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Dedicated to the memory of Andrzej Lasota

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REDUCTION AND CONTINUATION THEOREMS FOR BROUWER DEGREE AND APPLICATIONS TO NONLINEAR DIFFERENCE EQUATIONS

Abstract. The aim of this note is to describe the continuation theorem of [39,40] directly in the context of Brouwer degree, providing in this way a simple frame for multiple applications to nonlinear difference equations, and to show how the corresponding reduction property can be seen as an extension of the well-known reduction formula of Leray and Schauder [24], which is fundamental for their construction of Leray-Schauder's degree in normed vector spaces.

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1. INTRODUCTION

A continuation theorem introduced in [39] and developed in [40] in the frame of a degree theory for mappings of the type $L + N$ between normed vector spaces, with L Fredholm of index zero and N satisfying a suitable compactness property, has been often used, since 2000 in [57], for the study of various boundary value problems or periodic solutions of nonlinear difference equations (see e.g. [1–6, 9, 10, 12–23, 25–37, 43–56,58]). A fundamental result in proving this continuation theorem is the reduction of the Leray-Schauder degree of some compact perturbation of identity in a normed vector space to the Brouwer degree of the associated mapping in a finite-dimensional vector space (reduction property).

For nonlinear difference equations, numerous problems are reduced to proving the existence of a zero for a continuous mapping of a finite-dimensional vector space into a vector space of the same finite dimension, so that Brouwer degree applies directly. But the applications mentioned above show that the methodology of the continuation theorem in [39,40] remains fruitful, because it reduces the computation of the Brouwer degree of a mapping between spaces of finite but possibly large dimension to that of a related mapping between spaces of a much smaller dimension.

The aim of this note is to describe the continuation theorem of [39, 40] directly in the context of Brouwer degree, providing in this way a simple frame for many applications to nonlinear difference equations, and to show how the corresponding reduction property can be seen as an extension of the well-known reduction formula of Leray and Schauder [24], which is fundamental for their construction of Leray-Schauder's degree in normed vector spaces.

We only assume that the reader is familiar with the notion of Brouwer degree $d_B[f, D, z]$ of a continuous mapping f from the closure $\overline{D}$ of an open bounded set $D \subset \mathbb{R}^n$ into $\mathbb{R}^n$, such that $z \in \mathbb{R}^n \setminus f(\partial D)$, as well as with its fundamental properties. Among numerous others (see, e.g., [11, 38]), a simple approach can be found in [41].

2. BROUWER DEGREE FOR MAPPINGS BETWEEN FINITE-DIMENSIONAL TOPOLOGICAL VECTOR SPACES

To formulate our reduction and continuation theorems, it is convenient to recall the easy extension of Brouwer degree to continuous mappings between two oriented topological vector spaces of the same finite dimension.

Let A, B : $\mathbb{R}^n \to \mathbb{R}^n$ be linear isomorphisms, $D \subset \mathbb{R}^n$ an open bounded set, $g : \overline{D} \to \mathbb{R}^n$ continuous and $z \in \mathbb{R}^n \setminus Ag(\partial D)$. Then the mapping $f := A \circ g \circ B$ is continuous on $B^{-1}(\overline{D}) = \overline{B^{-1}(D)}$ and $z \in \mathbb{R}^n \setminus f(\partial B^{-1}(D)) = f(B^{-1}(\partial D))$, which is equivalent to $A^{-1}z \in \mathbb{R}^n \setminus g(\partial D)$. Consequently, both $d_B[f, B^{-1}(D), z]$ and $d_B[g, D, A^{-1}z]$ are defined. The following lemma relates those two Brouwer degrees, and we give a proof of this standard result for the reader's convenience.

Lemma 2.1. *Under the above assumptions,*

$$d_B[A \circ g \circ B, B^{-1}(D), z] = [sign\ det\ (AB)] \cdot d_B[g, D, A^{-1}z]. \tag{1}$$

Proof. From the definition of Brouwer degree for continuous mappings, Sard's lemma and the Weierstrass approximation theorem, without loss of generality, we may assume, that g is of class C^2 on D and that z is a regular value for $A \circ g \circ B$. Hence, $d_B[A \circ g \circ B, B^{-1}(D), z] = (\text{sign det } AB) \cdot d_B[g, D, A^{-1}z]$. □

Let X be an n-dimensional real topological vector space. It is well known that if $(\alpha^1, \cdots, \alpha^n)$ is a base in X, and $(e^1, \cdots, e^n)$ the canonical base in $\mathbb{R}^n$, the linear mapping

$$h : X \to \mathbb{R}^n, x = \sum_{j=1}^{n} x_j \alpha^j \mapsto h(x) = \sum_{j=1}^{n} x_j e^j$$

is a homeomorphism.

Let now $D \subset X$ be open and bounded, $f : \overline{D} \to X$ continuous and $z \in X \setminus f(\partial D)$. Then $h \circ f \circ h^{-1}$ is such a continuous mapping from the closure of the open bounded set $h(D) \subset \mathbb{R}^n$ to $\mathbb{R}^n$ that $h(z) \in \mathbb{R}^n \setminus h \circ f \circ h^{-1}(\partial h(D))$. Consequently, $d_B[h \circ f \circ h^{-1}, h(D), h(z)]$ is well defined. If now $(\beta_1, \cdots, \beta_n)$ is another base in X, and if

$$g : X \to \mathbb{R}^n, x = \sum_{j=1}^{n} x_j \beta^j \mapsto h(x) = \sum_{j=1}^{n} x_j e^j,$$

is the corresponding linear homeomorphism, then $d_B[g \circ f \circ g^{-1}, g(D), g(z)]$ is well defined. Now

$$g \circ f \circ g^{-1} = g \circ h^{-1} \circ h \circ f \circ h^{-1} \circ h \circ g^{-1},$$

so that, if we set $m = h \circ g^{-1}$, $m : \mathbb{R}^n \to \mathbb{R}^n$ is a linear homeomorphism and

$$g \circ f \circ g^{-1} = m^{-1} \circ (h \circ f \circ h^{-1}) \circ m.$$

We can therefore apply Lemma 2.1 to obtain

$$d_B[g \circ f \circ g^{-1}, g(D), g(z)] = d_B[h \circ f \circ h^{-1}, h(D), h(z)]. \tag{2}$$

This independence of the Brouwer degree with respect to the choice of the base justifies the following definition.

