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ASYMPTOTIC PROPERTIES OF SOLUTIONS OF SOME ITERATIVE FUNCTIONAL INEQUALITIES

Abstract. Continuous solutions of iterative linear inequalities of the first and second order are considered, belonging to a class $\mathcal{F}_T$ of functions behaving at the origin as a prescribed function T.

Keywords: functional inequalities, continuous solutions, test function, asymptotic behavior.

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1. INTRODUCTION

The iterative functional inequality of second order (studied, among others, in [6])

$$\psi[f^2(x)] \le (p(x) + q[f(x)])\psi[f(x)] - p(x)q(x)\psi(x), \tag{1}$$

where ψ is the unknown function, thanks to its specific form, is equivalent to the system consisting of the inequality of first order

$$z[f(x))] \le p(x)z(x) \tag{2}$$

(for the unknown z) and of the linear inhomogeneous functional equation

$$\varphi[f(x)] = q(x)\varphi(x) + z(x). \tag{3}$$

This means that every solution φ of equation (3) with a z satisfying inequality (2) is a solution of inequality (1) and vice versa: given a solution ψ of inequality (1), insert it to (3) in place of φ and calculate z from (3) to get a solution z of inequality (1).

The aim of this paper is to investigate these continuous solutions of inequalities (2) and (1) that behave at the fixed point of f like a prescribed "test" function T, in particular, like any of the functions p, q or f. Basic facts from the theory of iterative functional inequalities, originated by the first author in [1] (cf. also Chapter 12 in [5]) are recalled, as well as those on the asymptotic behavior of solutions of equation (3), found in the papers by the remaining authors: [2] and [3].

The functions f, p, q and T meet the following general assumptions:

(H_1) The function $f : I \to I$ is continuous and strictly increasing in an interval $I = [0, a|$ ($a > 0$ may belong to I or not). Moreover, $0 < f(x) < x$ for $x \in I^\star = I \setminus \{0\}$.
(H_2) The function $p : I \to \mathbf{R}$ is continuous in I and $p(x) > 0$ for $x \in I^\star$.
(H_3) The function $q : I \to \mathbf{R}$ is continuous in I and $q(x) \neq 0$ for $x \in I^\star$.
(H_4) The function $T : I \to \mathbf{R}$ is continuous in I and $T(x) \neq 0$ for $x \in I^\star, T(0) = 0$.

With f^n denoting the n-th iterate of the function f, hypotheses (H_1) imply (see [5]) that

$$\lim_{n\to\infty} f^n(x) = 0 \quad \text{for every} \quad x \in I. \tag{4}$$

Moreover, 0 is the only fixed point of f in I.

Assuming that (H_4) is satisfied, we introduce the following class of functions

$$\mathcal{F}_T := \left\{\varphi : I \to \mathbf{R} : \varphi \text{ is continuous on } I \text{ and the limit } L^T_\varphi := \lim_{x\to 0^+} \frac{\varphi(x)}{T(x)} \text{ is finite}\right\}.$$

Note that if $T(0) \neq 0$, the class $\mathcal{F}_T$ would consist of all functions continuous in I.

We are interested in solutions of inequalities (2) and (1) belonging to the class $\mathcal{F}_T$.

2. CONTINUOUS SOLUTIONS OF INEQUALITY (2)

It is known (see [5]) that the number of continuous solutions of inequality (2) as well as of the equation

$$\alpha[f(x)] = p(x)\alpha(x), \quad x \in I, \tag{5}$$

associated to inequality (2), depends on the behavior of the functional sequence

$$P_n(x) = \prod_{i=0}^{n-1} p[f^i(x)], \quad x \in I,\ n \in \mathbf{N}. \tag{6}$$

We shall need some results from [4] (cf. also [5], Section 3.1) on continuous solutions α of equation (5). They are quoted below as a lemma, and concern the following two cases:

(**A**) The limit

$$P(x) = \lim_{n\to\infty} P_n(x) \tag{7}$$

exists in I and P is continuous in I. Moreover, $P(x) > 0$ for $x \in I^\star$.

