Opuscula Mathematica • Vol. 28 • No. 4 • 2008

Professor Andrzej Lasota in Memoriam

Antoni Leon Dawidowicz, Anna Poskrobko

ON PERIODIC AND STABLE SOLUTIONS OF THE LASOTA EQUATION IN DIFFERENT PHASE SPACES

Abstract. We study properties of the Lasota partial differential equation in two different spaces: V_α (Hölder continuous functions) and L^p. The aim of this paper is to generalize the results of [1].

Keywords: partial differential equations, periodic solutions, stable solutions.

Mathematics Subject Classification: 35B10, 35B35, 37C75, 47D06.

1. INTRODUCTION

We consider the partial differential equation

$$\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = \lambda(x)u, \quad t \geq 0,\ 0 \leq x \leq 1 \tag{1.1}$$

with the initial condition

$$u(0,x) = v(x), \quad 0 \leq x \leq 1, \tag{1.2}$$

where v belongs to some normed vector space V of functions defined on $[0,1]$ and $\lambda : [0,1] \to \mathbb{R}$ is a given continuous function. Let a semidynamical system

$$T_t : V \to V$$

be given by the formula

$$(T_t v)(x) = u(t,x),$$

where u is the solution of (1.1), (1.2). It is clear that this unique solution is given by the formula

$$(T_t v)(x) = u(t,x) = \mathrm{e}^{g(x)}\mathrm{e}^{-g(x\mathrm{e}^{-t})}v(x\mathrm{e}^{-t}), \quad x \in [0,1], \tag{1.3}$$

where

$$g(x) = -\int_x^1 \frac{\lambda(s)}{s} ds$$

with the condition

$$\int_0^1 \frac{\lambda(s)}{s} ds = \infty. \tag{1.4}$$

We wish to investigate some properties of system (1.3): periodic solutions, strong and exponential stability.

Definition 1.1. *A function $v_0 \in V$ is a* periodic point *of the semigroup $(T_t)_{t\geq 0}$, with a period $t_0 \geq 0$ iff $T_{t_0}v_0 = v_0$. A number $t_0 > 0$, is called a* principal period *of a periodic point v_0 iff the set of all periods of v_0 is equal to $\mathbb{N}t_0$.*

Definition 1.2. *The semigroup $(T_t)_{t\geq 0}$ is* strongly stable *in V iff for every $v \in V$,*

$$\lim_{t\to\infty} T_t v = 0 \quad \text{in } V.$$

Definition 1.3. *The semigroup $(T_t)_{t\geq 0}$ is* exponentially stable *iff there exist $D < \infty$ and $\omega > 0$ such that*

$$\|T_t\| \leq De^{-\omega t}, \quad \text{for } t \geq 0.$$

The problem of the chaotic behaviour of a partial differential equation was considered by Lasota [5], Rudnicki [8], Łoskot [7] and Szarek [6]. In the papers [1–4] there were described properties of the partial differential equation, analogical to (1.1), but with a constant function λ:

$$\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = \gamma u, \quad t \geq 0,\ 0 \leq x \leq 1 \tag{1.5}$$

and with the initial condition

$$u(0,x) = v(x), \quad 0 \leq x \leq 1. \tag{1.6}$$

This work has been intended as an attempt at generalizing the results of [1]. In [1] there was described the chaotic and stability behaviour of the suitable semidynamical system

$$(\widetilde{T}_t v)(x) = \widetilde{u}(t,x) = \mathrm{e}^{\gamma t} v(x\mathrm{e}^{-t}), \quad x \in [0,1] \tag{1.7}$$

in different phase spaces V. All properties depended on the value γ. We are interested in finding a connection between this two equations. It is easy to check that if u and $\widetilde{u}$ are the solutions of equation (1.1) and (1.5), respectively, then

$$\widetilde{u}(t,x) = \kappa(x)u(t,x), \tag{1.8}$$

where

$$\kappa(x) = \mathrm{e}^{\int_0^x \frac{\lambda(0)-\lambda(s)}{s} ds} \quad \text{and} \quad \gamma = \lambda(0). \tag{1.9}$$

Hence the diagram

$$\begin{array}{ccc} V & \xrightarrow{T_t} & V \\ {\scriptstyle m_\kappa}\big\downarrow & & \big\downarrow{\scriptstyle m_\kappa} \\ V & \xrightarrow[\widetilde{T}_t]{} & V \end{array}$$

This substitution will be a useful tool. It will be used in the proofs of theorems on chaos and stability of system (1.3) in the spaces V_α and L^p.

