 0,$$

with the following coefficients $p_1(n)$ and $p_2(n)$:

$$p_1(n) = \frac{\lambda_n^2(q_{n+2} - \lambda) + \lambda_{n+1}^2(q_n - \lambda) - (q_n - \lambda)(q_{n+1} - \lambda)(q_{n+2} - \lambda)}{\lambda_{n+1}\lambda_{n+2}(q_n - \lambda)},$$

$$p_2(n) = \frac{\lambda_n\lambda_{n-1}(q_{n+2} - \lambda)}{\lambda_{n+1}\lambda_{n+2}(q_n - \lambda)}.$$

Denote:

$$\begin{cases} v_n := u_{2n-1}, \\ w_n := u_{2n}, \\ F_i(n) := p_i(2n), & i = 1, 2, \\ G_i(n) := p_i(2n+1), & i = 1, 2. \end{cases}$$

We get recurrence relations for odd and even components of the sequence $\{u_n\}$:

$$\begin{cases} v_{n+2} + F_1(n)v_{n+1} + F_2(n)v_n = 0, \\ w_{n+2} + G_1(n)w_{n+1} + G_2(n)w_n = 0, \end{cases} \tag{5}$$

and the coefficients $F_1(n)$, $F_2(n)$ and $G_1(n)$, $G_2(n)$ having the asymptotical expansions by the powers of $\frac{1}{n}$ with the following first terms

$$F_1(n), G_1(n) = (2 - b_1b_2) + \frac{(\lambda(b_1 + b_2) - 2)}{2n} + O\left(\frac{1}{n^2}\right),$$

$$F_2(n), G_2(n) = 1 - \frac{1}{n} + O\left(\frac{1}{n^2}\right).$$

This makes the Birkhoff-Adams Theorem applicable. Since $\mathbf{a}_0 = 2 - b_1 b_2$, $\mathbf{a}_1 = -1 + \frac{\lambda(b_1+b_2)}{2}$, $\mathbf{b}_0 = 1$ and $\mathbf{b}_1 = -1$, we get the equation $\alpha^2 - 2\alpha + 1 = 0$, its root $\alpha = 1$ and values $\beta = -\frac{3}{4}$, $\delta_1 = 2\sqrt{4 - \lambda(b_1 + b_2)} = -\delta_2$. The direct application of the Birkhoff-Adams Theorem yields the following result.

Lemma 1. *Let $b_1 b_2 = 4$, $b_1, b_2 > 0$ and $b_1 \neq b_2$. Equations* (5) *have bases of solutions v_n^+, v_n^- and w_n^+, w_n^- with the following asymptotics as $n \to \infty$:*

$$v_n^{\pm}, w_n^{\pm} = n^{-\frac{3}{4}} e^{\pm 2\sqrt{4-\lambda(b_1+b_2)}\sqrt{n}} \left(1 + O\Big(\frac{1}{\sqrt{n}}\Big)\right), \ i = 1, 2.$$

In order to return from odd and even components of the sequence $\{u_n\}$ back to the sequence itself, we should combine the obtained asymptotics properly. The following Lemma gives the answer.

Lemma 2. *Let $b_1 b_2 = 4$, $b_1, b_2 > 0$ and $b_1 \neq b_2$. Equation* (3) *has a basis of solutions u_n^+, u_n^- with the following asymptotics as $n \to \infty$:*

$$u_{2n-1}^{\pm} = \sqrt{b_2} n^{-\frac{3}{4}} e^{\pm 2\sqrt{4-\lambda(b_1+b_2)}\sqrt{n}} \left(1 + O\Big(\frac{1}{\sqrt{n}}\Big)\right),$$
$$u_{2n}^{\pm} = -\sqrt{b_1} n^{-\frac{3}{4}} e^{\pm 2\sqrt{4-\lambda(b_1+b_2)}\sqrt{n}} \left(1 + O\Big(\frac{1}{\sqrt{n}}\Big)\right).$$

