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ITERATION GROUPS, COMMUTING FUNCTIONS AND SIMULTANEOUS SYSTEMS OF LINEAR FUNCTIONAL EQUATIONS

Abstract. Let $(f^t)_{t\in\mathbb{R}}$ be a measurable iteration group on an open interval I. Under some conditions, we prove that the inequalies $g \circ f^a \leq f^a \circ g$ and $g \circ f^b \leq f^b \circ g$ for some $a, b \in \mathbb{R}$ imply that g must belong to the iteration group. Some weak conditions under which two iteration groups have to consist of the same elements are given. An extension theorem of a local solution of a simultaneous system of iterative linear functional equations is presented and applied to prove that, under some conditions, if a function g commutes in a neighbourhood of f with two suitably chosen elements f^a and f^b of an iteration group of f then, in this neighbourhood, g coincides with an element of the iteration group. Some weak conditions ensuring equality of iteration groups are considered.

Keywords: iteration group, commuting functions, functional equation, functional inequalities.

Mathematics Subject Classification: Primary 39B12, 39B72, 26A18

1. INTRODUCTION

Let $I \subset \mathbb{R}$ be an open interval. A family of continuous functions $(f^t)_{t\in\mathbb{R}}$ is said to be an iteration group of a function $f : I \to I$ if $f^1 = f$ and for all $s, t \in \mathbb{R}$, $f^t : I \to I$, and $f^s \circ f^t = f^{s+t}$.

An iteration group is called continuous (measurable) if, for every $x \in I$, the function $R \ni t \to f^t(x)$ is continuous (measurable). The number t in f^t is called the iterative index of f.

According to Zdun's theorem [5], every measurable iteration group $(f^t)_{t\in\mathbb{R}}$ is continuous and there exists a homeomorphic bijection $\varphi : I \to R$, referred to in the sequel as a generator of the iteration group, such that the following representation formula holds true

$$f^t(x) = \varphi^{-1}(\varphi(x) + t), \qquad x \in I, t \in \mathbb{R}. \tag{1.1}$$

It is proved in [3] that if a function $g : I \to I$ is continuous at at least one point and commutes with two elements f^a and f^b of the iteration group $(f^t)_{t\in\mathbb{R}}$ and $\frac{b}{a}$ is irrational, then g must be an element of the iteration group $(f^t)_{t\in\mathbb{R}}$ (cf. Theorem 2). In section 3, applying a recent result on a simultaneous system of linear functional inequalities [1] (cf. Lemma 1 in Section 2), we prove that if, additionally, $a < 0 < b$, then the commutativity of g with f^a and f^b can be replaced by the inequalities $g \circ f^a \le f^a \circ g$ and $g \circ f^b \le f^b \circ g$ (Theorem 3). In this section we also give some weak conditions under which two iteration groups have to consist of the same elements (Theorem 4).

In Section 4, applying Theorem 1 (a new result on an extension of a local solution of a simultaneous system of iterative an linear functional equations) we prove Theorem 5, which is a local version of Theorem 2. Here the commutativity is required in a neighbourhood of a fixed point of f.

The last section begins with a remark that every measurable iteration group $(f^t)_{t\in\mathbb{R}}$ is uniquely determined by any of its orbit $\{f^t(x_0) : t \in \mathbb{R}\}$. Let $(f^t)_{t\in\mathbb{R}}$ and $(g^t)_{t\in\mathbb{R}}$ be iteration groups. Theorem 6 gives simple conditions on the fixed numbers a, b, α, β under which the inequalities

$$f^\alpha \le g^\alpha, \qquad f^b \le g^\beta$$

imply the existence of a $p \in \mathbb{R}$ such that $g^t = f^{pt}$ for all $t \in \mathbb{R}$.

2. AUXILIARY RESULTS ON A SYSTEM OF SIMULTANEOUS FUNCTIONAL EQUATIONS AND INEQUALITIES

In the sequel we shall need the following result on an extension of solutions of a simultaneous system of iterative functional equations.