(**B**) There exists an interval $J \subset I$ such that, uniformly in J,

$$\lim_{n\to\infty} P_n(x) = 0.$$

Lemma 1. *Let hypotheses* (H_1) *and* (H_2) *be fulfilled.*

i) *In case* $(\mathbf{A})$ *all continuous solutions* $\alpha : I \to R$ *of equation* (5) *are given by*

$$\alpha(x) = \frac{c}{P(x)} \quad \text{for} \quad x \in I,$$

where c is any real number.

ii) *If, moreover,* $p(0) = 0$, *then in case* $(\mathbf{B})$ *equation* (5) *has in* I *continuous solution depending on an arbitrary function. (This means that for any* $x_0 \in I^\star$ *and every continuous function* $\alpha_0 : [f(x_0), x_0] \to \mathbf{R}$ *fulfilling the boundary condition* $\alpha_0[f(x_0)] = p(x_0)\alpha_0(x_0)$ *there is the unique continuous solution* $\alpha : I \to \mathbf{R}$ *of equation* (5) *such that*

$$\alpha(x) = \alpha_0(x) \quad \text{for} \quad x \in [f(x_0), x_0] \quad \text{and} \quad \alpha(0) = 0,$$

and every continuous solution to (5) *may be obtained in this manner.)*

For the sake of completeness, we now quote from [1] (cf. also [5], Chapter 12.) as next lemmas, some representation theorems which will be useful in the sequel. We start with the notions of an $\{f\}$-monotonic function.

Definition 1. *A continuous function* $\eta : I \longrightarrow \mathbf{R}$ *[resp.* $\vartheta : I \to \mathbf{R}$*] is said to be* $\{f\}$*-decreasing in* I *[resp.* $\{f\}$*-increasing in* I*], if*

$$\eta[f(x)] \le \eta(x), \quad x \in I \qquad \text{[resp.,} \quad \vartheta[f(x)] \ge \vartheta(x), \quad x \in I]. \tag{8}$$

The family of all continuous $\{f\}$*-decreasing [resp.,* $\{f\}$*-increasing] functions will be denoted by* $\mathcal{D}_f$ *[resp.,* $\mathcal{I}_f$*].*

Theorem 1. *Let hypotheses* (H_1), (H_2) *be satisfied and case* $(\mathbf{A})$ *occur. Then the general continuous solution* z *of inequality* (2) *is given by*

$$z(x) = \frac{\eta(x)}{P(x)}, \quad x \in I \tag{9}$$

where η *is an arbitrary function from* $\mathcal{D}_f$.

Note that, by Lemma 1*i*), formula (9) says that the function z is the product of a function from $\mathcal{D}_f$ and of a solution of equation (5). Similar assertion remains true for case $(\mathbf{B})$, but in a narrower class of solutions than that of continuous ones, namely, solutions called *regular.*

Definition 2. *A continuous solution* z *of inequality* (2) *is said to be regular [a* CR*-solution, for short] if there exists a continuous solution* α *of equation* (5) *such that* $\alpha(x) \le z(x), x \in I$, *and the function* α_z *defined by formula*

$$\alpha_z(x) = \lim_{n\to\infty} \frac{z[f^n(x)]}{P_n(x)} \quad \text{for} \quad x \in I^\star, \quad \alpha_z(0) = 0, \tag{10}$$

is a continuous solution of equation (5) *in* I.

Some necessary and sufficient conditions for a solution of (2) to be *regular* are collected in [1], cf. also [5], Section 12.4. A representation theorem based on the results found therein reads:

Theorem 2. *Let hypotheses* (H_1),(H_2) *be satisfied,* $p(0) = 0$*, and let case* $(\boldsymbol{B})$ *occur.*

i) *The general* CR*-solution* z *of inequality* (2) *such that (cf.* (10)*)*

$$\alpha_z(x) \neq 0, \quad x \in I^\star, \tag{11}$$

is given by the formula (valid for $x \in I^\star$*)*

$$z(x) = \begin{cases} \eta(x)\alpha(x), & \text{if} \quad \alpha_z(x) > 0, x \in I^\star, \\ \vartheta(x)\alpha(x), & \text{if} \quad \alpha_z(x) < 0, x \in I^\star, \end{cases} \tag{12}$$

where α *is an arbitrary continuous solution of equation* (5) *vanishing at* $x = 0$ *only,* $\eta \in \mathcal{D}_f$ *and* $\vartheta \in \mathcal{I}_f$*, both are arbitrary and such that*

$$\eta(0) = 1, \quad \vartheta(0) = -1. \tag{13}$$

ii) *The general* CR*-solution* z *of inequality* (2) *such that*

$$\alpha_z(x) = 0 \quad \text{for} \quad x \in I, \tag{14}$$

is given by the formula

$$z(x) = \eta(x)\alpha_0(x), \quad x \in I, \tag{15}$$

where $\eta \in \mathcal{D}_f$ *is arbitrary with* $\eta(0) = 0$ *and* α_0 *is a positive on* $I^\star$ *continuous solution of* (2)*.*

Finally, we put

$$p_T(x) = \frac{p(x)T(x)}{T[f(x)]}, \quad x \in I^\star, \tag{16}$$

and introduce an auxiliary equation.

Lemma 2. *Let hypotheses* $(H_1), (H_2)$ *and* (H_4) *be fulfilled and let the limit*

$$p_T(0) := \lim_{x \to 0^+} p_T(x) \tag{17}$$

exist. Then there is a one-to-one correspondence between the continuous solutions $\beta : I \to \mathbf{R}$ *of the auxiliary equation*

$$\beta[f(x)] = p_T(x)\beta(x), \quad x \in I, \tag{18}$$

where p_T *is given by* (16) *and* (17)*, and the solutions* α *of equation* (5)*, belonging to* $\mathcal{F}_T$*.*

Proof. The equivalence claimed in the lemma is established as follows. If a function $\alpha \in \mathcal{F}_T$ satisfies (5), then it is easily verified that the function $\beta : I \longrightarrow \mathbf{R}$ given by

$$\beta(x) = \frac{\alpha(x)}{T(x)}, \quad x \in I^\star, \qquad \beta(0) = L_\alpha^T \tag{19}$$

(cf. the definition of $\mathcal{F}_T$), is a continuous in I solution of (18). Conversely, if a continuous $\beta : I \to \mathbf{R}$ satisfies (18) then $\alpha := T \cdot \beta$ is a solution to (5), belonging to $\mathcal{F}_T$. □

3. SOLUTIONS OF INEQUALITY (2) ASYMPTOTICALLY COMPARABLE WITH T

We start with introducing the functional sequence corresponding to (6) (with p replaced by p_T), defined by

$$P_n^T(x) = \prod_{i=0}^{n-1} p_T[f^i(x)], \quad x \in I, n \in \mathbf{N} \tag{20}$$

It is easy to verify the following formula

$$P_n^T(x) = \frac{T(x)P_n(x)}{T[f^n(x)]}, \quad x \in I^\star, \qquad P_n^T(0) = [p_T(0)]^n. \tag{21}$$

Case (A)

We note that $p(0) = 1$ in this case (see [3]) and, consequently, $P(0) = 1$, where the function P is defined by (6). Moreover, since P_n^T tends to ∞ when n does (cf. (4) and $T(0) = 0$), the zero function (defined on I) is the only continuous solution of equation (18) in I, cf. [4]. Thus, by virtue of Lemma 2, this zero function is also the only solution of equation (2) in the class $\mathcal{F}_T$.

As a simple consequence of Theorem 2i) we obtain

Theorem 3. *Let hypotheses* (H_1), (H_2) *be satisfied and let case* (**A**) *occur. Then the general solution* $z \in \mathcal{F}_T$ *of inequality* (2) *is given by formula* (9), *where* $\eta \in \mathcal{D}_f \cap \mathcal{F}_T$ *is arbitrary.*

Proof. Refer to (9), (6) and (7) (with $P(0) = 1$), and the definition of the class $\mathcal{F}_T$ to obtain the relations

$$\lim_{x\to 0^+} \frac{\eta(x)}{T(x)} = \lim_{x\to 0^+} \frac{z(x)P(x)}{T(x)} = \lim_{x\to 0^+} \frac{z(x)}{T(x)},$$

whenever the functions z and η have the properties stated in the theorem. □

Remark 1. *Note that if* $k : I \to \mathbf{R}$ *is any nonnegative function from* $\mathcal{D}_f$*, such that* $\lim_{x\to 0^+} k(x)$ *exists (in particular, if* k *is increasing), then the function* $\eta := k \cdot T \in \mathcal{D}_f \cap \mathcal{F}_T$.