Definition 2.1. *Let X be a n-dimensional topological vector space, $D \subset X$ open and bounded, $f : \overline{D} \to X$ continuous and $z \in X \setminus f(\partial D)$. The* ***Brouwer degree*** $d_B[f, D, z]$ *is defined by the formula*

$$d_B[f, D, z] = d_B[h \circ f \circ h^{-1}, h(D), h(z)]$$

where

$$h : X \to \mathbb{R}^n, x = \sum_{j=1}^{n} x_j \alpha_j \mapsto \sum_{j=1}^{n} x_j e^j$$

is the linear homeomorphism associated with a base $(\alpha^1, \cdots, \alpha^n)$ of X and the canonical base $(e^1, \cdots, e^n)$ of $\mathbb{R}^n$.

From this definition, it is easy to show, that the degree in space X has all the properties of degree in $\mathbb{R}^n$.

Suppose now that X and Z are two n-dimensional topological vector spaces, $D \subset X$ is open and bounded, $f : \overline{D} \to Z$ is continuous and $z \in Z \setminus f(\partial D)$. Choosing bases $(\alpha^1, \cdots, \alpha^n)$ and $(\beta^1, \cdots, \beta^n)$ in X and Z respectively, and denoting by $h : X \to \mathbb{R}^n$ and $g : Z \to \mathbb{R}^n$ linear homeomorphisms constructed as above, we see that the Brouwer degree $d_B[g \circ f \circ h^{-1}, h(D), g(z)]$ is well defined. If we change bases, i.e., homemorphisms, then, with $\widetilde{h} : X \to \mathbb{R}^n$ and $\widetilde{g} : Z \to \mathbb{R}^n$,

$$g^{-1} \circ g \circ f \circ h^{-1} \circ h = f = \widetilde{g}^{-1} \circ \widetilde{g} \circ f \circ \widetilde{h}^{-1} \circ \widetilde{h},$$

and hence

$$\widetilde{g} \circ f \circ \widetilde{h}^{-1} = m \circ g \circ f \circ h^{-1} \circ \widetilde{m},$$

where $m := \widetilde{g} \circ g^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ and $\widetilde{m} := h \circ \widetilde{h}^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ are linear homeomorphisms. Then, by Lemma 2.1, as above, we obtain,

$$\begin{aligned} & d_B[\widetilde{g} \circ f \circ \widetilde{h}^{-1}, \widetilde{h}(D), \widetilde{h}(z)] = \\ = \quad & \text{sign } (\det m \cdot \det \widetilde{m}) \cdot d_B[g \circ f \circ h, h(D), h(z)], \end{aligned}$$

and this relation can be interpreted as defining a Brouwer degree for f between the *oriented* topological vector spaces X and Z.

The Brouwer index can be extended to this more general situation. Let X, Z be two n-dimensional topological vector spaces, which we suppose oriented if they are different. Let $D \subset X$ be an open bounded set.

Definition 2.2. *Let $f : \overline{D} \to Z$ be continuous, $z \in Z$, and y be an isolated element of $D \cap f^{-1}(z)$. The **Brouwer index of f at** y is defined by*

$$i_B[f, y] = d_B[f, B(y, r), z] = d_B[f, B(y, r), f(y)], \tag{3}$$

where $r > 0$ is such that $\{y\} = B(y, r) \cap f^{-1}(z)$.

It easily follows from the excision property of degree that the right-hand member of formula (3) does not depend upon the choice of r.

Example 2.1. *If $L : X \to Z$ is linear and invertible, then*

$$i_B[L, 0] = \textit{sign det } gLh^{-1}, \tag{4}$$

where $h : X \to \mathbb{R}^n$ and $g : Z \to \mathbb{R}^n$ are linear homeomorphisms of the type introduced above.

3. REDUCTION FORMULAS

Suppose now that X is an n-dimensional topological vector space, $Y \subset X$ a vector subspace of dimension $m < n$, $D \subset X$ is open and bounded, $c : \overline{D} \to Y$ is continuous and $z \in Y \setminus c(\partial D)$.

Any solution of equation $x - c(x) = z$ is such that $x = c(x) + z \in Y$, and hence a relation could be expected between $d_B[I - c, D, z]$ and $d_B[(I - c)|_Y, D \cap Y, z]$. This is the conclusion of the **first reduction formula** due to Leray and Schauder [24]. See [11] or [38] for a proof.

Lemma 3.1. *Let X be an n-dimensional topological vector space, $Y \subset X$ a vector subspace of dimension $1 \le m < n$, $D \subset X$ be open and bounded, $c : \overline{D} \to Y$ continuous and $z \in Y \setminus c(\partial D)$. Then*

$$d_B[I - c, D, z] = d_B[(I - c)|_Y, D \cap Y, z]. \tag{5}$$

From Lemma 3.1, we deduce the **second reduction formula**, proved in a more general setting in [40].