2. PROPERTIES OF THE DYNAMICAL SYSTEM $(T_t)_{t\geq 0}$ IN THE SPACE V_α

Let v be a continous function on $[0,1]$ such that $v(0)=0$. For every interval $A \subset [0,1]$ and for every $\alpha \in (0,1]$, define

$$H_{A,\alpha}(v) = \sup_{x,y\in A, x\neq y} \frac{|v(x)-v(y)|}{|x-y|^\alpha}.$$

A function v for which $H_{A,\alpha}(v) < \infty$ is called a Hölder continuous on the interval A with exponent α. Write

$$H_\alpha = H_{[0,1],\alpha}.$$

Definition 2.1. *Denote by V_α the space of all Hölder continuous functions v on $[0,1]$ with exponent α, vanishing at zero and satisfying the following condition*

$$\lim_{x\to 0} H_{[0,x],\alpha}(v) = 0.$$

Certain properties of system (1.7) in the space V_α have been estabilished. For $\gamma > \alpha$, there exist periodic solutions of problem (1.5) and the set of all periodic points is dense in V_α. Strong and exponential stability take place, provided that $\gamma \leq \alpha$ and $\gamma < \alpha$, respectively (see [1] for more details).

Theorem 2.2. *Let*

$$|\lambda(0)-\lambda(x)| \leq Cx, \quad C>0, \quad x\in[0,1] \tag{2.1}$$

hold. Then the function $u \in V_\alpha$ if and only if $\widetilde{u} \in V_\alpha$.

Proof. The assumption $u \in V_\alpha$ means that u is a Hölder continuous function with exponent α, vanishing at zero and $\lim_{a\to 0} H_{[0,a],\alpha}(u) = 0$. This gives

$$\begin{aligned} H_{[0,a],\alpha}(\widetilde{u}) &= \sup_{x,y\in[0,a],x\neq y} \frac{|\widetilde{u}(t,x)-\widetilde{u}(t,y)|}{|x-y|^\alpha} = \\ &= \sup_{x,y\in[0,a],x\neq y} \frac{|\kappa(x)u(t,x)-\kappa(y)u(t,y)|}{|x-y|^\alpha} \leq \\ &\leq \sup_{x\in[0,a]} |\kappa(x)| \cdot \sup_{x,y\in[0,a],x\neq y} \frac{|u(t,x)-u(t,y)|}{|x-y|^\alpha} + \\ &\quad + \sup_{y\in[0,a]} |u(t,y)| \cdot \sup_{x,y\in[0,a],x\neq y} \frac{|\kappa(x)-\kappa(y)|}{|x-y|^\alpha}. \end{aligned}$$

Using (1.9) and (2.1), we obtain

$$|\kappa(x)| \le e^{\int_0^x \frac{|\lambda(0)-\lambda(s)|}{s} ds} \le e^{\int_0^x C ds} = e^{Cx}.$$

It remains to show that the function κ is Hölder continous. If $\kappa^{\frac{1}{\alpha}}$ apears the Lipschitz function, it will mean that κ is Hölder continuous with exponent α. For $x \in [0, a]$

$$|(\kappa^{\frac{1}{\alpha}}(x))'| = \left| e^{\frac{1}{\alpha}\int_0^x \frac{\lambda(0)-\lambda(s)}{s} ds} \cdot \frac{1}{\alpha} \cdot \frac{\lambda(0)-\lambda(x)}{x} \right| \le \left| e^{\frac{1}{\alpha}\int_0^x C ds} \cdot \frac{C}{\alpha} \right| =$$
$$= \left| e^{\frac{Cx}{\alpha}} \cdot \frac{C}{\alpha} \right| \le e^{\frac{Ca}{\alpha}} \cdot \frac{C}{\alpha}.$$

Summarizing,

$$H_{[0,a],\alpha}(\widetilde{u}) \le e^{Ca} \left(H_{[0,a],\alpha}(u) + \sup_{y \in [0,a]} |u(t,y)| \cdot \left(\frac{C}{\alpha}\right)^{\alpha} \right).$$

Therefore, $\lim_{a \to 0} H_{[0,a],\alpha}(\widetilde{u}) = 0$ and finally $\widetilde{u} \in V_\alpha$. The rest of the proof runs as before. We can draw the same conclusion for the funcition u, assuming that $\widetilde{u} \in V_\alpha$. □

Assumption (2.1) will be needed throughout this section and the above theorem will be crucial for next results.