Case (B)

We start with a theorem which describes all CR-solutions $z \in \mathcal{F}_T$ of (2) in the case of the function α_z defined by (10) vanishing at zero only.

Theorem 4. *Let hypotheses* (H_1), (H_2) *be satisfied,* $p(0) = 0$ *and let case* $(\mathbf{B})$ *occur. Then the general* CR*-solution* $z \in \mathcal{F}_T$ *of inequality* (2) *such that* (11) *holds is given by formula* (12)*, where* $\alpha \in \mathcal{F}_T$ *is an arbitrary continuous solution of* (5) *vanishing at* $x = 0$ *only and* $\eta \in \mathcal{D}_f$ *if* $\alpha_z > 0$ *[resp.* $\theta \in \mathcal{I}_f$ *if* $\varphi_z < 0$*] is an arbitrary function satisfying* (13).

When $\alpha_z = 0$ in I we have only a sufficient condition for z to be in $\mathcal{F}_T$:

Theorem 5. *Let hypotheses* (H_1), (H_2) *be satisfied,* $p(0) = 0$ *and let case* $(\mathbf{B})$ *occur. Moreover let* α_0 *be a continuous solution of* (5) *such that* (14) *holds and let* $\eta \in \mathcal{D}_f$ *be arbitrary with* (13)*. If either* $\alpha_0 \in \mathcal{F}_T$ *or* $\eta \in \mathcal{F}_T$*, then the function* z *defined by formula* (15) *is a* CR*-solution of* (2) *in the class* $\mathcal{F}_T$*.*

Theorems 5 and 6 follow directly from formula (12), resp. (16), and the definition of $\mathcal{F}_T$.

Finally we consider the following two subcases of $(\mathbf{B})$:

$(\mathbf{BA^T})$ The case $(\mathbf{B})$ occurs, the limit

$$P^T(x) = \lim_{n\to\infty} P_n^T(x) \tag{22}$$

exists in I and P^T is continuous in I. Moreover, $P^T(x) > 0$, for $x \in I^\star$.

$(\mathbf{BB^T})$ The case $(\mathbf{B})$ occurs and there exists an $x_0 \in I^\star$ such that, uniformly in $[f(x_0), x_0]$,

$$\lim_{n\to\infty} P_n^T(x) = 0.$$

We shall present the following

Theorem 6. *Let hypotheses* (H_1), (H_2) *be fulfilled and let the limit* (17) *exist. In cases* $(\mathbf{BA^T})$ *or* $(\mathbf{BB^T})$*, every* CR*-solution* z *of inequality* (2) *such that* (11) *or* (14) *holds belongs to the class* $\mathcal{F}_T$*.*

Moreover, in case $(\mathbf{BA^T})$*, the general* CR*-solution* $z \in \mathcal{F}_T$ *of inequality* (2) *such that* (11) *is satisfied, is given by the formulae (for* $x \in I$*)*

$$z(x) = c \cdot \frac{\eta(x)T(x)}{P^T(x)} \quad \textit{with} \quad c > 0; \qquad z(x) = c \cdot \frac{\vartheta(x)T(x)}{P^T(x)} \quad \textit{with} \quad c < 0 \tag{23}$$

where P_T *is defined by* (22)*,* $\eta \in \mathcal{D}_f$ *and* $\vartheta \in \mathcal{I}_f$ *satisfy condition* (13).