Theorem 1. *Let $I \subseteq \mathbb{R}$ be an open interval, let $\xi \in \mathbb{R} \cup \{-\infty, +\infty\}$ be one of the endpoints of I, and let $I_\xi \subset I$ be a nonempty open interval with the endpoint ξ. Let the functions $f_1, f_2 : I \to \mathbb{R}$ be continuous, increasing and such that one of the following conditions is fulfilled:*

1. *if ξ is finite then*
$$0 < \frac{f_i(x) - \xi}{x - \xi} < 1, \qquad x \in I, i = 1, 2;$$
2. *if $\xi = -\infty$ then*
$$f_i(x) < x, \qquad x \in I, i = 1, 2;$$
3. *if $\xi = +\infty$ then*
$$f_i(x) > x, \qquad x \in I, i = 1, 2.$$

Let the functions $g_1, g_2, h_1, h_2 : I \to \mathbb{R}$ be such that $g_1(x)g_2(x) \neq 0$ for all $x \in I$.

Suppose that a function $\gamma : I_\xi \to \mathbb{R}$ satisfies the simultaneous system of functional equations
$$\gamma[f_i(x)] = g_i(x)\gamma(x) + h_i(x), \qquad x \in I_\xi,\ i = 1, 2. \tag{2.1}$$

If the functions f_1, f_2 commute, i.e.

$$f_1 \circ f_2 = f_2 \circ f_1, \tag{2.2}$$

and

$$g_1(x)g_2[f_1(x)] = g_2(x)g_1[f_2(x)], \qquad x \in I, \tag{2.3}$$

$$g_1[f_2(x)]h_2(x) + h_1[f_2(x)] = g_2[f_1(x)]h_1(x) + h_2[f_1(x)], \qquad x \in I, \tag{2.4}$$

then there exists exactly one function $\Phi : I \to \mathbb{R}$ such that

$$\Phi\,|I_\xi = \gamma$$

and

$$\Phi[f_i(x)] = g_i(x)\Phi(x) + h_i(x), \qquad x \in I, i = 1, 2.$$

Proof. Assume that the functions f_1 and f_2 satisfy condition 1, that is ξ is finite. Without any loss of generality, we can assume that $I = (0, \infty)$, $\xi = 0$ and $I_\xi = (0, \delta)$ for some $\delta > 0$. Let $\gamma : I_\xi \to \mathbb{R}$ be a solution of system (2.1). For each $i \in \{1, 2\}$ there exists a unique function $\Phi_i : (0, \infty) \to \mathbb{R}$ such that

$$\Phi_i[f_i(x)] = g_i(x)\Phi_i(x) + h_i(x), \qquad x \in (0, \infty),\ i = 1, 2, \tag{2.5}$$

and

$$\Phi_i\,\big|_{(0,a)} = \gamma$$

(cf. M. Kuczma [2], p. 246–247). Let

$$c := \sup\left\{b > 0 : \Phi_1(x) = \Phi_2(x) \quad \text{for all } x \in (0, b)\right\}.$$

Obviously $c \geq \delta$. To prove the theorem it is enough to show that $c = +\infty$. Assume for the contrary that $c < +\infty$. Then

$$\Phi_1(x) = \Phi_2(x), \qquad x \in (0, c). \tag{2.6}$$

The assumptions on f_1 and f_2 imply that there exists a $d > c$ such that

$$f_i(x) < c, \qquad x \in (0, d), i = 1, 2.$$

Hence, as $f_i \circ f_j(x) \in (0, c)$ for $i, j = 1, 2$ and $x \in (0, d)$, applying in turn: (2.5) for $i = 1$; (2.6); (2.5) for $i = 2$; (2.2) and (2.6); (2.5) for $i = 1$; (2.6); (2.5) for $i = 2$; and finally (2.3) and (2.4), we get

$$\begin{aligned}
\Phi_1(x) &= \frac{1}{g_1(x)}\left\{\Phi_1[f_1(x)] - h_1(x)\right\} = \frac{1}{g_1(x)}\left\{\Phi_2[f_1(x)] - h_1(x)\right\} = \\
&= \frac{1}{g_1(x)}\left\{\frac{1}{g_2[f_1(x)]}\left\{\Phi_2[f_2(f_1(x))] - h_2[f_1(x)]\right\} - h_1(x)\right\} = \\
&= \frac{1}{g_1(x)}\left\{\frac{1}{g_2[f_1(x)]}\left\{\Phi_1[f_1(f_2(x))] - h_2[f_1(x)]\right\} - h_1(x)\right\} =
\end{aligned}$$