Theorem 2.3. *If $\lambda(0) > \alpha \in (0, 1]$, then for any t_0 there exists such $v_0 \in V_\alpha$ that*

$$T_{t_0} v_0 = v_0 \tag{2.2}$$

and

$$T_t v_0 = v_0 \quad \text{if and only if} \quad t = nt_0 \text{ for some positive integer } n. \tag{2.3}$$

Proof. Let w be an arbitrary Hölder continuous function with the exponent α defined on the interval $[e^{-t_0}, 1]$ and satisfying the following conditions:

$$e^{-g(e^{-t_0})} w(e^{-t_0}) = w(1), \tag{2.4}$$

$$\forall t \in (0, t_0) \quad e^{-g(e^{-t})} w(e^{-t}) \ne w(1). \tag{2.5}$$

Consider the following function v on the interval $(0, 1]$:

$$v(x) = e^{g(x)} e^{-g(xe^{nt_0})} w(xe^{nt_0}) \quad \text{for} \quad x \in [e^{-(n+1)t_0}, e^{-nt_0}].$$

The function v is defined on the whole interval $(0, 1] = \bigcup_{n=0}^{\infty} (e^{-(n+1)t_0}, e^{-nt_0}]$ and come into being by squeezing the graph of the function w into each interval $(e^{-(n+1)t_0}, e^{-nt_0}]$.
By the assumption of the continuity of w on $[e^{-t_0}, 1]$, its boundedness follows, i.e.,

there is $M > 0$ such that $|w(x)| \leq M$ for each $x \in [\mathrm{e}^{-t_0}, 1]$. By the above, for $x \in [\mathrm{e}^{-(n+1)t_0}, \mathrm{e}^{-nt_0}]$, the following estimate holds:

$$|v(x)| = \mathrm{e}^{g(x)}\mathrm{e}^{-g(x\mathrm{e}^{nt_0})}|w(x\mathrm{e}^{nt_0})| \leq M\mathrm{e}^{g(x)} \cdot \sup_{x\in[\mathrm{e}^{-t_0},1]} \mathrm{e}^{-g(x)} \leq M_1\mathrm{e}^{g(x)},$$

where $M_1 = M \cdot \sup_{x\in[\mathrm{e}^{-t_0},1]} \mathrm{e}^{-g(x)}$. From assumption (1.4), $\lim_{x\to 0} \mathrm{e}^{g(x)} = 0$, so we deduce that $v(0) = 0$. We obtain the continuous function v defined on the whole interval $[0, 1]$. Property (2.2) follows from (2.4), while property (2.3) from (2.5).

The assumption $\lambda(0) > \alpha$ yields $\widetilde{v} \in V_\alpha$, see [1] for details. Under Theorem 2.2, we see at once that $v \in V_\alpha$. □

Theorem 2.4. *For $\lambda(0) > \alpha$, the set of periodic points of* (1.1) *is dense in V_α.*

Proof. Let v be an arbitrary function belonging to V_α. Define

$$w(x) = \mathrm{e}^{g(x)}\left(\mathrm{e}^{-g(x\mathrm{e}^{nt_0})}v(x\mathrm{e}^{nt_0}) - \sum_{k=n+1}^{\infty} m\mathrm{e}^{-g(x\mathrm{e}^{kt_0})}\right), \tag{2.6}$$

where $m = \mathrm{e}^{-g(\mathrm{e}^{-t_0})}v(\mathrm{e}^{-t_0}) - v(1)$ and $x \in [\mathrm{e}^{-(n+1)t_0}, \mathrm{e}^{-nt_0}]$. To show the correctness of this definition, it is sufficient to make the following observation