Proof. Case $(\mathbf{BA^T})$. If $z \in \mathcal{F}_T$ is a CR-solution of inequality (2) and (11) holds, then by Theorem 4 it is given by formula (12), where $\alpha \in \mathcal{F}_T$ is any continuous solution of equation (5) vanishing at $x = 0$ only and η and ϑ are some $\{f\}-$monotonic functions described in the assertion of the theorem. Define β by (19). Lemma 2 says that this β is a continuous solution of equation (18). Consequently from Lemma 1, applied to equation (18), we obtain $\beta(x) = c/P^T(x)$, where $c \neq 0$. Thanks to $(\mathbf{BA^T})$, (12) and (19), we get (23). Of course, the function z defined by (23) is a CR-solution of inequality (2). Moreover, $z \in \mathcal{F}_T$, because (we take the first equality in (23); for the other, the proof is the same)

$$\lim_{x\to 0^+} \frac{z(x)}{T(x)} = \lim_{x\to 0^+} c \cdot \frac{\eta(x)}{P^T(x)} = c\,\eta(0) = c.$$

Case ($\mathbf{BB^T}$). Given a continuous solution α of equation (5) vanishing at $x = 0$ only, in virtue of Theorem 5, it is enough to check whether it belongs to $\mathcal{F}_T$. For, we know that a continuous solution of both equations (5) and (18) depends on an arbitrary function. Let us then take an $x_0 \in I^\star$ and define

$$\beta_0(x) = \frac{\alpha(x)}{T(x)} \quad \text{for} \quad x \in [f(x_0), x_0].$$

Since $\alpha[f(x_0)] = p(x_0)\alpha(x_0)$, the relation $\beta_0[f(x_0)] = p_T(x_0)\beta_0(x_0)$ also holds (see (19) and (16)). Consequently, there exists the (unique) continuous solution β of (18) such that $\beta(x) = \beta_0(x)$ for $x \in [f(x_0), x_0]$ and $\beta(0) = 0$. Obviously, $\alpha^\star := \beta \cdot T$ satisfies (5), is continuous, vanishes at $x = 0$ only and is in $\mathcal{F}_T$. And $\alpha^\star(x) = \alpha(x)$ for $x \in [f(x_0), x_0]$. This means that $\alpha^\star = \alpha$, i.e., $\alpha \in \mathcal{F}_T$, as claimed. □

To conclude the section, we supply some examples concerning solutions of inequality (2) that behave at the origin like the given functions p or f, occurring in (2).

Example 1. Take $I = [0, 1)$ and consider the inequality

$$z(x^2) \le \frac{1}{1+x} z(x). \tag{24}$$

Here

$$P_n(x) = \prod_{i=0}^{n-1} (1 + x^{2^i})^{-1} = \frac{1-x}{1-x^{2^n}}, \quad \text{whence} \quad P(x) = 1 - x.$$

Consequently, case ($\mathbf{A}$) occurs and the continuous solutions of inequality (24) are of the form $z(x) = \eta(x)/(1 - x)$, $x \in I$, where $\eta \in \mathcal{D}_f$ is arbitrary. Let $T(x) = f(x) = x^2$, $x \in I$. By Theorem 3, $z \in \mathcal{F}_f$ if and only if there exists the finite limit $\lim_{x\to 0^+} x^{-2}\eta(x)$ (which is the case, for instance, when $\eta(x) = x^\gamma, \gamma \ge 2$ or $\eta(x) = e^{x^2} - 1$).

Example 2. Consider the inequality

$$z(\frac{x}{2}) \le \frac{1}{2} z(x) \tag{25}$$

in the interval $I = [0, 1)$ and take $T(x) = f(x) = x/2$, $x \in I$. Since $\lim_{n\to\infty} P_n(x) = \lim_{n\to\infty} 2^{-n} = 0$ and $P_n^T(x) = 1, x \in I$, case ($\mathbf{BA^T}$) occurs. In virtue of Theorem 6, the general CR-solution $z \in \mathcal{F}_f$ has the form

$$z(x) = x\eta(x), \quad \eta(0) > 0 \quad \text{or} \quad z(x) = x\vartheta(x), \quad \vartheta(0) < 0.$$