$$= \frac{g_1[f_2(x)]\Phi_1[f_2(x)] + h_1[f_2(x)] - h_2[f_1(x)] - h_1(x)g_2[f_1(x)]}{g_1(x)g_2[f_1(x)]} =$$
$$= \frac{g_1[f_2(x)]\left\{g_2(x)\Phi_2(x) + h_2(x)\right\} + h_1[f_2(x)] - h_2[f_1(x)] - h_1(x)g_2[f_1(x)]}{g_1(x)g_2[f_1(x)]} =$$
$$= \frac{g_1[f_2(x)]g_2(x)}{g_1(x)g_2[f_1(x)]}\Phi_2(x) +$$
$$+ \frac{g_1[f_2(x)]h_2(x) + h_1[f_2(x)] - h_2[f_1(x)] - h_1(x)g_2[f_1(x)]}{g_1(x)g_2[f_1(x)]} = \Phi_2(x)$$

which contradicts the definition of d. It completes the proof. □

A more general result for system of functional equations of higher orders will be presented in [4]. By $\mathbb{Q}$ we denote the set of all rational numbers. Let us quote the following

Lemma 1 ([1], Theorem 1)**.** *Let $a, b, \alpha, \beta \in \mathbb{R}$ be such that*

$$a < 0 < b, \qquad \frac{b}{a} \notin \mathbb{Q}, \qquad \frac{\alpha}{a} \geq \frac{\beta}{b}.$$

If a function $\gamma : \mathbb{R} \to \mathbb{R}$ continuous at at least one point satisfies the simultaneous system of functional inequalities

$$\gamma(x + a) \leq \gamma(x) + \alpha, \qquad \gamma(x + b) \leq \gamma(x) + \beta, \qquad x \in \mathbb{R},$$

then

$$\gamma(x) = \frac{\alpha}{a}x + \gamma(0), \qquad x \in \mathbb{R}.$$

3. COMMUTING FUNCTIONS

Remark 1. *Functions $\psi, \varphi : I \to \mathbb{R}$ are generators of the same iteration group $(f^t)_{t\in\mathbb{R}}$ iff there exists a constant $a \in \mathbb{R}$ such that $\psi = \varphi + a$.*

In [3] we have proved the following

Theorem 2. *Let $f : I \to I$ be strictly increasing, onto and without fixed points in an open interval I. Suppose that $(f^t)_{t\in\mathbb{R}}$ is a measurable iteration group of f. If a function $g : I \to I$ is continuous at at least one point and commutes with two functions f^a and f^b such that $\frac{b}{a}$ is irrational, then there exists a $c \in \mathbb{R}$ such that $g = f^c$.*

Assuming that $a < 0 < b$, the commutativity assumption may be weakened. Namely, we prove the following

Theorem 3. *Let $f : I \to I$ be strictly increasing, onto and without fixed points in an open interval I. Suppose that $(f^t)_{t\in\mathbb{R}}$ is a measurable iteration group of f. If a function $g : I \to I$ is continuous at at least one point and*

$$g \circ f^a \leq f^a \circ g, \qquad g \circ f^b \leq f^b \circ g,$$

where $a, b \in \mathbb{R}\backslash\{0\}$ are such that $\frac{b}{a}$ is irrational and $a < 0 < b$, then there exists a $c \in \mathbb{R}$ such that $g = f^c$.

Proof. According to Zdun's theorem, $(f^t)_{t\in\mathbb{R}}$ has to be a continuous iteration group (cf. [5]). Consequently, there exists a continuous and strictly monotonic function $\varphi : I \to \mathbb{R}$ with $\varphi(I) = \mathbb{R}$ such that

$$f^t(x) = \varphi^{-1}(\varphi(x) + t), \qquad x \in I,\ t \in \mathbb{R}.$$

Without any loss of generality, we may assume that φ is increasing. By the assumption we hence get

$$g\left(\varphi^{-1}(\varphi(x) + a)\right) \le \varphi^{-1}(\varphi(g(x)) + a), \qquad x \in I,$$

and

$$g\left(\varphi^{-1}(\varphi(x) + b)\right) \le \varphi^{-1}(\varphi(g(x)) + b), \qquad x \in I.$$