Proof. From Lemma 1, we know that any solution of recurrence relation (3) can be written in the following form:

$$u_{2k-1} = c_- v_k^- + c_+ v_k^+, \ u_{2k} = d_- w_k^- + d_+ w_k^+$$

with some constants c_-, c_+, d_- and d_+. Let us substitute the solution u of this form into the recurrence relation written for the number $2k$ in order to obtain relations between these constants.

$$\begin{aligned}(2k-1)u_{2k-1} &+ (b_2(2k) - \lambda)u_{2k} + (2k)u_{2k+1} = \\ &= (2k^{\frac{1}{4}}) \Big[e^{2\sqrt{4-\lambda(b_1+b_2)}\sqrt{k}} (c_+ + b_2 d_+ + c_+ + o(1)) + \\ &\qquad + e^{-2\sqrt{4-\lambda(b_1+b_2)}\sqrt{k}} (c_- + b_2 d_- + c_- + o(1)) \Big] = 0.\end{aligned}$$

From the condition of cancellation in the main order of this equation, we can conclude that

$$d_{\pm} = -\frac{2c_{\pm}}{b_2} = -\sqrt{\frac{b_1}{b_2}} c_{\pm},$$

which means that any solution of the recurrence relation is a linear combination of the following two solutions u_n^+ and u_n^-:

$$\begin{cases} u_{2k-1}^{\pm} := -\sqrt{b_2} v_k^{\pm}, \\ u_{2k}^{\pm} := \sqrt{b_1} w_k^{\pm}, \end{cases}$$

which have asymptotics of the form stated. □

Now that we know the asymptotics of generalized eigenvectors, we apply the subordinacy theory and arrive at the result regarding the spectrum of the operator J.

Theorem 1. *Let $b_1 b_2 = 4$, $b_1, b_2 > 0$. The spectrum of the operator J is purely absolutely continuous (of local multiplicity* 1 *a.e.) on the interval* $(\frac{4}{b_1+b_2}; +\infty)$ *and pure point on the interval* $(-\infty; \frac{4}{b_1+b_2})$.

Remark 2. *We can use the same arguments to derive asymptotics of generalized eigenvectors in the other situations, i.e. when $b_1 b_2 \neq 4$, and obtain the answers concerning the spectral structure of the operator J. The answers will be the same as in the papers mentioned above, which is self-understood.*

3. DISCRETENESS OF THE PART OF THE SPECTRUM IN THE BOUNDARY CASE $b_1 b_2 = 4$

In this section, we focus on the pure point part of the spectrum in the case considered. We investigate into this part of spectrum showing its discreteness and and analyzing the accumulation of eigenvalues to the edge of the essential spectrum $\sigma_{ess}(J) = [\frac{4}{b_1+b_2}; +\infty)$. Towards the end of the Section, we consider the particular case of $b_1 = b_2 = 2$. This case should be developed separately and we will acquire a simple result.

3.1. SEMI-BOUNDEDNESS OF THE OPERATOR

First we prove the semi-boundedness of the operator, which is not an idle question for us. The fact of semi-boundeness makes the investigation into the discrete spectrum much easier. We want to stress this property of the operator in our case and, although we could derive this fact from the estimates for the eigenvalues that we obtain later, it follows immediately from the simple estimate of the quadratic form, which we show below.

Theorem 2. *If $b_1 b_2 = 4$ and $b_1, b_2 > 0$, then the operator J is semi-bounded from below by the constant $\frac{1}{2}\min\{b_1, b_2\}$.*

Proof. Write out the quadratic form of the operator,

$$2(Ju, u) = 2\sum_{n=1}^{\infty} n b_n |u_n|^2 + 2\sum_{n=1}^{\infty} n\left(u_n \overline{u_{n+1}} + u_{n+1}\overline{u_n}\right) =$$

and substitute the values of the sequence $\{b_n\}$ into it,

$$= 2\sum_{k=1}^{\infty}\left[(2k-1)b_1|u_{2k-1}|^2 + 2k b_2 |u_{2k}|^2\right] +$$
$$+ 2\sum_{k=1}^{\infty}\left[(2k-1)(u_{2k-1}\overline{u_{2k}} + u_{2k}\overline{u_{2k-1}}) + (2k)(u_{2k}\overline{u_{2k+1}} + u_{2k+1}\overline{u_{2k}})\right] =$$