$$\mathrm{e}^{g(\mathrm{e}^{-(n+1)t_0})}\left(\mathrm{e}^{-g(\mathrm{e}^{-(n+1)t_0}\mathrm{e}^{nt_0})}v(\mathrm{e}^{-(n+1)t_0}\mathrm{e}^{nt_0}) - \sum_{k=n+1}^{\infty} m\mathrm{e}^{-g\left(\mathrm{e}^{-(n+1)t_0}\mathrm{e}^{kt_0}\right)}\right) =$$

$$= \mathrm{e}^{g(\mathrm{e}^{-(n+1)t_0})}\left(v(1) + m - \sum_{k=n+1}^{\infty} m\mathrm{e}^{-g\left(\mathrm{e}^{-(n+1)t_0}\mathrm{e}^{kt_0}\right)}\right) =$$

$$= \mathrm{e}^{g(\mathrm{e}^{-(n+1)t_0})}\left(\mathrm{e}^{-g\left(\mathrm{e}^{-(n+1)t_0}\mathrm{e}^{(n+1)t_0}\right)}v(\mathrm{e}^{-(n+1)t_0}\mathrm{e}^{(n+1)t_0}) - \right.$$
$$\left. - \sum_{k=n+2}^{\infty} m\mathrm{e}^{-g\left(\mathrm{e}^{-(n+1)t_0}\mathrm{e}^{kt_0}\right)}\right).$$

The function w is continuous and vanishes at 0, which is a consequence of (2.6). Let $\epsilon > 0$. Since $v \in V_\alpha$ and $w \in V_\alpha$, there exists such t_0 that $H_{[0,\mathrm{e}^{-t_0}],\alpha}(v) < \frac{\epsilon}{4}$ and $H_{[0,\mathrm{e}^{-t_0}],\alpha}(w) < \frac{\epsilon}{4}$. From (1.8) we know that $w(x) = \frac{\widetilde{w}(x)}{\kappa(x)}$, where $\widetilde{w}$ is the periodic solution of (1.5). For $\lambda(0) > \alpha$ the set of periodic points of (1.5) is dense in V_α (see [1]), so $H_\alpha(v - \widetilde{w}) < \frac{\epsilon}{4}$ and $H_\alpha(w - \widetilde{w}) < \frac{\epsilon}{4}$.
Thus

$$\begin{aligned} H_\alpha(v - w) &\leq H_{[\mathrm{e}^{-t_0},1],\alpha}(v - w) + H_{[0,\mathrm{e}^{-t_0}],\alpha}(v - w) \leq \\ &\leq H_\alpha(v - \widetilde{w}) + H_\alpha(\widetilde{w} - w) + H_{[0,\mathrm{e}^{-t_0}],\alpha}(v) + H_{[0,\mathrm{e}^{-t_0}],\alpha}(w) < \\ &< \epsilon. \end{aligned}$$

This completes the proof. □

Theorem 2.5. *If $\lambda(0) \le \alpha$ and $v \in V_\alpha$, then*

$$\lim_{t\to\infty} H_\alpha(T_t v) = 0.$$

Moreover, if $\lambda(0) < \alpha$, then the semigroup $(T_t)_{t\ge 0}$ is exponentially stable.

Proof. Take any $v \in V_\alpha$. Using Theorem 2.2 and proceeding analogously as in its proof, we compute

$$\begin{aligned} H_\alpha(T_t v) &= \sup_{x,y\in[0,1],x\neq y} \frac{|u(t,x)-u(t,y)|}{|x-y|^\alpha} = \sup_{x,y\in[0,1],x\neq y} \frac{\left|\frac{\widetilde{u}(t,x)}{\kappa(x)} - \frac{\widetilde{u}(t,y)}{\kappa(y)}\right|}{|x-y|^\alpha} \le \\ &\le \sup_{x\in[0,1]} \left|\frac{1}{\kappa(x)}\right| \cdot \sup_{x,y\in[0,1],x\neq y} \frac{|\widetilde{u}(t,x)-\widetilde{u}(t,y)|}{|x-y|^\alpha} + \\ &\quad + \sup_{y\in[0,1]} |\widetilde{u}(t,y)| \cdot \sup_{x,y\in[0,1],x\neq y} \frac{\left|\frac{1}{\kappa(x)} - \frac{1}{\kappa(y)}\right|}{|x-y|^\alpha} \le \\ &\le e^C \cdot \left(H_\alpha(T_t\widetilde{v}) + \sup_{y\in[0,1]} |\widetilde{u}(t,y)| \cdot \left(\frac{C}{\alpha}\right)^\alpha \right). \end{aligned}$$