It follows that the function $\gamma := \varphi \circ g \circ \varphi^{-1}$ satisfies the simultaneous system of functional inequalities

$$\gamma(t + a) \le \gamma(t) + a, \qquad \gamma(t + b) \le \gamma(t) + b, \qquad t \in \mathbb{R}.$$

Since the function γ is continuous at at least one point, there is a $c \in \mathbb{R}$ such that $\gamma(t) = t + c$ for all $t \in \mathbb{R}$ (cf. Theorem 1 in [1]). Thus $\varphi \circ g \circ \varphi^{-1}(t) = t + c$ for all $t \in \mathbb{R}$, whence $g = \varphi^{-1}(\varphi(x) + c)$ for all $x \in I$ which, by representation formula (RF), means that $g = f^c$. □

Theorem 4. *Let $f, g : I \to I$ be strictly increasing, onto and without fixed points in an open interval I. Suppose that $(f^t)_{t\in\mathbb{R}}$ and $(g^t)_{t\in\mathbb{R}}$ are measurable iteration groups of f and g, respectively. If for some noncommensurable $\alpha, \beta \in \mathbb{R}$, each of the functions g^α and g^β commutes with two functions of the iteration group $(f^t)_{t\in\mathbb{R}}$ of the noncommensurable iterative indices, then*

$$g^t = f^{pt}, \qquad t \in \mathbb{R},$$

for some $p \in \mathbb{R}$, $p \ne 0$; in particular, $\{g^t : t \in \mathbb{R}\} = \{f^t : t \in \mathbb{R}\}$.

Proof. In view of Theorem 1 there are $a, b \in \mathbb{R}$ such that $g^\alpha = f^a$ and $g^\beta = f^b$. Hence, by induction, $g^{m\alpha} = f^{ma}$, $g^{n\beta} = f^{nb}$ for all integers $m, n \in \mathbb{Z}$, whence,

$$g^{m\alpha + n\beta} = f^{ma+nb}, \qquad m, n \in \mathbb{Z}. \tag{3.1}$$

Since, by the representation formula,

$$f^t(x) = \varphi^{-1}(\varphi(x) + t), \qquad g^t(x) = \psi^{-1}(\psi(x) + t), \qquad x \in I, t \in \mathbb{R},$$

for some homeomorphic functions $\varphi, \psi : I \to \mathbb{R}$, we get

$$\psi^{-1}(\psi(x) + m\alpha + n\beta) = \varphi^{-1}(\varphi(x) + ma + nb), \qquad x \in I; n, m \in \mathbb{Z}. \tag{3.2}$$

Take an arbitrary $t \in \mathbb{R}$. The density of set $\{m\alpha + n\beta : m, n \in \mathbb{Z}\}$ in $\mathbb{R}$ implies that there exist two sequences of integers (m_k), (n_k) such that

$$t = \lim_{k\to\infty} (m_k\alpha + n_k\beta). \tag{3.3}$$

The noncommensurabilty of α and β implies that $\alpha\beta \neq 0$ and

$$\lim_{k\to\infty} |m_k| = \infty = \lim_{k\to\infty} |n_k| .$$

Therefore, from (3.3),

$$\lim_{k\to\infty} \left(\frac{m_k}{n_k}\alpha + \beta\right) = \lim_{k\to\infty} \frac{m_k\alpha + n_k\beta}{n_k} = \lim_{k\to\infty} \frac{t}{n_k} = 0,$$

which implies that

$$\lim_{k\to\infty} \frac{m_k}{n_k} = -\frac{\beta}{\alpha}.$$

From (3.2), for an arbitrarily fixed $x \in I$,

$$m_k a + n_k b = \varphi\left[\psi^{-1}\left(\psi(x) + m_k\alpha + n_k\beta\right)\right] - \varphi(x), \qquad k \in \mathbb{N},$$

whence, by the continuity of $\varphi, \varphi^{-1}, \psi, \psi^{-1}$,

$$\lim_{k\to\infty} (m_k a + n_k b) = \varphi\left[\psi^{-1}\left(\psi(x) + t\right)\right] - \varphi(x).$$