Theorem 3.1. *Let X and Z be n-dimensional topological vector spaces, $L : X \to Z$ a linear mapping with $N(L) \neq \{0\}$, $Y \subset Z$ a vector subspace such that $Z = Y \oplus R(L)$, $D \subset X$ an open bounded set, $r : \overline{D} \to Y$ a continuous mapping such that $0 \notin (L + r)(\partial D)$. Then, for each isomorphism $J : N(L) \to Y$, and each projector $P : X \to X$ such that $R(P) = N(L)$, there is*

$$d_B[L + r, D, 0] = i_B[L + JP, 0] \cdot d_B[J^{-1}r|_{N(L)}, D \cap N(L), 0]. \tag{6}$$

Proof. Let $Q : Z \to Z$ be the projector such that $R(Q) = Y$ and $N(Q) = R(L)$. We first notice that if $(L + JP)x = 0$, then, by applying Q and $I - Q$ to the equation, we obtain the equivalent system

$$Lx = 0, \quad JPx = 0$$

which immediately implies that $x = 0$. Thus $L + JP : X \to Z$, one-to-one, is onto and $i_B[L + JP, 0]$ is well defined and has the absolute value one. Furthermore, if $z \in Y$, then, again by projecting on Y and $R(L)$, one gets

$$(L + JP)(x) = z \Leftrightarrow Lx = 0,\ JPx = z \Leftrightarrow x = Px = J^{-1}z,$$

so that

$$(L + JP)^{-1}z = J^{-1}z.$$

Consequently,

$$\begin{aligned} L + r = L + JP + r - JP &= (L + JP)[I + (L + JP)^{-1}(r - JP)] = \\ &= (L + JP)(I - P + J^{-1}r). \end{aligned}$$

Using Lemmas 2.1 and 3.1, and the definitions above, we get

$$\begin{aligned} &d_B[L + r, D, 0] = d_B[g \circ (L + r) \circ h^{-1}, h(D), 0] = \\ &= d_B[g \circ (L + JP) \circ h^{-1} \circ h \circ (I - P + J^{-1}r) \circ h^{-1}, h(D), 0] = \\ &= \text{sign det } [g \circ (L + JP) \circ h^{-1}] \cdot d_B[h \circ (I - P + J^{-1}r) \circ h^{-1}, h(D), 0] = \\ &= i_B[L + JP, 0] \cdot d_B[I - P + J^{-1}r, D, 0] = \\ &= i_B[L + JP, 0] \cdot d_B[(I - P + J^{-1}r)|_{N(L)}, D \cap N(L), 0] = \\ &= i_B[L + JP, 0] \cdot d_B[J^{-1}r|_{N(L)}, D \cap N(L), 0]. \end{aligned}$$

□

Remark 3.1. *Formula* (6) *implies in particular that*

$$|d_B[L + r, D, 0]| = \left|d_B[r|_{N(L)}, D \cap N(L), 0]\right|.$$

Remark 3.2. *In the special case of*

$$X = Z = N(L) \oplus R(L),$$

(which is in particular the case when L is symmetrical), one can take $Q = P$ and $J = I$, and formula (6) *becomes*

$$d_B[L + r, D, 0] = \textit{sign det}(L|_{R(L)}) \cdot d_B[r|_{N(L)}, D \cap N(L), 0]. \tag{7}$$

Remark 3.3. *We can also see how the Leray-Schauder reduction formula follows from the second reduction formula. With the notations of Lemma 3.1, let* $Q : X \to Y$ *be a projector and write the equation* $x - c(x) = z$ *as*

$$(I - Q)x + Qx - c(x) - z = 0, \tag{8}$$

which has the form

$$Lx + r(x) = 0,$$

if we set

$$L = I - Q, \quad r(\cdot) = Q - c(\cdot) - z.$$

Now, trivially

$$r : \overline{D} \to Y, \quad Y = N(L), \quad X = Y \oplus R(L) = N(L) \oplus R(L),$$

so that formula (7) *gives*

$$\begin{aligned} d_B[I - c, D, z] &= d_B[I - c - z, D, 0] = \\ &= \mathit{sign}\ \mathit{det}\ \left[(I - Q)|_{R(I-Q)}\right] \cdot d_B[(Q - c - z)|_Y, D \cap Y, 0] = \\ &= d_B[(I - c)|_Y, D \cap Y, z]. \end{aligned}$$

4. CONTINUATION THEOREMS

The following finite-dimensional version of **Leray-Schauder's continuation theorem** [24] is a consequence of Brouwer degree theory and Whyburn's lemma.

Let X and Z be topological vector spaces of the same finite dimension n.

Lemma 4.1. *Let* $\mathcal{D} \subset X \times [a, b]$ *be open and bounded,* $\mathcal{F} \in C(\overline{\mathcal{D}}, Z)$ *and the following conditions hold:*

(a) $z \in Z \setminus \mathcal{F}(\partial\mathcal{D})$,
(b) $d_B[\mathcal{F}(\cdot, \overline{\lambda}), \mathcal{D}_{\overline{\lambda}}, z] \neq 0$ *for some* $\overline{\lambda} \in [a, b]$, *where*

$$(\mathcal{F}^{-1}(z))_\lambda = \{x \in X : (x, \lambda) \in \mathcal{F}^{-1}(z)\}.$$

Then there exists a compact connected component $\mathcal{C}$ *of* $\mathcal{F}^{-1}(z)$ *along which* λ *takes all values in* $[a, b]$.

Lemma 4.1 implies the following **continuation theorem for semilinear equations.**

Theorem 4.1. *Let* $L : X \to Z$ *be a linear mapping,* Y *a direct summand of* $R(L)$ *in* Z, $\mathcal{D} \subset X \times [0, 1]$ *an open bounded set, and* $\mathcal{N} : \overline{\mathcal{D}} \to Z$ *a continuous mapping. Assume that the following conditions hold:*

1. $\mathcal{N}(\mathcal{D}_0 \times \{0\}) \subset Y$.
2. $Lx + \mathcal{N}(x, \lambda) \neq 0$ *for each* $(x, \lambda) \in \partial\mathcal{D}$.

3. $0 \in \mathcal{D}_0$ *or* $d_B[\mathcal{N}(\cdot,0)|_{N(L)}, \mathcal{D}_0 \cap N(L), 0] \neq 0$, *according to whether* $N(L) = \{0\}$ *or* $N(L) \neq \{0\}$, *respectively.*

Then

$$\mathcal{S} := \{(x,\lambda) \in \overline{\mathcal{D}} : Lx + \mathcal{N}(x,\lambda) = 0\}.$$

contains a compact connected component $\mathcal{C}$ *along which* λ *takes all values in* $[0,1]$. *In particular, the equation*

$$Lx + \mathcal{N}(x,1) = 0$$

has at least one solution in $\mathcal{D}_1$.