then group the items of the same type and take out the linear terms,

$$= \Bigg(\sum_{k=1}^{\infty} [(2k-1)(b_1|u_{2k-1}|^2 + b_2|u_{2k}|^2 +$$
$$+ 2(u_{2k-1}\overline{u_{2k}} + u_{2k}\overline{u_{2k-1}}))] + \sum_{k=1}^{\infty} b_2|u_{2k}|^2 \Bigg) +$$
$$+ \Bigg(b_1|u_1|^2 + \sum_{k=1}^{\infty} [(2k)(b_1|u_{2k+1}|^2 + b_2|u_{2k}|^2 +$$
$$+ 2(u_{2k}\overline{u_{2k+1}} + u_{2k+1}\overline{u_{2k}}))] + \sum_{k=1}^{\infty} b_1|u_{2k+1}|^2 \Bigg) =$$

and finally use the fact that $b_1 b_2 = 4$,

$$= \sum_{n=1}^{\infty} b_n|u_n|^2 + \sum_{n=1}^{\infty} n\left|\sqrt{b_n}u_n + \sqrt{b_{n+1}}u_{n+1}\right|^2.$$

We arrive at the equality that is useful in itself,

$$2(Ju,u) = \sum_{n=1}^{\infty} \left[n\left|\sqrt{b_n}u_n + \sqrt{b_{n+1}}u_{n+1}\right|^2 + b_n|u_n|^2 \right]. \tag{6}$$

A very rough estimate yields

$$2(Ju,u) \geq \sum_{n=1}^{\infty} b_n|u_n|^2 \geq \min\{b_1, b_2\}\|u\|^2,$$

which completes the proof. □

3.2. NON-EMPTINESS OF THE PURE POINT PART OF THE SPECTRUM

Now we show the non-emptiness of the part of the spectrum lying below the point $\frac{4}{b_1+b_2}$. To this end, we find an appropriate vector of norm 1 (with the real components from l_{fin}), which makes the quadratic form of the operator less than $\frac{4}{b_1+b_2}$. We need the following technical lemma first.

For a finite vector $\{v_n\}_{n=1}^{\infty}$, denote

$$X(\{v_n\}) := \frac{\sum\limits_{n=1}^{\infty} n(v_n - v_{n+1})^2}{\left|\sum\limits_{n=1}^{\infty} v_n^2(-1)^n\right|}$$

(this expression can take the value ∞, when the denominator turns to zero).

Lemma 3.

$$\inf_{\sum\limits_{n=1}^{\infty} v_n^2(-1)^n<0} X(\{v_n\}) = \inf_{\sum\limits_{n=1}^{\infty} v_n^2(-1)^n>0} X(\{v_n\}) = 0.$$

We will prove this lemma a little later and now we just use its result to prove the following theorem concerning the operator J.

Theorem 3. *Let $b_1 b_2 = 4$, $b_1, b_2 > 0$. If $b_1 \neq b_2$, then the spectrum of the operator J in the interval $\left(-\infty; \frac{4}{b_1+b_2}\right)$ is non-empty.*

Proof. We need to construct a vector $\{u_n\}$ that makes the following expression negative:

$$2\left((Ju, u) - \frac{4}{b_1+b_2}\|u\|^2\right) = \sum_{n=1}^{\infty}\left[n\left|\sqrt{b_n}u_n + \sqrt{b_{n+1}}u_{n+1}\right|^2 + b_n|u_n|^2 - \frac{8}{b_1+b_2}|u_n|^2\right].$$