Repeating now the above argument we infer that

$$\lim_{k\to\infty} \frac{m_k}{n_k} = -\frac{b}{a},$$

and, consequently $\frac{b}{a} = \frac{\beta}{\alpha}$, whence $\frac{a}{\alpha} = \frac{b}{\beta}$. Setting

$$p := \frac{a}{\alpha} = \frac{b}{\beta}$$

we get

$$a = p\alpha, \qquad b = p\beta,$$

which implies that a and b are not commensurable and, by (3.1),

$$g^{m\alpha+n\beta} = f^{p(m\alpha+n\beta)}, \qquad m, n \in \mathbb{Z},$$

whence, in particular,

$$g^{m_k\alpha+n_k\beta} = f^{p(m_k\alpha+n_k\beta)}, \qquad k \in \mathbb{N}.$$

Letting here $k \to \infty$, we obtain

$$g^t = f^{pt},$$

which completes the proof. □

Applying Theorem 2 and Theorem 3, we obtain

Theorem 5. *Let $f, g : I \to I$ be strictly increasing, onto and without fixed points in an open interval I. Suppose that $(f^t)_{t\in\mathbb{R}}$ and $(g^t)_{t\in\mathbb{R}}$ are measurable iteration groups of f and g, respectively. If for some noncommensurable $\alpha, \beta \in \mathbb{R}$ there are two pairs (a, b) and (c, d) of noncomensurable real numbers such that*

$$a < 0 < b, \qquad g^{\alpha} \circ f^{a} \leq f^{a} \circ g^{\alpha}, \quad g^{\alpha} \circ f^{b} \leq f^{b} \circ g^{\alpha},$$

and

$$c < 0 < d, \qquad g^{\beta} \circ f^{c} \leq f^{c} \circ g^{\beta}, \quad g^{\beta} \circ f^{d} \leq f^{d} \circ g^{\beta},$$

then

$$g^t = f^{pt}, \qquad t \in \mathbb{R},$$

for some $p \in \mathbb{R}$, $p \neq 0$; in particular, $\{g^t : t \in \mathbb{R}\} = \{f^t : t \in \mathbb{R}\}$.

Remark 2. *Note that the function f^{pt} in the previous result can be treated as the p-th iterate of f^t. Indeed, let $\varphi : I \to \mathbb{R}$ be a generator of the iteration group of f. If $n \in \mathbb{N}$, then, by induction,*

$$\left(f^t\right)^{\frac{m}{n}}(x) = \varphi^{-1}\left(\varphi(x) + \frac{m}{n}t\right) = f^{\frac{m}{n}t}, \qquad n \in \mathbb{N}, m \in \mathbb{Z}, t \in \mathbb{R}, x \in I,$$

which means that

$$\left(f^t\right)^{p}(x) = \varphi^{-1}\left(\varphi(x) + pt\right) = f^{pt}, \qquad p \in \mathbb{Q}, t \in \mathbb{R}, x \in I.$$

The continuity of the iteration group and the functions φ and φ^{-1} implies that this relation can be uniquely extended onto all $p \in \mathbb{R}$.

Corollary 1 ([3]). *Let $f, g : I \to I$ be strictly increasing, onto and without fixed points in an open interval I. If the measurable iteration groups $(f^t)_{t\in\mathbb{R}}$ and $(g^t)_{t\in\mathbb{R}}$ of f and g, respectively, are commuting in pairs, i.e.,*

$$f^t \circ g^t = g^t \circ f^t, \qquad t \in \mathbb{R},$$

then, for every $t \in \mathbb{R}$, $g^t = f^{tp}$ for some $p \in \mathbb{R}$, $p \neq 0$.

4. LOCALLY COMMUTING FUNCTIONS

In this section we prove the following local counterpart of Theorem 2.