Proof. The mapping $\mathcal{F} : \overline{\mathcal{D}} \to Z$ defined by $\mathcal{F}(x,\lambda) = Lx + \mathcal{N}(x,\lambda)$ satisfies Assumption (a) of Lemma 4.1 with $z = 0$. Now, if $N(L) \neq \{0\}$, Assumptions 2 and 3, and Theorem 3.1 imply that

$$d_B[L + \mathcal{N}(\cdot,0), \mathcal{D}_0, 0] = d_B[\mathcal{N}(\cdot,0)|_{N(L)}, \mathcal{D}_0 \cap N(L), 0] \neq 0,$$

so that Assumption (b) of Lemma 4.1 holds with $\overline{\lambda} = 0$. If $N(L) = \{0\}$, Assumption 1 implies that $\mathcal{N}(\cdot,0) = 0$, and hence

$$d_B[L + \mathcal{N}(\cdot,0), \mathcal{D}_0, 0] = d_B[L, \mathcal{D}_0, 0] = \text{sign det } L,$$

so that Assumption (b) of Lemma 4.1 holds, again. □

5. SEMILINEAR EQUATIONS HOMOTOPIC TO LINEAR ONES

Some special cases of Theorem 4.1, obtained from homotopies to linear mappings, are of interest.

Corollary 5.1. *Let* $L : X \to Z$ *be a linear mapping,* $\mathcal{D} \subset X \times [0,1]$ *an open bounded set, and* $N : \overline{\mathcal{D}} \to Z$ *a continuous mapping. Assume that there exists a linear* $A : X \to Z$ *such that the following conditions hold:*

(i) $N(L + A) = \{0\}$.
(ii) $Lx + (1-\lambda)Ax + \lambda N(x) \neq 0$ *for each* $(x,\lambda) \in \partial\mathcal{D}$.
(iii) $0 \in \mathcal{D}_0$.

Then

$$\mathcal{S}_A = \{(x,\lambda) \in \overline{\mathcal{D}} : Lx + (1-\lambda)Ax + \lambda N(x) = 0\}$$

contains a compact connected component $\mathcal{C}_A$ *along which* λ *takes all values in* $[0,1]$. *In particular, the equation*

$$Lx + N(x) = 0 \tag{9}$$

has at least one solution in $\mathcal{D}_1$.

Proof. This follows immediately from Theorem 4.1 with L replaced by $L + A$ and $\mathcal{N}(x, \lambda) = \lambda[N(x) - Ax]$. $\square$

A useful special case of Corollary 5.1 goes as follows.

Corollary 5.2. *Let $L : X \to Z$ be a linear mapping, $P : X \to X$, $Q : Z \to Z$ projectors such that*

$$R(P) = N(L), \quad N(Q) = R(L), \tag{10}$$

$J : N(L) \to R(Q)$ an isomorphism, $\mathcal{D} \subset X \times [0, 1]$ an open bounded set, and $N : \overline{\mathcal{D}} \to Z$ a continuous mapping. Assume that the following conditions hold:

(A) $Lx + (1 - \lambda)JPx + \lambda N(x) \neq 0$ *for each* $(x, \lambda) \in \partial\mathcal{D}$.
(B) $0 \in \mathcal{D}_0$.

Then

$$\mathcal{S}_{JP} = \{(x, \lambda) \in \overline{\mathcal{D}} : Lx + (1 - \lambda)JPx + \lambda N(x) = 0\}$$

contains a compact connected component $\mathcal{C}_{JP}$ along which λ takes all values in $[0, 1]$. *In particular, equation* (9) *has at least one solution in $\mathcal{D}_1$.*

Proof. There is

$$\begin{aligned}(L + JP)x = 0 \quad &\Leftrightarrow \quad Lx = 0,\ JPx = 0 \ \Leftrightarrow\ x \in N(L),\ Px = 0 \\ &\Leftrightarrow \quad x = 0,\end{aligned}$$

and the result follows from Corollary 5.1 with $A = JP$. $\square$

6. AN APPLICATION TO SECOND ORDER FUNCTIONAL DIFFERENCE EQUATIONS

Following [34], we consider the existence of periodic solutions of the second order nonlinear functional difference equation

$$\Delta^2 x(n - 1) = f(n, x(n), x(n - \tau_1(n)), \dots, x(n - \tau_m(n))) \quad (n \in \mathbb{Z}) \tag{11}$$

where $\tau_j : \mathbb{Z} \to \mathbb{Z}$ is T-periodic for some integer $T \geq 1$ and $j = 1, \dots, m$,

$$f : \mathbb{Z} \times \mathbb{R}^{m+1} \to \mathbb{R}, (n, x_0, x_1, \dots, x_m) \mapsto f(n, x_0, x_1, \dots, x_m)$$

is T-periodic with respect to n for each $x_0, x_1, \dots, x_m \in \mathbb{R}^{m+1}$ and continuous with respect to $(x_0, x_1, \dots, x_m)$ for each $n \in \mathbb{Z}$, and

$$\Delta^2 x(n - 1) = \Delta[x(n) - x(n - 1)] = x(n + 1) - 2x(n) + x(n - 1) \quad (n \in \mathbb{Z}).$$

The vector space

$$X := \{x : \mathbb{Z} \to \mathbb{R} : x(n + T) = x(n) \ for\ all\ n \in \mathbb{Z}\}$$

has the finite dimension T and will be endowed with the Hölder norm

$$\|x\|_r := \left(\sum_{n=1}^{T} |x(n)|^r \right)^{\frac{1}{r}} \tag{12}$$

for some $r \geq 1$. We also use the maximum norm $\|x\|_\infty = \max_{1 \leq n \leq T} |x(n)|$. The following result improves, in several directions and with a much simpler proof, a theorem of Yuji Liu [34].