If we take u_n of the form $u_n = \frac{(-1)^n v_n}{\sqrt{b_n}}$, we get

$$2\left((Ju, u) - \frac{4}{b_1+b_2}\|u\|^2\right) = \sum_{n=1}^{\infty}\left[n(v_n - v_{n+1})^2 + v_n^2 - \frac{8}{b_1+b_2}\frac{v_n^2}{b_n}\right] =$$

using the fact that $\frac{4}{b_n} = b_{n+1}$,

$$= \sum_{n=1}^{\infty}\left[n(v_n - v_{n+1})^2 + \frac{b_n - b_{n+1}}{b_1+b_2}v_n^2\right] =$$

taking into account that $b_n - b_{n+1} = (-1)^n(b_2 - b_1)$,

$$= \sum_{n=1}^{\infty}\left[n(v_n - v_{n+1})^2 + \frac{b_2 - b_1}{b_1+b_2}(-1)^n v_n^2\right] =$$

$$= \left[\sum_{n=1}^{\infty}(-1)^n v_n^2\right]\left(\frac{b_1 - b_2}{b_1+b_2} + \frac{\sum\limits_{n=1}^{\infty} n(v_n - v_{n+1})^2}{\sum\limits_{n=1}^{\infty}(-1)^n v_n^2}\right).$$

Lemma 3 guarantees that we can choose such finite vector $\{v_n\}$ that will make the latter expression negative. The choice depends on the sign and the value of the constant $\frac{b_1-b_2}{b_1+b_2}$. □

Now, when it is clear how to apply Lemma 3, we can prove it.

Proof of Lemma 3. Let us construct the sequences $\{v_n\}$ in both cases.

1) In the case of $\sum\limits_{n=1}^{\infty} v_n^2(-1)^{n+1} > 0$, there is

$$X(\{v_n\}) = \frac{\sum\limits_{n=1}^{\infty} n(v_n - v_{n+1})^2}{\sum\limits_{n=1}^{\infty} v_n^2(-1)^{n+1}} =$$

(choose $v_{2k} = v_{2k+1} =: w_{k+1}$ for $k \geq 1$ and $w_1 := v_1$ with some sequence $\{w_n\}$)

$$= \frac{\sum\limits_{n=1}^{\infty} (2n-1)(w_{n+1} - w_n)^2}{w_1^2} =$$

(denote $s_n := w_n - w_{n+1}$ for $n \geq 1$)

$$= \frac{\sum\limits_{n=1}^{\infty} (2n-1)s_n^2}{\left(\sum\limits_{n=1}^{\infty} s_n\right)^2}.$$

We set

$$s_n^{(N)} = \begin{cases} \frac{1}{n \ln n}, & n \leq N, \\ 0, & n > N, \end{cases}$$

$w_{N+1}^{(N)} = 0$ and reconstruct sequences $\{w_n^{(N)}\}$ and $\{v_n^{(N)}\}$. Due to the fact that

$$\begin{cases} \sum\limits_{n=1}^{\infty} \frac{1}{n \ln n} = \infty, \\ \sum\limits_{n=1}^{\infty} \frac{1}{n \ln^2 n} < \infty, \end{cases}$$

we can claim that choosing the number N large enough we can make the value of X at the sequence $\{v_n^{(N)}\}$ very close to zero:

$$X(\{v_n^{(N)}\}) \to 0 \text{ as } N \to \infty.$$

2) On the other hand, in the case of $\sum\limits_{n=1}^{\infty} v_n^2(-1)^n > 0$, there is

$$X(\{\widetilde{v}_n\}) = \frac{\sum\limits_{n=1}^{\infty} n(\widetilde{v}_n - \widetilde{v}_{n+1})^2}{\sum\limits_{n=1}^{\infty} \widetilde{v}_n^2(-1)^n} =$$

and taking now $\widetilde{v}_{2k+1} =: \widetilde{w}_{k+2}$, $\widetilde{v}_{2k} =: \widetilde{w}_{k+1}$ for $k \geq 1$, but $\widetilde{w}_1 := \widetilde{v}_1$, we get

$$= \frac{(\widetilde{w}_2 - \widetilde{w}_1)^2 + \sum\limits_{n=1}^{\infty} 2n(\widetilde{w}_{n+2} - \widetilde{w}_{n+1})^2}{\widetilde{w}_2^2 - \widetilde{w}_1^2} =$$