Theorem 6. *Let $f : I \to I$ be strictly increasing, onto and without fixed points in an open interval I. Suppose that $(f^t)_{t\in\mathbb{R}}$ is a measurable iteration group of f. Let ξ be one of the endpoints of I and $I_\xi \subset I$ be an arbitrary nonempty open interval with the endpoint ξ. If a function $g : I_\xi \to I$ is continuous at at least one point and there are $a, b \in \mathbb{R}$, $a \neq 0$, $\frac{b}{a}$ irrational and such that*

$$f^a(I_\xi) \subset I_\xi, \qquad f^b(I_\xi) \subset I_\xi \tag{4.1}$$

and

$$g \circ f^a = f^a \circ g, \qquad g \circ f^b = f^b \circ g, \tag{4.2}$$

then $g = f^c \big|_{I_\xi}$ *for some* $c \in \mathbb{R}$.

Proof. We shall only consider the case of ξ finite, as in the opposite case the argument is analogous. Without any loss of generality, we may assume that $\xi = 0$ is the left-end point of I and $I_\xi = (0, \delta)$ for some $\delta > 0$. Since

$$f^t(x) = \varphi^{-1}(\varphi(x) + t), \qquad x \in I, t \in \mathbb{R},$$

for a homeomorphic function $\varphi : I \to \mathbb{R}$, then

$$f^a(x) = \varphi^{-1}(\varphi(x) + a), \qquad f^b(x) = \varphi^{-1}(\varphi(x) + b), \qquad x \in I.$$

Assume that, for instance, φ is increasing. Then, by (4.1), both a and b must be negative, and by (4.2),

$$g\left[\varphi^{-1}(\varphi(x) + a)\right] = \varphi^{-1}(\varphi(g(x)) + a), \qquad x \in (0, \delta),$$

$$g\left[\varphi^{-1}(\varphi(x) + b)\right] = \varphi^{-1}(\varphi(g(x)) + b), \qquad x \in (0, \delta).$$

Since $\varphi((0, \delta)) = (-\infty, \varphi(\delta))$, the function $\gamma := \varphi \circ g \circ \varphi^{-1}$ is defined on $(-\infty, \varphi(\delta))$ and satisfies the simultaneous system of functional equations

$$\gamma(t + a) = \gamma(t) + a, \qquad \gamma(t + b) = \gamma(t) + b, \qquad t < \varphi(\delta).$$

Assuming $I = \mathbb{R}$, $\xi = -\infty$, $I_\xi = (-\infty, \varphi(\delta))$, $f_1(t) = t + a$, $f_2(t) = t + b$, $g_1 = g_2 = 1$ it is easy to verify that the conditions of Theorem 1 are fulfilled. Therefore there exists a unique function $\Phi : \mathbb{R} \to \mathbb{R}$ such that $\Phi \big|_{(-\infty, \varphi(\delta))} = \gamma$ and

$$\Phi(t + a) = \Phi(t) + a, \qquad \Phi(t + b) = \Phi(t) + b, \qquad t \in \mathbb{R}.$$

Since the function Φ is continuous at at least one point, applying Theorem 1 of [3], we conclude that, for some $c \in \mathbb{R}$,

$$\Phi(t) = t + c, \qquad t \in \mathbb{R}.$$

It follows that $\gamma(t) := \varphi \circ g \circ \varphi^{-1}(t) = t + c$ for all $t < \varphi(\delta)$, whence, by representation formula (RF)

$$g(x) = \varphi^{-1}(\varphi(x) + c)) = f^c(x), \qquad x \in (0, \delta).$$

This completes the proof. □

5. EQUALITY OF ITERATION GROUPS

We begin this section with the following

Proposition 1. *Let $f, g : I \to I$ be strictly increasing, onto and without fixed points in an open interval I. Suppose that $(f^t)_{t\in\mathbb{R}}$ and $(g^t)_{t\in\mathbb{R}}$ are measurable iteration groups of f and g, respectively. If for some $x_0 \in I$,*

$$f^t(x_0) = g^t(x_0) \quad \text{for all } t \in \mathbb{R},$$

then

$$f^t = g^t \quad \text{for all } t \in \mathbb{R}.$$

Proof. Let $\varphi, \psi : I \to \mathbb{R}$ be the generators of the iteration groups $(f^t)_{t\in\mathbb{R}}$ and $(g^t)_{t\in\mathbb{R}}$, respectively. By the assumption,

$$\varphi^{-1}(\varphi(x_0) + t) = \psi^{-1}(\psi(x_0) + t), \qquad t \in \mathbb{R},$$

whence, after simple calculations, we obtain

$$\psi(x) = \varphi(x) + \psi(x_0) - \varphi(x_0),$$

and Remark 1 completes the proof. □

Remark 3. *According to the above proposition, every measurable iteration group is uniquely determined by any of its orbits $\{f^t(x_0) : t \in \mathbb{R}\}$.*