Theorem 6.1. *Assume that $f = g + h$, where $g, h : \mathbb{Z} \times \mathbb{R}^{m+1} \to \mathbb{R}$ have the same periodicity and regularity properties as f and verify the following conditions:*

1. *There exists $\beta > 0$ and $\theta \geq 1$ such that*

$$g(n, x_0, x_1, \ldots, x_m)x_0 \geq \beta |x_0|^{\theta+1} \tag{13}$$

for all $n \in \mathbb{Z}$ and $(x_0, x_1, \ldots, x_m) \in \mathbb{R}^{m+1}$.

2. *There exist T-periodic mappings $p_i : \mathbb{Z} \to \mathbb{R}^+$ $(1 \leq i \leq m)$ and $r : \mathbb{Z} \to \mathbb{R}^+$ such that*

$$|h(n, x_0, x_1, \ldots, x_m)| \leq \sum_{i=0}^{m} p_i(n)|x_i|^\theta + r(n) \tag{14}$$

for all $n \in \mathbb{Z}$ and $(x_0, x_1, \ldots, x_m) \in \mathbb{R}^{m+1}$.

3.

$$\|p_0\|_\infty + T \sum_{i=1}^{m} \|p_i\|_\infty < \beta. \tag{15}$$

Then equation (11) has at least one T-periodic solutions.

Proof. Let

$$\begin{aligned}
L &: X \to X, (x(n))_{n\in\mathbb{Z}} \mapsto (\Delta^2 x(n-1))_{n\in\mathbb{Z}}, \\
A &: X \to X, (x(n))_{n\in\mathbb{Z}} \mapsto -(x(n))_{n\in Z}, \\
N &: X \to X, (x(n))_{n\in\mathbb{Z}} \mapsto (f(n, x(n), x(n-\tau_1(n)), \ldots, x(n-\tau_m(n))))_{n\in Z}, \\
\mathcal{F} &: X \times [0,1] \to X, (x, \lambda) \mapsto Lx + (1-\lambda)Ax + \lambda N(x)).
\end{aligned}$$

Let $\lambda \in [0,1]$ and $x = (x(n))_{n\in\mathbb{Z}}$ be a possible zero of $\mathcal{F}(\cdot, \lambda)$. Then,

$$\begin{aligned}
0 &= \sum_{n=1}^{T} x(n)\Delta^2 x(n-1) - (1-\lambda)\sum_{n=1}^{T} x(n)^2 - \\
&- \lambda \sum_{n=1}^{T} x(n) f(n, x(n), x(n-\tau_1(n)), \ldots, x(n-\tau_m(n))).
\end{aligned} \tag{16}$$

Now,

$$
\begin{aligned}
&2\sum_{n=1}^{T} x(n)\Delta^2 x(n-1) = \\
&= 2\sum_{n=1}^{T} x(n)\Delta x(n) - 2\sum_{n=1}^{T} x(n)\Delta x(n-1) = \\
&= 2\sum_{n=1}^{T}[x(n)x(n+1) - x(n)^2] - 2\sum_{n=1}^{T}[x(n)^2 - x(n)x(n-1)] = \\
&= \sum_{n=1}^{T}\{-[x(n+1)-x(n)]^2 + x(n+1)^2 + x(n)^2 - 2x(n)^2\} + \\
&\quad + \sum_{n=1}^{T}\{-[x(n)-x(n-1)]^2 + x(n)^2 + x(n-1)^2 - 2x(n)^2\} = \\
&= -\sum_{n=1}^{T}[x(n+1)-x(n)]^2 + x(T+1)^2 - x(1)^2 - \\
&\quad - \sum_{n=1}^{T}[x(n)-x(n-1)]^2 + x(0)^2 - x(T)^2.
\end{aligned}
$$

Hence, using the T-periodicity of $x(n)$, we obtain

$$
\sum_{n=1}^{T} x(n)\Delta^2 x(n-1) \leq 0. \tag{17}
$$

Now, if $\mathcal{F}(x,0) = 0$, then, using (17), we get

$$
0 = \sum_{n=1}^{T} x(n)[\Delta^2 x(n-1) - x(n)] \leq -\sum_{n=1}^{T} x(n)^2
$$

and hence $x = 0$. Thus Assumption (i) of Corollary 5.1 holds. Using (17) in (16), we obtain

$$
\lambda \sum_{n=1}^{T} x(n) f(n, x(n), x(n-\tau_1(n)), \ldots, x(n-\tau_m(n))) \leq 0
$$

and hence, using Assumptions 1 and 2, for $\lambda \in]0,1]$, we get

$$
\begin{aligned}
\beta \sum_{n=1}^{T} |x(n)|^{\theta+1} &\leq -\sum_{n=1}^{T} x(n) h(n, x(n), x(n-\tau_1(n)), \ldots, x(n-\tau_m(n))) \leq \\
&\leq \sum_{n=1}^{T} |x(n)||h(n, x(n), x(n-\tau_1(n)), \ldots, x(n-\tau_m(n)))| \leq \qquad (18)\\
&\leq \sum_{n=1}^{T}\left[p_0(n)|x(n)|^{\theta+1} + \sum_{i=1}^{m} p_i(n)|x(n-\tau_i(n))|^{\theta}|x(n)| + r(n)|x(n)|\right].
\end{aligned}
$$

