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SOME REMARKS ON THE OPTIMIZATION OF EIGENVALUE PROBLEMS INVOLVING THE p-LAPLACIAN

Abstract. Given a bounded domain $\Omega \subset \mathbb{R}^n$, numbers $p > 1$, $\alpha \geq 0$ and $A \in [0, |\Omega|]$, consider the optimization problem: find a subset $D \subset \Omega$, of measure A, for which the first eigenvalue of the operator $u \mapsto -\mathrm{div}(|\nabla u|^{p-2}\nabla u) + \alpha\chi_D|u|^{p-2}u$ with the Dirichlet boundary condition is as small as possible. We show that the optimal configuration D is connected with the corresponding positive eigenfunction u in such a way that there exists a number $t \geq 0$ for which $D = \{\, u \leq t \,\}$. We also give a new proof of symmetry of optimal solutions in the case when Ω is Steiner symmetric and $p = 2$.

Keywords: p-Laplacian, the first eigenvalue, Steiner symmetry.

Mathematics Subject Classification: 35P30, 35J65, 35J70.

1. INTRODUCTION

In this paper we obtain some results closely connected with those of [9], concerning the optimal pairs of an eigenvalue problem involving the p-Laplacian. The paper [9] is available online at `www.im.uj.edu.pl/actamath`. For the reader's convenience we shall recall the basic notation and terminology of [9], which in turn follow those of [1]. The paper [1] has originated research in the optimizaton of eigenvalues for the linear case of $p = 2$.

Let Ω be a bounded domain (i.e. open and connected set) in the space $\mathbb{R}^n$ $(n \geq 1)$ with the closure $\overline{\Omega}$ and boundary $\partial\Omega$. We denote by $|\Omega|$ the Lebesgue measure of Ω. Given numbers $p > 1$, $\alpha \geq 0$ and a measurable subset D of Ω, we shall be concerned with the eigenvalue problem of the form

$$\begin{cases} -\Delta_p(u) + \alpha\chi_D\,\varphi_p(u) = \lambda\varphi_p(u) & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Δ_p is the p-Laplacian, χ_D is the characteristic function of D, while φ_p is a function defined by

$$\varphi_p(u) := \begin{cases} |u|^{p-2}u, & \text{if } u \neq 0, \\ 0, & \text{if } u = 0. \end{cases}$$

The p-Laplacian is a nonlinear differential operator of the form

$$\Delta_p(u) = \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \operatorname{div}(\varphi_p(\nabla u)),$$

which coincides with the Laplacian Δ for $p = 2$.

In this paper we deal with real function spaces only. In paticular, we use standard Sobolev spaces $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$, with $1 < p < \infty$. It is customary to use solutions of (1) in a weak sense. Any nontrivial function $u\colon \Omega \to \mathbb{R}$ is said to be an eigenfunction of problem (1) if and only if $u \in W_0^{1,p}(\Omega)$ and

$$\int_\Omega \varphi_p(\nabla u)\nabla v + \alpha \int_\Omega \chi_D\, \varphi_p(u)\, v = \lambda \int_\Omega \varphi_p(u)\, v, \quad \text{for} \quad \text{any} \quad v \in W_0^{1,p}(\Omega).$$

Let $\lambda(\alpha, D)$ stand for the lowest eigenvalue λ of problem (1). It is known that $\lambda(\alpha, D)$ is positive and its eigenfunction is unique up to a scalar multiple (see, e.g., [3, 10] and the references therein). Let us fix $A \in [0, |\Omega|]$ and define

$$\Lambda(\alpha, A) := \inf\{\, \lambda(\alpha, D)\colon D \subset \Omega,\ |D| = A \,\}. \tag{2}$$

Any minimizer in (2) is called an *optimal configuration.* If u is an eigenfunction of problem (1) with $\lambda = \Lambda(\alpha, A)$ and with an optimal configuration D, then (u, D) is said to be an *optimal pair* (or *optimal solution*).

If u is an eigenfunction corresponding to the first eigenvalue of problem (1), then u does not change sign in Ω (see, e.g., [3, 10] and the references therein). From now on it will be chosen positive in Ω.