(denote $\widetilde{s}_n := -\widetilde{w}_n + \widetilde{w}_{n+1}$ for $n \geq 1$)

$$= \frac{\widetilde{s}_1^2 + \sum\limits_{n=1}^{\infty} (2n)\widetilde{s}_{n+1}^2}{\widetilde{s}_1 \left(\widetilde{s}_1 + 2 \sum\limits_{n=1}^{\infty} \widetilde{s}_n \right)}.$$

Setting again

$$\widetilde{s}_n^{(N)} = \begin{cases} \frac{1}{n \ln n}, & n \leq N, \\ 0, & n > N, \end{cases}$$

and $\widetilde{w}_{N+1}^{(N)} = 0$, reconstructing sequences $\{\widetilde{w}_n^{(N)}\}$ and $\{\widetilde{v}_n^{(N)}\}$ and using the same fact

$$\begin{cases} \sum\limits_{n=1}^{\infty} \frac{1}{n \ln n} = \infty, \\ \sum\limits_{n=1}^{\infty} \frac{1}{n \ln^2 n} < \infty, \end{cases}$$

we find that for sequences $\{\widetilde{v}_n^{(N)}\}$,

$$X(\{\widetilde{v}_n^{(N)}\}) \to 0 \text{ as } N \to \infty,$$

which completes the proof.

□

3.3. DISCRETENESS OF THE PURE POINT PART OF THE SPECTRUM AND THE ESTIMATE FOR THE EIGENVALUES

Now we prove the discreteness of the pure point part of the spectrum, which is non-empty as we have shown above. Using the Glazman Lemma we obtain an estimate characterizing the speed of possible accumulation of eigenvalues to the point $\frac{4}{b_1+b_2}$.

Theorem 4. *Let $b_1 b_2 = 4$ and $b_1, b_2 > 0$. If $b_1 \neq b_2$, then the spectrum of the operator J in the interval $(-\infty; \frac{4}{b_1+b_2})$ is discrete and for any $\varepsilon > 0$ the number of eigenvalues in the interval $(-\infty; \frac{4}{b_1+b_2} - \varepsilon)$ obeys the estimate*

$$\# \left\{ \lambda_n : \lambda_n < \frac{4}{b_1 + b_2} - \varepsilon \right\} < \frac{(b_2 - b_1)^2 (\sqrt{b_1} + \sqrt{b_2})^2}{16 \varepsilon (b_1 + b_2)^2} + 1.$$

Proof. We use the following simple consequence of the Glazman Lemma (cf. [1]). If for every finite vector with the first N components being zeros the estimate $(Ju,u) > C\|u\|^2$ holds, then the spectrum of the operator J on the interval $(-\infty; C)$ is discrete and the operator has at most N eigenvalues on this interval.

Hence in order to prove the assertion of the Theorem we need to show that the quadratic form

$$(Ju,u) - \left(\frac{4}{b_1+b_2} - \varepsilon\right)\|u\|^2$$

is positive on every finite vector with a number of initial components being zeros.

First estimate the quadratic form of the operator using formula (6)

$$2(Ju,u) = \sum_{n=1}^{\infty}(n|\sqrt{b_n}u_n + \sqrt{b_{n+1}}u_{n+1}|^2 + b_n|u_n|^2),$$

the Cauchy inequality and introducing notation $v_n := \sqrt{b_n}|u_n|$:

$$(Ju,u) \geq \frac{1}{2}\sum_{n=1}^{\infty}(n(v_n - v_{n+1})^2 + v_n^2).$$

Rewrite also the norm of the vector $\{u_n\}$

$$\|u\|^2 = \sum_{n=1}^{\infty}\frac{v_n^2}{b_n} = \frac{1}{4}\sum_{n=1}^{\infty}v_n^2 b_{n+1},$$

which can be further rewritten in more symmetric fashion:

$$\|u\|^2 = \frac{1}{8}v_1^2 b_2 + \frac{1}{8}\sum_{n=1}^{\infty}(v_n^2 b_{n+1} + v_{n+1}^2 b_n).$$