Then, using Hölder's inequality repeatedly, we obtain

$$\begin{aligned}
\beta \sum_{n=1}^{T} |x(n)|^{\theta+1} &\leq \|p_0\|_\infty \sum_{n=1}^{T} |x(n)|^{\theta+1} + \\
&+ \sum_{i=1}^{m} \left(\sum_{n=1}^{T} \left[p_i(n) |x(n-\tau_i(n))|^{\theta} \right]^{\frac{\theta+1}{\theta}} \right)^{\frac{\theta}{\theta+1}} \left(\sum_{n=1}^{T} |x(n)|^{\theta+1} \right)^{\frac{1}{\theta+1}} + \\
&+ \left(\sum_{n=1}^{T} r(n)^{\frac{\theta+1}{\theta}} \right)^{\frac{\theta}{\theta+1}} \left(\sum_{n=1}^{T} |x(n)|^{\theta+1} \right)^{\frac{1}{\theta+1}} \leq \\
&\leq \|p_0\|_\infty \sum_{n=1}^{T} |x(n)|^{\theta+1} + \\
&+ \left(\sum_{i=1}^{m} \|p_i\|_\infty \right) \left(\sum_{n=1}^{T} |x(n-\tau_i(n))|^{\theta+1} \right)^{\frac{\theta}{\theta+1}} \left(\sum_{n=1}^{T} |x(n)|^{\theta+1} \right)^{\frac{1}{\theta+1}} + \\
&+ \left(\sum_{n=1}^{T} r(n)^{\frac{\theta+1}{\theta}} \right)^{\frac{\theta}{\theta+1}} \left(\sum_{n=1}^{T} |x(n)|^{\theta+1} \right)^{\frac{1}{\theta+1}} \leq \\
&\leq \|p_0\|_\infty \sum_{n=1}^{T} |x(n)|^{\theta+1} + \\
&+ \left(\sum_{i=1}^{m} \|p_i\|_\infty \right) T \left(\sum_{n=1}^{T} |x(n)|^{\theta+1} \right)^{\frac{\theta}{\theta+1}} \left(\sum_{n=1}^{T} |x(n)|^{\theta+1} \right)^{\frac{1}{\theta+1}} + \\
&+ \left(\sum_{n=1}^{T} r(n)^{\frac{\theta+1}{\theta}} \right)^{\frac{\theta}{\theta+1}} \left(\sum_{n=1}^{T} |x(n)|^{\theta+1} \right)^{\frac{1}{\theta+1}} = \\
&= \|p_0\|_\infty \sum_{n=1}^{T} |x(n)|^{\theta+1} + T \left(\sum_{i=1}^{m} \|p_i\|_\infty \right) \left(\sum_{n=1}^{T} |x(n)|^{\theta+1} \right) + \\
&+ \left(\sum_{n=1}^{T} r(n)^{\frac{\theta+1}{\theta}} \right)^{\frac{\theta}{\theta+1}} \left(\sum_{n=1}^{T} |x(n)|^{\theta+1} \right)^{\frac{1}{\theta+1}}.
\end{aligned}$$

Therefore, using (15),

$$\left(\sum_{n=1}^{T} |x(\theta))|^{\theta+1} \right)^{\frac{1}{\theta+1}} \leq \frac{\left(\sum_{n=1}^{T} r(n)^{\frac{\theta+1}{\theta}} \right)^{\frac{1}{\theta+1}}}{\left(\beta - \|p_0\|_\infty - T \sum_{i=1}^{m} \|p_i\|_\infty \right)^{\frac{1}{\theta}}} := R_0. \tag{19}$$

Let us take some $R > R_0$ and $\mathcal{D} = B(0,R) \times [0,1]$. Then assumptions (ii) and (iii) of Corollary 5.1 hold and $F(\cdot, 1) = L + N$ has at least one zero in $B(0,R)$. $\square$

Remark 6.1. *Theorem 6.1 improves the result of* [34] *in several ways, by suppressing Assumption (B), allowing* $\theta = 1$, *correcting the last inequality on p. 69, and substantially simplifying the proof by remaining in the frame of Brouwer degree and using the simpler Corollary 5.1 instead of Corollary 7.1.*

7. SEMILINEAR EQUATIONS HOMOTOPIC TO SOME NONLINEAR ONES AND A POINCARÉ-BOHL THEOREM

We can now combine Theorem 3.1 with Theorem 4.1 to obtain another useful continuation theorem.

Corollary 7.1. *Let* $L : X \to Z$ *be a linear noninvertible mapping,* $P : X \to X$, $Q : Z \to Z$ *projectors such that* (10) *holds,* $\mathcal{D} \subset X \times [0,1]$ *an open bounded set, and* $N : \overline{\mathcal{D}} \to Z$ *a continuous mapping. Assume that the following conditions hold:*

(a) $Lx + \lambda N(x) \neq 0$ *for each* $x \in (\partial\mathcal{D})_\lambda$ *and each* $\lambda \in]0,1]$.
(b) $QN(x) \neq 0$ *for each* $x \in (\partial\mathcal{D})_0 \cap N(L)$.
(c) $d_B[QN|_{N(L)}, (\mathcal{D})_0 \cap N(L), 0] \neq 0$.

Then

$$\mathcal{S}_{QN} = \{(x,\lambda) \in \overline{\mathcal{D}} : Lx + \lambda N(x) = 0\}$$

contains a compact connected component $\mathcal{C}_{QN}$ *along which* λ *takes all values in* $[0,1]$. *In particular, equation* (9) *has at least one solution in* $\mathcal{D}_1$.

Proof. Define $\mathcal{N} : \overline{\mathcal{D}} \to Z$ by

$$\mathcal{N}(x,\lambda) = (1-\lambda)QN(x) + \lambda N(x).$$

For $\lambda = 0$, there is

$$\begin{aligned} Lx + \mathcal{N}(x,0) = 0 \quad &\Leftrightarrow \quad Lx + QN(x) = 0 \quad \Leftrightarrow \quad Lx = 0, \quad QN(x) = 0 \\ &\Leftrightarrow \quad x \in N(L), \quad QN(x) = 0. \end{aligned}$$

Consequently, Assumptions (a) and (b) imply that $0 \in Z \setminus \mathcal{F}(\partial\mathcal{D})$. On the other hand, it follows from Theorem 3.1 and Assumption (c) that

$$\begin{aligned} d_B[L + \mathcal{N}(\cdot,0), \mathcal{D}_0, 0] &= d_B[L + QN(\cdot,0), \mathcal{D}_0, 0] = \\ &= \pm d_B[QN(\cdot,0)|_{N(L)}, \mathcal{D}_0 \cap N(L), 0] \neq 0. \end{aligned}$$

The result follows from Theorem 4.1. $\square$

Now recall two classical results. The first one is a version of **Poincaré-Bohl's theorem** [7, 42].