Take the function $z : I \to R$, given by $z(x) = e^x - 1,\ x \in I$, which is a particular continuous solution of (25), and the identity function α_0 (on I), which is a particular continuous solution of the equation

$$\alpha(\frac{x}{2}) = \frac{1}{2}\alpha(x), \tag{26}$$

corresponding to inequality (25). Since $\lim_{x\to 0^+}[(e^x-1)/x] = 1$, then $z \in \mathcal{F}_f$. By Theorem 12.4.3 from [5], z is an CR-solution of (25) and $\alpha_z(x) = x$ for $x \in I$. The function η given by

$$\eta(x) = \frac{e^x - 1}{x}, \quad x \in I^\star, \quad \eta(0) = 1,$$

belongs to $\mathcal{D}_f$ and our solution z of (25) is of the form (12) with η as above and $\alpha(x) = x$ for $x \in I$.

For another continuous solution z of (25) given by $z(x) = ax^2, a > 0$, we calculate $\lim_{x\to 0^+}\big(z[f^n(x)]/P_n(x)\big) = \lim_{x\to 0^+}(ax^2 2^{-n}) = 0$. Consequently, this solution z is *regular* with $\alpha_z(x) = 0$ for $x \in I$; moreover, it is actually given by formula (9) with $\eta \in \mathcal{D}_f,\ \eta(x) = ax,\ x \in I$, and a positive in $I^\star$ continuous solution α_0 of (26) defined by $\alpha_0(x) = x,\ x \in I$.

4. SOLUTIONS OF INEQUALITY (1) ASYMPTOTICALLY COMPARABLE WITH T

Thanks to the equivalence of inequality (1) and the system consisting of inequality (2) and equation (3), we may use a uniqueness result from [4], which is adapted to equation (3) and quoted below as the last lemma.

Given a continuous function $r : I^\star \to \mathbf{R}$, define on $I^\star$ the (continuous) functions q_T and z_T as follows

$$q_T(x) = \frac{q(x)T(x)}{T[f(x)]}, \quad \text{and} \quad z_T(x) = \frac{z(x)}{T[f(x)]}, \quad \text{for} \quad x \in I^\star. \tag{27}$$

Now, we are in a position to formulate

Lemma 3. *Assume that hypotheses* $(H_1) - (H_4)$ *are fulfilled, and that there exist the finite limits*

$$q_T(0) := \lim_{x\to 0^+} q_T(x); \qquad z_T(0) := \lim_{x\to 0^+} z_T(x). \tag{28}$$

i) *If*

$$|q_T(0)| > 1, \tag{29}$$

then equation (3) *has the unique solution* φ *in* $\mathcal{F}_T$ *which is given by the formula*

$$\varphi(x) = -\sum_{n=0}^{\infty} \frac{z[f^n(x)]}{Q_{n+1}(x)}, \quad x \in I^\star; \qquad \varphi(0) = 0, \tag{30}$$

where (cf. (6)*)*

$$Q_{n+1} = \prod_{i=0}^{n} q \circ f^i, \quad n \in N \cup \{0\}. \tag{31}$$

ii) *If* $|q_T(0)| < 1$*, then for any* $x_0 \in I^\star$ *every continuous function* $\varphi_0 : [f(x_0), x_0] \to \mathbf{R}$ *satisfying the condition:* $\varphi(f(x_0)) = q(x_0\varphi(x_0) + z(x_0))$ *can be uniquely extended to a solution* $\varphi : I \to \mathbf{R}$ *of* (3) *belonging to* $\mathcal{F}_T$*.*

Since (2) with (3) are equivalent to (1), directly from Lemma 3*i*) we get the following

Theorem 7. *Assume that hypotheses* $(H_1) - (H_4)$ *are fulfilled, inequality* (2) *has a continuous solution* $z : I \to \mathbf{R}$ *and there exist limits* (28) *and condition* (29) *is satisfied.*

Then all solutions $\psi \in \mathcal{F}_T$ (3) *(with this* z*, thus also of inequality* (1)*) are given by the formula*

$$\psi(x) = -\sum_{n=0}^{\infty} \frac{z[f^n(x)]}{Q_{n+1}(x)}, \quad x \in I^\star; \qquad \psi(0) = 0. \tag{32}$$

In the case of z a CR-solution (cf. Definition 2) to (2), formula (32) may be written in another form, see [6].