Substitute this into the expression that we desire to estimate:

$$\begin{aligned}(Ju,u) - \left(\frac{4}{b_1+b_2} - \varepsilon\right)\|u\|^2 \geq \frac{1}{2}\sum_{n=1}^{\infty} n(v_n - v_{n+1})^2 \frac{1}{2(b_1+b_2)}\sum_{n=1}^{\infty}(b_n + b_{n+1})v_n^2 - \\ - \frac{1}{2(b_1+b_2)}\sum_{n=1}^{\infty}2b_{n+1}v_n^2 + \\ + \frac{\varepsilon}{8}v_1^2 b_2 + \frac{\varepsilon}{8}\sum_{n=1}^{\infty}(v_n^2 b_{n+1} + v_{n+1}^2 b_n).\end{aligned}$$

Taking into account that

$$b_n - b_{n+1} = (-1)^n(b_2 - b_1)$$

and making another kind of symmetrization

$$2\sum_{n=1}^{\infty}(-1)^n v_n^2 = -v_1^2 + \sum_{n=1}^{\infty}(-1)^n(v_n^2 - v_{n+1}^2),$$

we obtain

$$(Ju,u)-\Big(\frac{4}{b_1+b_2}-\varepsilon\Big)\|u\|^2 \ge \frac{1}{2}\sum_{n=1}^{\infty} n(v_n-v_{n+1})^2+$$
$$+\frac{b_2-b_1}{4(b_1+b_2)}\sum_{n=1}^{\infty}(-1)^n(v_n^2-v_{n+1}^2)+$$
$$+\frac{\varepsilon}{8}\sum_{n=1}^{\infty}(v_n^2b_{n+1}+v_{n+1}^2b_n)+\frac{\varepsilon}{8}v_1^2b_2-\frac{b_2-b_1}{4(b_1+b_2)}v_1^2.$$

We need to show that for large n the following expression

$$n(v_n-v_{n+1})^2+\frac{\varepsilon}{4}(v_n^2b_{n+1}+v_{n+1}^2b_n) > \frac{|b_1-b_2|}{2(b_1+b_2)}|v_n^2-v_{n+1}^2|$$

is positive (then we will be able to take zeros for initial components and obtain an estimate for the quadratic form). To this end, denote

$$\begin{cases} y_n := v_n - v_{n+1}, \\ x_n := \sqrt{v_n^2 b_{n+1} + v_{n+1}^2 b_n}, \\ B := \frac{|b_1-b_2|(\sqrt{b_1}+\sqrt{b_2})}{2(b_1+b_2)}. \end{cases}$$

Noticing that

$$v_n+v_{n+1} < \frac{\sqrt{b_1}+\sqrt{b_2}}{\sqrt{b_1b_2}}x_n = \frac{\sqrt{b_1}+\sqrt{b_2}}{2}x_n,$$

we get

$$n(v_n-v_{n+1})^2+\frac{\varepsilon}{4}(v_n^2b_{n+1}+v_{n+1}^2b_n)-\frac{|b_1-b_2|}{2(b_1+b_2)}|v_n^2-v_{n+1}^2| > \frac{\varepsilon}{4}x_n^2-\frac{B}{4}x_ny_n+ny_n^2.$$

And the latter expression is positive for any $n > N(\varepsilon) := \frac{B^2}{16\varepsilon}$, because

$$\left(\frac{x_n}{y_n}\right)^2-\frac{B}{\varepsilon}\left(\frac{x_n}{y_n}\right)+\frac{4n}{\varepsilon} = \left(\frac{x_n}{y_n}-\frac{B}{2\varepsilon}\right)^2-\frac{B^2}{4\varepsilon^2}+\frac{4n}{\varepsilon} > 0$$

for such n. Now it is clear that if we assume the first $N(\varepsilon)+1$ components of the vector $\{u_n\}$ to equal zero, we get the desired estimate of the quadratic form. That completes the proof. □

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