Lemma 7.1. *Let X be a finite-dimensional Hilbert space with the inner product $\langle \cdot, \cdot \rangle$, $\rho > 0$ and $N : \overline{B(\rho)} \subset X \to X$ be continuous and such that*

$$\langle Nx, x \rangle \geq 0 \quad (\textit{resp.}, \quad \langle Nx, x \rangle \leq 0), \quad \textit{whenever} \quad \|x\| = \rho. \tag{20}$$

Then N has at least one zero in $\overline{B(\rho)}$.

The second one is **Brouwer's fixed point theorem** [8].

Lemma 7.2. *Let X be a finite-dimensional normed vector space, $R > 0$ and $N : \overline{B(R)} \subset X \to X$ be continuous and such that*

$$\|N(x)\| \leq R, \quad \textit{whenever} \quad \|x\| \leq R. \tag{21}$$

Then N has at least one fixed point in $\overline{B(R)}$.

We use Theorem 4.1 to obtain a single statement containing and connecting Lemmas 7.1 and 7.2.

Theorem 7.1. *Let X be a normed vector space and Z a Hilbert space of the same finite dimension, $L : X \to Z$ be linear, $P : X \to X$, $Q : Z \to Z$ be projectors such that*

$$N(L) = R(P), \quad R(L) = N(Q),$$

$J : N(L) \to R(Q)$ an isomorphism, and let $\alpha > 0$ be such that

$$\|L(I-P)x\|_Z \geq \alpha \|(I-P)x\|_X \quad \textit{for all} \quad x \in X. \tag{22}$$

Let $\rho > 0,\ R > 0$,

$$D_{\rho,R} := \{x \in X : \|Px\|_X < \rho, \quad \|(I-P)x\|_X < R\},$$

and $N : \overline{D_{\rho,R}} \to Z$ be continuous. Assume that the following conditions hold:

(i) $\|(I-Q)N(x)\|_Z \leq \alpha R$ *for all* $x \in \overline{D_{\rho,R}}$.
(ii) $\langle QN(x), JPx \rangle \geq 0$ *whenever* $\|Px\|_X = \rho$ *and* $\|(I-P)x\|_X \leq R$.

Then $L + N$ has at least one zero in $\overline{D_{\rho,R}}$.

Proof. If $L + N$ has a zero such that $\|Px\|_X \leq \rho$ and $\|(I-P)x\|_X = R$, or has a zero such that $\|Px\|_X = \rho$ and $\|(I-P)x\|_X \leq R$, then the theorem is proved. Then we can assume that

$$Lx + Nx = 0 \quad \Rightarrow \quad \|(I-P)x\|_X < R \quad \text{and} \quad \|Px\|_X < \rho. \tag{23}$$

We apply Corollary 5.2 with $\mathcal{D} = D_{\rho,R} \times [0,1]$. Let $\lambda \in [0,1]$ and $x \in \overline{D_{\rho,R}}$ be a possible zero of $L + (1-\lambda)JPx + \lambda N$. Then

$$L(I-P)x + \lambda(I-Q)N(x) = 0, \tag{24}$$

and

$$(1-\lambda)JPx + \lambda QN(x) = 0. \tag{25}$$

From (24), (22) and Assumption (i), we deduce that, for $\lambda \in [0,1[$

$$\alpha\|(I-P)x\|_X \leq \|L(I-P)x\|_Z = \|\lambda N(x)\|_Z < \alpha R,$$

and hence

$$\|(I-P)x\|_X < R. \tag{26}$$

By (23), we can assume that (26) also holds for $\lambda = 1$. From (25), we obtain

$$(1-\lambda)\|JPx\|_Z^2 + \lambda\langle QN(x), JPx\rangle = 0,$$

and Assumption (ii) and (23) imply that $\|Px\|_X \neq \rho$, so that $\|Px\|_X < \rho$. Consequently, for each $\lambda \in [0,1]$, each possible zero of $L + (1-\lambda)JPx + \lambda N$ belongs to $D_{\rho,R}$, and the result follows from Corollary 5.2. □

Remark 7.1. *If $X = Z$ is a finite-dimensional Hilbert space, $L = 0$, then $P = Q = I$, $D_{\rho,R} = B(\rho)$, condition* (22) *and Assumption (i) are trivially satisfied, and Assumption (ii) becomes condition* (20) *if we choose $J = I$ or $J = -I$. Hence we recover Poincaré-Bohl's theorem.*

Remark 7.2. *If L is invertible, then $P = Q = 0$, $D_{\rho,R} = B(R)$, Assumption (ii) is trivially satisfied, and the remaining assumptions*

$$\|Lx\| \geq \alpha\|x\| \quad \forall\, x \in X, \quad \|N(x)\| \leq \alpha R \quad \forall\, x \in \overline{B(R)}$$

imply the existence of at least one zero of $L + N$ in $\overline{B(R)}$. This is a slight extension of Brouwer fixed point theorem, which refers to $X = Z$ a finite-dimensional normed space, $L = -I$ and $\alpha = 1$.

8. AN APPLICATION TO PLANAR SYSTEMS OF FIRST ORDER DIFFERENCE EQUATIONS OCCURING IN POPULATION DYNAMICS

Let $T \geq 1$ be an integer, $a, b, c, d : \mathbb{Z} \to \mathbb{R}^+$ be T-periodic, non identically zero and let $f, g : \mathbb{R} \to \mathbb{R}_0^+$ be increasing and such that

$$\lim_{s\to-\infty} f(s) = \lim_{s\to-\infty} g(s) = 0, \quad \lim_{s\to+\infty} f(s) = \lim_{s\to+\infty} g(s) = +\infty. \tag{27}$$

We consider the system

$$\begin{aligned} \Delta u(n) &= a(n) - b(n)f(v(n)), \\ \Delta v(n) &= -c(n) + d(n)g(u(n)) \quad (n \in \mathbb{Z}), \end{aligned} \tag{28}$$

which comes from population dynamics (the Lotka-Volterra discrete model when $f(s) = g(s) = \exp s$), and study the existence of its T-periodic solutions.