Theorem 7. *Let $f, g : I \to I$ be strictly increasing, onto and without fixed points in an open interval I. Suppose that $(f^t)_{t\in\mathbb{R}}$ and $(g^t)_{t\in\mathbb{R}}$ are measurable iteration groups of f and g, respectively. If*

$$f^a \le g^\alpha, \qquad f^b \le g^\beta,$$

for some $a, b, \alpha, \beta \in \mathbb{R}$ such that

$$a < 0 < b, \qquad \frac{b}{a} \notin \mathbb{Q}, \qquad \frac{\alpha}{a} \ge \frac{\beta}{b},$$

then

$$g^t = f^{pt}, \qquad t \in \mathbb{R},$$

for some $p \in \mathbb{R}$, $p \ne 0$; in particular, $\{g^t : t \in \mathbb{R}\} = \{f^t : t \in \mathbb{R}\}$.

Proof. Let $\varphi, \psi : I \to \mathbb{R}$ be the generators of the iteration groups $(f^t)_{t\in\mathbb{R}}$ and $(g^t)_{t\in\mathbb{R}}$, respectively. By the assumption,

$$\varphi^{-1}(\varphi(x) + a) \le \psi^{-1}(\psi(x) + \alpha), \qquad x \in I,$$

and

$$\varphi^{-1}(\varphi(x) + b) \le \psi^{-1}(\psi(x) + \beta), \qquad x \in I.$$

Setting $\gamma := \psi \circ \varphi^{-1}$we hence get

$$\gamma(t + a) \le \gamma(t) + \alpha, \qquad \gamma(t + b) \le \gamma(t) + \beta, \qquad t \in \mathbb{R}.$$

Applying Lemma 1 we obtain $\gamma(t) = \frac{1}{p}t + q$ for all $t \in \mathbb{R}$, where $p := \frac{a}{\alpha}$ and $q = \gamma(0)$, that is $\psi \circ \varphi^{-1}(t) = \frac{1}{p}t + q$ $(t \in \mathbb{R})$ whence

$$\psi(x) = p\varphi(x) + q, \qquad t \in \mathbb{R}.$$

Now, by simple calculations, we obtain

$$g^t(x) = \psi^{-1}\left(\psi(x) + t\right) = \varphi^{-1}\left(\varphi(x) + pt\right) = f^{pt}(x)$$

for all $x \in I$ and $t \in \mathbb{R}$. □

If $\alpha = a$ and $\beta = b$ in the above theorem, then $p = 1$ and we obtain the following

Corollary 2. *Let $f, g : I \to I$ be strictly increasing, onto and without fixed points in an open interval I. Suppose that $(f^t)_{t\in\mathbb{R}}$ and $(g^t)_{t\in\mathbb{R}}$ are measurable iteration group of f and g, respectively. If*

$$f^{\alpha} \leq g^a, \qquad f^b \leq g^b,$$

for some $a, b \in \mathbb{R}$ such that

$$a < 0 < b, \qquad \frac{b}{a} \notin \mathbb{Q},$$

then

$$f^t = g^t, \qquad t \in \mathbb{R}.$$

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REFERENCES

[1] D. Krassowska, J. Matkowski, *A pair of linear functional inequalities and a characterization of L^p-norm*, Ann. Polon. Math. **85** (2005) 1, 1–10.

[2] M. Kuczma, *Functional equations in a single variable*, Monografie Mat. **46**, Polish Scientific Publishers, Warsaw, 1968.

[3] J. Matkowski, *Cauchy functional equation on restricted domain and commuting functions*, in Iteration Theory and Its Functional Equations, Proc., Schloss Hofen 1984, Springer Verlag **1163** (1985), 101–106.

[4] J. Matkowski, *On extension of solutions of a simultaneous system of linear functional equations* (in peparation).

[5] M.C. Zdun, *Continuous and differentiable iteration groups*, Prace Matematyczne, Silesian University, 1979.

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