Theorem 8. *Assume that hypotheses* $(H_1) - (H_4)$ *are fulfilled,* z *is a* CR*- solution to* (2)*, case* (**A**) *or* (**B**) *occurs and there exist the second limit in* (28) *and a finite limit*

$$t := \lim_{x \to 0^+} \frac{T(x)}{T[f(x)]}. \tag{33}$$

Moreover,

$$|q(0)\ t| > 1. \tag{34}$$

If $\psi \in \mathcal{F}_T$ *solves equation* (3) *(with this* z*, hence also of inequality* (1)*), then*

$$\psi(x) = -S(x) \sum_{n=0}^{\infty} \frac{\zeta[f^n(x)] P_n(x)}{Q_{n+1}(x)}, \quad x \in I^\star, \qquad \psi(0) = 0, \tag{35}$$

where:

a) *in case* (**A**) *there is* $S = 1/P$ *(a continuous solution of equation* (5) *when* P *is defined by* (7)*, cf. Lemma 1), and* $\zeta = \eta \in \mathcal{D}_f \cap \mathcal{F}_T, \eta(0) = 1$*;*

b) *in case* (**B**)*, when* $\alpha_z \neq 0$ *for* $x \in I^\star$ *(cf. Definition 2), there is* $S = \alpha \in \mathcal{F}_T$ *solving* (5) *and vanishing at zero only, and* $\zeta = \eta \in \mathcal{D}_f, \eta(0) = 1$ *(when* $\alpha_z > 0$*), whereas* $\zeta = \vartheta \in \mathcal{I}_f, \vartheta(0) = -1$ *(when* $\alpha_z < 0$*);*

c) *in case* (**B**)*, when* $\alpha_z = 0$ *for* $x \in I$*, there is* $S = \alpha_0$*, which is a continuous solution of equation* (5)*, positive in* $I^\star$ *(provided it does exist),* $\zeta = \eta \in \mathcal{D}_f, \eta(0) = 1$*, and either* α_0 *or* η *belongs to* $\mathcal{F}_T$*.*

Proof. Note first that because of (27) and (33), for $z \in \mathcal{F}_T$, from (28) we derive

$$q_T(0) = \lim_{x \to 0^+} \big(q(x)\frac{T(x)}{T[f(x)]}\big) = q(0)t; \quad z_T(0) = \lim_{x \to 0^+} \big(\frac{z(x)}{T(x)}\frac{T(x)}{T[f(x)]}\big) = L_z^T t.$$

Thus, thanks to inequality (34), Lemma 3 applies and formula (30) determines the solutions $\varphi \in \mathcal{F}_T$ of equation (3), whence, as $\psi = \varphi$, those of inequality (1).

It remains to check formulae (35). Observe first that if α is a solution to (5), then it also satisfies, for every $n \in N$, the equations below, resulting from equation (5) on iterating it n times,

$$\alpha[f^n(x)] = P_n(x)\alpha(x), \quad x \in I. \tag{36}$$

a) Because of Theorem 3, the solutions $z \in \mathcal{F}_T$ of (2) are given by (9), and the function $\alpha = 1/P$ satisfies (3). Using (9) and (36) in (30), we get (35).
b) This results from Theorem 4 and formula (12) with (36), when used in (30).
c) By Theorem 5, we obtain formula (15) for z and (35) follows from (36).