The above results were discussed in detail in our paper [9]. We shall also use the following lemmas:

Lemma 1. *Let $u \in W_{loc}^{1,1}(\Omega)$ and $t \in \mathbb{R}$. Then $\nabla u(x) = 0$ for almost every $x \in \{u = t\}$.*

In this connection refer to [5], Lemma 7.7, or [8], Theorem 6.19. We use the notation $\{u = t\} := \{x \in \Omega\colon u(x) = t\}$.

Lemma 2. *Assume that $u \in W_{loc}^{1,p}(\Omega)$ is a weak solution of the equation*

$$-\Delta_p(u) = f \qquad \text{in } \Omega$$

with $p > 1$, $f \in L^q(\Omega)$, $q > \dfrac{n}{p}$, $q \geq 2$. Let

$$Z := \{x \in \Omega\colon \nabla u(x) = 0\}.$$

Then $|\nabla u|^{p-1} \in W_{loc}^{1,2}(\Omega)$ and $f(x) = 0$ for almost every $x \in Z$.

This result comes from [7] and is quoted in [2].

2. OPTIMAL PAIRS

We recall that Ω is any bounded domain in $\mathbb{R}^n$ and $p \in (1, \infty)$. It is worth noting that we need no additional assumptions concerning the regularity of the boundary $\partial\Omega$.

Theorem 3. *For any $\alpha \geq 0$ and $A \in [0, |\Omega|]$ there exists an optimal pair.*

A proof of this result can be found in our paper [9].

Theorem 4. *Every optimal pair (u, D) has the following properties:*

(a) *$u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and ∇u is locally Hölder continuous, i.e., for every compact $K \subset \Omega$ there exists $\beta \in (0, 1)$ such that $\nabla u \in C^{0,\beta}(K)$,*

(b) *there is a number $t \geq 0$ such that (up to a set of measure zero)*

$$D = \{u \leq t\}. \tag{3}$$

As usual, we write $\{u < t\}$ instead of $\{x \in \Omega \colon u(x) < t\}$ and similarly we put $\{u \leq t\} := \{x \in \Omega \colon u(x) \leq t\}$.

Proof. The regularity properties of eigenfunctions, stated in assertion (a), are rather well known. In this connection see [9] and the references therein. Equality (3) in the case of $p = 2$ was stated in [1]. A lack of higher regularity of eigenfunctions is a source of difficulty in obtaining more general results.

We now claim that (3) holds for arbitrary $p > 1$. For $p \neq 2$ this is a new result. Let (u, D) be an optimal solution, corresponding to the optimal eigenvalue $\lambda_1 = \Lambda(\alpha, A)$ with $A > 0$ (the case of $A = 0$ is obvious). Note that according to [9], Theorem 1, there exists a number $t > 0$ such that

$$\{u < t\} \subset D \subset \{u \leq t\}. \tag{4}$$

In fact

$$t = \sup\{s \colon |\{u < s\}| \leq A\}.$$

In view of (4), it is sufficient to show that

$$|D^c \cap \{u = t\}| = 0,$$

where $D^c := \Omega \setminus D$. To begin with, let us introduce the critical set

$$Z := \{x \in \Omega \colon \nabla u(x) = 0\}.$$

According to Lemma 1, $\{u = t\} \subset Z$ and hence we see that $D^c \cap \{u = t\} \subset Z$. By Lemma 2,

$$-\Delta_p(u) = 0 \qquad \text{in } Z, \tag{5}$$

(i.e. this equality holds almost everywhere in Z). On the other hand,

$$-\Delta_p(u) = (\lambda_1 - \alpha\chi_D)\varphi_p(u) = \lambda_1\varphi_p(u) \qquad \text{in } D^c. \tag{6}$$

It now follows from (5) and (6) that

$$\lambda_1 \varphi_p(u) = 0 \qquad \text{in } D^c \cap \{u = t\}.$$

Note that $\lambda_1 \neq 0$ and $\varphi_p(u) = t^{p-1} \neq 0$ in $\{u = t\}$. Thus

$$\lambda_1 \varphi_p(u) \neq 0 \qquad \text{in } D^c \cap \{u = t\}.$$

This is only possible when $|D^c \cap \{u = t\}| = 0$, as desired. □

Remark 5. *In a similar way, we can conclude that all level sets $\{u = s\}$ (with $s > 0$) have measure zero, provided that $\alpha \neq \Lambda(\alpha, A)$.*

Remark 6. *According to (3), our optimization of eigenvalue problem is equivalent to finding the smallest eigenvalue and an associated eigenfunction of the problem*

$$\begin{cases} -\Delta_p(u) + \alpha\chi_{\{u \leq t\}}\varphi_p(u) = \lambda\varphi_p(u) & \text{in } \Omega, \\ u = 0 \qquad \text{on } \partial\Omega, \\ |\{u \leq t\}| = A \end{cases}$$

with free variables u and t.