The vector space of T-periodic mappings $[u, v] : \mathbb{Z} \to \mathbb{R}^2$ has the finite dimension $2T$. We denote by $\overline{e}$ the average of the T-periodic mapping $e : \mathbb{Z} \to \mathbb{R}$ over a single period, namely

$$\overline{e} := \frac{1}{T} \sum_{n=1}^{T} e(n).$$

Our assumptions upon a, b, c, d can be written equivalently as:

$$\overline{a} > 0, \quad \overline{b} > 0, \quad \overline{c} > 0, \quad \overline{d} > 0. \tag{29}$$

Theorem 8.1. *If assumption* (29) *holds, system* (28) *has at least one T-periodic solution.*

Proof. Define $L : X \to X$ and $N : X \to X$, respectively, by

$$\begin{aligned} L[u, v] &= [\Delta u(n), \Delta v(n)]_{n \in \mathbb{Z}}, \\ N[u, v] &= [b(n)f(v(n)) - a(n), -d(n)g(u(n)) + c(n)]_{n \in \mathbb{Z}}, \end{aligned}$$

so that the problem consists in solving equation $L[u, v] + N[u, v] = 0$ in X, to which we apply Corollary 7.1. Let $\lambda \in]0, 1]$ and $[u, v]$ be a possible zero of $L + \lambda N$. Then, summing the equations from 1 to T and using the fact that

$$\sum_{n=1}^{T} \Delta u(n) = u(T+1) - u(1) = 0, \quad \sum_{n=1}^{T} \Delta v(n) = v(T+1) - v(1) = 0,$$

we obtain

$$\sum_{n=1}^{T} b(n)f(v(n)) = T\overline{a}, \quad \sum_{n=1}^{T} d(n)g(u(n)) = T\overline{c}. \tag{30}$$

Hence, if

$$\begin{aligned} u_L &:= \min_{1 \le n \le T} u(n), \ v_L := \min_{1 \le n \le T} v(n), \\ u_M &:= \max_{1 \le n \le T} u(n), \ v_M := \max_{1 \le n \le T} v(n), \end{aligned}$$

from (30) and the increasing character of f and g, we deduce that

$$\overline{b}f(v_L) \le \overline{a}, \quad \overline{d}g(u_L) \le \overline{c}, \quad \overline{b}f(v_M) \ge \overline{a}, \quad \overline{d}g(u_M) \ge \overline{c}$$

and hence

$$v_L \le f^{-1}\left(\overline{a}/\overline{b}\right), \quad u_L \le g^{-1}\left(\overline{c}/\overline{d}\right), \quad v_M \ge f^{-1}\left(\overline{a}/\overline{b}\right), \quad u_M \ge g^{-1}\left(\overline{c}/\overline{d}\right). \tag{31}$$

On the other hand, we deduce from the system and (30), we deduce that

$$\begin{aligned} \sum_{n=1}^{T} |\Delta u(n)| &= \lambda \sum_{n=1}^{T} |a(n) - b(n)f(v(n))| \le \sum_{n=1}^{T} b(n)f(v(n)) + T\overline{a} = 2T\overline{a}, \\ \sum_{n=1}^{T} |\Delta v(n)| &= \lambda \sum_{n=1}^{T} |c(n) - d(n)g(u(n))| \le \sum_{n=1}^{T} d(n)g(u(n)) + T\overline{c} = 2T\overline{c}, \end{aligned}$$

which, together with the inequalities

$$u_M - u_L \le \sum_{n=1}^{T} |\Delta u(n)|, \quad v_M - v_L \le \sum_{n=1}^{T} |\Delta v(n)|,$$

implies that

$$u_M - u_L \le 2T\overline{a}, \quad v_M - v_L \le 2T\overline{c}. \tag{32}$$

Combining (31) with (32), we obtain the estimates

$$\begin{aligned} g^{-1}\left(\overline{c}/\overline{d}\right) - 2T\overline{a} \le u_L \le u_M \le g^{-1}\left(\overline{c}/\overline{d}\right) + 2T\overline{a}, \\ f^{-1}\left(\overline{a}/\overline{b}\right) - 2T\overline{c} \le v_L \le v_M \le f^{-1}\left(\overline{a}/\overline{b}\right) + 2T\overline{c}. \end{aligned}$$

Take

$$\begin{aligned} r_1 < g^{-1}\left(\overline{c}/\overline{d}\right) - 2T\overline{a} \le g^{-1}\left(\overline{c}/\overline{d}\right) + 2T\overline{a} < R_1, \\ r_2 < f^{-1}\left(\overline{a}/\overline{b}\right) - 2T\overline{c} \le f^{-1}\left(\overline{a}/\overline{b}\right) + 2T\overline{c} < R_2, \end{aligned}$$

and consider the open bounded set

$$\Omega := \{[u,v] \in X : r_1 < u(n) < R_1, \quad r_2 < v(n) < R_2 \quad (n \in \mathbb{Z})\}.$$

Hence, any possible zero $[u,v]$ of $L + \lambda N$ ($\lambda \in]0,1]$) belongs to Ω. Now $N(L) \simeq \mathbb{R}^2$ and the mapping $QN : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$QN(x,y) = [\overline{b}f(y) - \overline{a}, -\overline{d}g(x) + \overline{c}].$$

It is easy to see that QN is a one-to-one map of $\mathbb{R}_0^+ \times \mathbb{R}_0^+$ onto itself and its unique zero

$$[x,y] = \left[g^{-1}\left(\overline{c}/\overline{d}\right), f^{-1}\left(\overline{a}/\overline{b}\right)\right]$$

belongs to $\Omega \cap \mathbb{R}^2$. Consequently,

$$d_B[QN, \Omega \cap N(L), 0] = \pm 1$$

and the result follows from Corollary 7.1. □

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