We know that $T_t\widetilde{v} \to 0$ in V_α for every $\widetilde{v} \in V_\alpha$. The claim $\lim_{t\to\infty} H_\alpha(T_t\widetilde{v}) = 0$ is based on the results of paper [1]. From the same source, we have derived the estimate $\|T_t\widetilde{v}\| \le e^{(\gamma-\alpha)t}\|\widetilde{v}\|$.

$$|\widetilde{u}(t,y)| = |(T_t\widetilde{v})(y)| = |(T_t\widetilde{v})(y) - (T_t\widetilde{v})(0)| \le H_\alpha(T_t\widetilde{v})y^\alpha \le H_\alpha(T_t\widetilde{v}),$$

hence, since $\lambda(0) < \alpha$, there follows the exponential stability of the semigroup $(T_t)_{t\ge 0}$ with $D = e^C \left(1 + \left(\frac{C}{\alpha}\right)^\alpha\right)$ and $\omega = \alpha - \lambda(0)$. □

3. CHAOS AND STABILITY OF THE SYSTEM $(T_t)_{t\ge 0}$ IN THE SPACE L^p

Theorem 3.1. *Assume that*

$$\exists C, q > 0 \quad \forall x \in [0,1] \quad |\lambda(0) - \lambda(x)| \le Cx^q. \tag{3.1}$$

The function u belongs to the space L^p if and only if $\widetilde{u} \in L^p$.

Proof. Using (3.1), substitution (1.8) discussed in Section 1 and assuming that $u \in L^p$, we obtain

$$\begin{aligned} \|\widetilde{u}(t,x)\|^p &= \int_0^1 |\kappa(x)u(t,x)|^p dx \le \int_0^1 e^{p\int_0^x \frac{|\lambda(0)-\lambda(s)|}{s}ds} |u(t,x)|^p dx \le \\ &\le \int_0^1 e^{\frac{Cp}{q}x^q} |u(t,x)|^p dx \le e^{\frac{Cp}{q}} \int_0^1 |u(t,x)|^p dx = \\ &= e^{\frac{Cp}{q}} \|u(t,x)\|^p < \infty \end{aligned}$$

In the same manner, we can establish the inverse implication. □

The above theorem will be significant in obtaining the next one. It enables the results of [1] to be used and generalized. From now on, we assume (3.1).

Theorem 3.2. *For* $\lambda(0) > -\frac{1}{p}$ *there exists a periodic solution of* (1.1).

Proof. For any t_0, define the following function v :

$$v(x) = \mathrm{e}^{g(x)}\mathrm{e}^{-g(x\mathrm{e}^{nt_0})}w(x\mathrm{e}^{nt_0}), \quad x \in [\mathrm{e}^{-(n+1)t_0}, \mathrm{e}^{-nt_0}], \tag{3.2}$$

$$v(0) = 0,$$

where w is an arbitrary function from the space L^p. We have shown that such function v is a periodic solution, so it is sufficient to prove that $v \in L^p$. The function v is the solution of equation (1.1), so we can express it using the function $\widetilde{v}$, the solution of (1.5), $v(x) = \frac{\widetilde{v}(x)}{\kappa(x)}$. Brzeźniak and one of the authors [1] showed that $\widetilde{v}$ is a periodic solution and belongs to L^p when $\gamma > -\frac{1}{p}$. Our assumption and Theorem 3.1 guarantee the same conclusion for the function v. □

Theorem 3.3. *If* $\lambda(0) > -\frac{1}{p}$, *then the set of periodic points is dense in* L^p.