The applicability of Theorems 7 and 8 in the case of the "test function" $T = q$ or $T = f$ will be shown in next two examples. □

Example 3. Take $I = [0, 1)$ and consider the inequality

$$\psi(x^4) \le \big(\frac{1}{1+x} + 2x^2\big)\ \psi(x^2) - \frac{2x}{1+x}\ \psi(x), \quad x \in I. \tag{37}$$

This inequality is of the form (1), with $f(x) = x^2,\ p(x) = (1+x)^{-1},\ q(x) = 2x$; all for $x \in I$. These functions fulfil hypotheses $(H_1) - (H_3)$. The function $\eta : I \to \mathbf{R}$, given by $\eta(x) = \ln(1+x^2),\ x \in I$, satisfies (8), so that $\eta \in \mathcal{D}_f$, and it produces the solution $z : I \to \mathbf{R}$;

$$z(x) = \frac{\ln(1+x^2)}{1-x}, \quad x \in I, \tag{38}$$

of inequality (24) (cf. (2)), related to (37). For $T(x) = q(x) = 2x$, the limits (28) are:

$$q_T(0) = \lim_{x \to 0^+} \frac{[q(x)]^2}{q[f(x)]} = 2; \quad z_T(0) = \lim_{x \to 0^+} \frac{z(x)}{q[f(x)]} = \lim_{x \to 0^+} \frac{1}{2}\frac{\ln(1+x^2)}{x^2} = \frac{1}{2},$$

so that condition (29) is fulfilled. Consequently, Theorem 7 works, and in the class $\mathcal{F}_q$ inequality (37) has the solution $\psi : I \to \mathbf{R}$ given by formula (32), in which $f^n(x) = x^{2^n}$,

$$Q_{n+1}(x) = \prod_{i=0}^{n} (2x^{2^i}) = 2^{n+1} x^{2^{n+1}-1}$$

and z is given by (38). Thus (35) now takes the form

$$\psi(x) = -\sum_{n=0}^{\infty} 2^{-n-1} x^{1-2^{n+1}} \frac{\ln(1+x^{2^{n+1}})}{1-x^{2^n}}, \quad \psi(0) = 0.$$

Example 4. Take $I = [0, 1)$ and consider the inequality

$$\psi\Big(\frac{1}{4}x\Big) \leq \Big(\frac{1}{2} + \cos\Big(\frac{1}{4}x\Big)\Big)\psi\Big(\frac{1}{2}x\Big) - \frac{1}{2}\cos\Big(\frac{1}{2}x\Big)\psi(x), \quad x \in I. \tag{39}$$

The given functions $f(x) = \frac{1}{2}x$, $p(x) = \frac{1}{2}$, $q(x) = \cos(\frac{1}{2}x)$, and $T(x) = f(x) = \frac{1}{2}x$, $x \in I$, satisfy hypotheses $(H_1) - (H_4)$. We note that the function $z : I \to \mathbf{R}$ given by $z(x) = e^x - 1, x \in I$, is a regular solution of inequality (25) (related to (39), cf. (2)) and $\alpha_z(x) = x, x \in I^*$. Moreover, $z \in \mathcal{F}_f$ (see Example 2). In turn, since $P_n(x) = 2^{-n}$, case **(B)** occurs. Finally, we have $f^i(x) = 2^{-i}x$, whence

$$Q_{n+1}(x) = \prod_{i=0}^{n} \cos(2^{-i}x) = \frac{\sin x \cos x}{2^n \sin(2^{-n}x)}.$$

(see (31)). We calculate the limits (see (28) and (33)):

$$t = \lim_{x\to 0^+} \frac{f(x)}{f^2(x)} = 2; \quad z_T(0) = \lim_{x\to 0^+} \frac{z(x)}{f^2(x)} = 4 \lim_{x\to 0^+} \frac{e^x - 1}{x} = 4.$$

Since $q(0) = 1$, (34) holds. By Theorem 8 formula (35) represents all solutions $\psi \in \mathcal{F}_f$ of inequality (39). With $S = \alpha_z = id|_I$ (which satisfies equation (25) and vanishes at zero only), formula (35) now takes the form

$$\psi(x) = -\frac{x}{\sin x \cos x} \sum_{n=0}^{\infty} \eta(2^{-n}x) \sin(2^{-n}x), \quad x \in I^*, \quad \psi(0) = 0,$$

where $\eta \in \mathcal{D}_f, \eta(0) = 1$, is arbitrary. We may take $\eta(x) = (e^x - 1)/x, x \in I^*, \eta(0) = 1$ (cf. Example 2), to get a particular solution $\psi \in \mathcal{F}_f$ of inequality (1).

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