3. STEINER SYMMETRY

In this section we consider the linear case with $p = 2$ and we give another proof of a known result concerning the symmetry of optimal solutions (see [1], Theorem 4). Some ideas of [6] are adopted here.

From now on we shall assume that Ω satisfies the exterior cone condition at each point $x \in \partial\Omega$, which means that there exists a finite right circular cone $V = V_x$ with vertex x such that $\overline{\Omega} \cap V_x = \{x\}$.

Let us recall that a domain G of $\mathbb{R}^n$ is Steiner symmetric with respect to a hyperplane P iff for any point $x = (x_1, \ldots, x_n) \in G$ the segment connecting x and the point x^* reflected with respect to P is contained in G.

The next theorem is a key result of interesting book [4]. Theorem 3.6 of [4] may be stated as follows:

Theorem 7. *Let Ω be bounded, connected and Steiner symmetric relative to the hyperplane $P = \{x = (x_1, x') \colon x_1 = 0\}$. Assume that $u \in C(\overline{\Omega}) \cap C^1(\Omega)$ is a positive weak solution of the boundary value problem*

$$\begin{cases} -\Delta u = f_1(u) + f_2(u) & \text{in } \Omega, \\ u = 0 \qquad \text{on } \partial\Omega, \end{cases}$$

where $f_1 \colon [0, \infty) \to \mathbb{R}$ is locally Lipschitz continuous, while $f_2 \colon [0, \infty) \to \mathbb{R}$ is non-decreasing and is identically zero on an interval $[0, h]$ for some $h > 0$. Then

$$u(-x_1, x') = u(x_1, x') \qquad \text{for } (x_1, x') \in \Omega.$$

Moreover,

$$\frac{\partial u}{\partial x_1}(x_1, x') < 0 \qquad \textit{if } (x_1, x') \in \Omega \textit{and } x_1 > 0.$$

A proof of Theorem 7 is an essential part of book [4]. We are now in a position to prove the following theorem.

Theorem 8. *Let $p = 2$. If the domain Ω is Steiner symmetric with respect to a hyperplane P, then for any optimal pair (u, D) both u and D are symmetric with respect to P, and D^c is Steiner symmetric with respect to P.*

Proof. Without loss of generality, we may assume that

$$P = \{x = (x_1, x')\colon x_1 = 0\}.$$

Let (u, D) be an optimal solution. It follows from assertion (a) of Theorem 4 that $u \in C^1(\Omega)$. Next, the assumption that Ω satisfies the exterior cone condition at each point of $\partial\Omega$ yields $u \in C(\overline{\Omega})$ (see, e. g., [5], Theorem 8.30). By statement (b) of Theorem 4, there exists t such that

$$D = \{u \le t\} = \{u - t \le 0\}.$$

Suppose that $t > 0$. Using the Heaviside function $H\colon \mathbb{R} \to \mathbb{R}$ defined by

$$H(s) := \begin{cases} 0 & \text{if } s < 0, \\ 1 & \text{if } s \ge 0, \end{cases}$$

we observe that

$$\chi_D = H(t - u) \qquad \text{in } \Omega.$$

Since

$$\begin{cases} -\Delta u + \alpha\chi_D u = \lambda_1 u & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $\lambda_1 = \Lambda(\alpha, A)$, we see that the eigenfunction u is a weak solution of the problem

$$\begin{cases} -\Delta u = \lambda_1 u - \alpha H(t - u)u & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases}$$

An application of Theorem 7 with

$$f_1(u) = (\lambda_1 - \alpha)u, \qquad f_2(u) = \alpha\,(1 - H(t - u))\,u$$

and $h = t$ gives the desired result. In the case of $t = 0$, corresponding to the assumption that $A = 0$, there is $|D| = 0$ and thus

$$\begin{cases} -\Delta u = \lambda_1 u & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

so that Theorem 7 may be applied again. This completes the proof. □

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