Proof. Let $w \in L^p$ and $\epsilon > 0$. Fix t_0 such that

$$\left[\int_0^{\mathrm{e}^{-t_0}} |w(x)|^p dx\right]^{\frac{1}{p}} < \frac{\epsilon}{2}$$

and

$$\mathrm{e}^{\frac{C}{q}}\|\widetilde{v}\| < \frac{\epsilon}{2}, \quad C > 0$$

where $\widetilde{v}$ is a periodic solution of (1.5).
The function v is defined by formula (3.2). The function v belongs to the set of periodic points due to Theorem 3.2. It is sufficient to estimate $||v - w||$. Since $v(x) = w(x)$ for $x \in [\mathrm{e}^{-t_0}, 1]$, it is obvious that $||v - w|| = ||(v - w)1_{[0,\mathrm{e}^{-t_0}]}||$, where $1_{[0,\mathrm{e}^{-t_0}]}$ denotes the indicator of the set $[0, \mathrm{e}^{-t_0}]$. Applying the estimate from Theorem 3.1 and substitution (1.8), we can assert that

$$\|v - w\| \leq \|v1_{[0,\mathrm{e}^{-t_0}]}\| + \|w1_{[0,\mathrm{e}^{-t_0}]}\| < \epsilon.$$

□

Theorem 3.4. *If* $\lambda(0) \leq -\frac{1}{p}$, *and* $v \in L^p$ *then*

$$\lim_{t\to\infty} \|T_t v\| = 0.$$

Moreover, for $\lambda(0) < -\frac{1}{p}$ *the semigroup* $(T_t)_{t\geq 0}$ *is exponentially stable on* L^p.

Proof. Let $v \in L^p$ be an arbitrary function.

$$\|T_t v\|^p = \int_0^1 |u(t,x)|^p dx = \int_0^1 \left|\frac{\widetilde{u}(t,x)}{\kappa(x)}\right|^p dx =$$
$$= \int_0^1 \left|\frac{1}{\kappa(x)}(T_t\widetilde{v})(x)\right|^p dx \le \mathrm{e}^{\frac{Cp}{q}} \|T_t\widetilde{v}\|^p,$$

where $C > 0$. We know that $\|T_t\widetilde{v}\| \to 0$, as $t \to \infty$ (see [1]), which proves the first part of the Theorem. From [1] we know that $\|T_t\widetilde{v}\|^p \le \mathrm{e}^{(\gamma p+1)t}\|\widetilde{v}\|^p$. It gives the exponential stability of the semigroup $(T_t)_{t\ge 0}$ with $D = \mathrm{e}^{\frac{C}{q}}$ and $\omega = -\frac{1}{p}(\lambda(0)p + 1)$. □

Acknowledgements

The second author acknowledges the support from Białystok Technical University (Grant No. S/WI/1/08).

REFERENCES

[1] Z. Brzeźniak, A.L. Dawidowicz, *On the periodic solution to the von Foerster-Lasota equation*, to appear in Semigroup Forum.

[2] A.L. Dawidowicz, *On the existence of an invariant measure for a quasi-linear partial differential equation*, Zeszyty Naukowe UJ, Prace Matematyczne 23 (1982), 117–123.

[3] A.L. Dawidowicz, *On the existence of an invariant measure for the dynamical system generated by partial differential equation*, Ann. Polon. Math. XLI (1983), 129–137.

[4] A.L. Dawidowicz, N. Haribash, *On the periodic solutions of von Foerster type equation*, Universitatis Iagellonicae Acta Mathematica (1999) 37, 321–324.

[5] A. Lasota, G. Pianigiani, *Invariant measures on topological spaces*, Boll. Un. Mat. Ital. (5) 15-B (1977), 592–603.

[6] A. Lasota, T. Szarek, *Dimension of measures invariant with respect to Ważewska partial differential equation*, J. Differential Equations **196** (2004) 2, 448–465.

[7] K. Łoskot, *Turbulent solutions of first order partial differential equation*, J. Differential Equations **58** (1985) 1, 1–14.

[8] R. Rudnicki, *Invariant measures for the flow of a first order partial differential equation*, Ergodic Theory and Dynamical Systems, **5** (1985) 3, 437–443.

Antoni Leon Dawidowicz
Antoni.Leon.Dawidowicz@im.uj.edu.pl

Jagiellonian University
Institute of Mathematics
ul. Prof. Łojasiewicza, 30-348 Kraków, Poland

Anna Poskrobko
aposkrobko@wi.pb.edu.pl

Białystok Technical University
Institute of Mathematics and Physics
ul. Wiejska 45A, 15-351 Białystok, Poland

Received: March 3, 2008.
Accepted: October 10, 2008.