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LOCAL SUBDIFFERENTIALS AND MULTIVARIATIONAL INEQUALITIES IN BANACH AND FRECHET SPACES

Abstract. Some functional-topological concepts of subdifferential and locally subdifferential maps in Frechet spaces are established. Multivariational inequalities with an operator of the pseudo-monotone type, connected with subdifferential maps, are considered.

Keywords: local subdifferential, multi-variational inequality, Frechet space.

Mathematics Subject Classification: 47J20, 49J40, 58E30, 54C05, 47H04.

1. INTRODUCTION AND NOTATION

Subdifferential maps play an important role in the non-smooth analysis and the optimization theory [1–3], in nonlinear boundary value problems for partial differential equations, the theory of control of the distributed systems [4,5], as well as the theory of differential games and mathematical economy [6,7]. For basic properties of such maps we refer the reader to [2,3,8]. In this paper we will generalize basic properties of subdifferentials and local subdifferentials known for Banach spaces to the case of Frechet spaces.

Let X be a Frechet space, X^* its topologically dual (adjoint) space. For $x \in X$ and $f \in X^*$, as usual, the symbol $\langle f, x \rangle$ stands for the bilinear pairing between X and X^*. Assume that X is endowed with the topology τ generated by a family of seminorms $\{\rho_i\}_{i=1}^{\infty}$ separating points of X. Recall that the topology τ is Hausdorff and metrizable by the metric

$$d(x, y) = \sum_{i=1}^{\infty} 2^{-i} \frac{\rho_i(x-y)}{1+\rho_i(x-y)}. \tag{1}$$

Observe that $d(x+h, y+h) = d(x, y)$, $d(\alpha x, \alpha y) < |\alpha| d(x, y)$ if $|\alpha| > 1$ and $d(\alpha x, \alpha y) \geq |\alpha| d(x, y)$ if $|\alpha| \leq 1$.

Let Y be a locally convex linear space and $T : Y \to X$ be a linear continuous map. Recall that the adjoint (dual) transformation $T^* : X^* \to Y^*$ is given by a formula $\langle x^*, Ty\rangle = \langle T^*x^*, y\rangle$ for $y \in Y$, $x^* \in X^*$. For the existence and uniqueness of such transformation, see [10] .

Throughout the paper, F stands for the functional $F : X \to \mathbb{R} \cup \{+\infty\}$ and the symbol dom F denotes the set $\{x \in X \mid F(x) < +\infty\}$.

Given a functional F and a convex body U such that $intU \subset domF$, a *local subdifferential* of F at the point $x_0 \in U \cap domF$ is, by definition, the set

$$\partial F(x_0; U) = \{\xi \in X^* \mid \langle \xi, x - x_0\rangle_X \le F(x) - F(x_0) \text{ for all } x \in U\}$$

Observe that $\partial F(x_0; U_1) \supset \partial F(x_0; U_2)$, if $U_1 \subset U_2$. In particular, $\partial F(x_0; X) = \partial F(x_0) \subset \partial F(x_0; U)$. The last set is called the *subdifferential* of F at the point x_0.

2. RESULTS

Proposition 1. *Let a functional $F : X \to \mathbb{R} \cup \{+\infty\}$ be given. Assume that there are a convex body U and a point $x_0 \in int\, U$ such that $\partial F(x_0; U) \neq \emptyset$. Then the functional F is weakly lower semicontinuous at x_0. Moreover, if $\partial F(x_0; U) \neq \emptyset$ for every $x \in U$, then F is convex on U.*

Proof. Let $\{x_\alpha\}$ be a net converging to $\{x_0\}$ and $W \subset U$ be a neighborhood of $\{x_0\}$. Obviously there exists α_0 such that $x_\alpha \in W$ for $\alpha \succeq \alpha_0$. Let $x^* \in \partial F(x_0, U)$. For $\alpha \succeq \alpha_0$, there is $\langle x^*, x_\alpha - x_0\rangle \le F(x_\alpha) - F(x_0)$. Passing with x_α to x_0, we deduce that $\varliminf_\alpha F(x_\alpha) \ge F(x_0)$.

Now suppose that $\partial F(x_0; U) \neq \emptyset$ for an arbitrary $x_0 \in U$. Fix $x_0 \in U$. For $x^* \in \partial F(x_0; U)$ and x_1, $x_2 \in U$, there is $F(x_1) - F(x_0) \ge \langle x^*, x_1 - x_0\rangle$, and $F(x_2) - F(x_0) \ge \langle x^*, x_2 - x_0\rangle$ for all $x_1, x_2 \in U$.

Let $t \in [0, 1]$. Adding the first inequality multiplied by t to the second one multiplied by $1 - t$, we obtain

$$tF(x_1) + (1 - t)F(x_2) \ge F(x_0) + \langle x^*, tx_1 + (1 - t)x_2 - x_0\rangle.$$

Since U is a convex set we can take $x_0 = tx_1 + (1 - t)x_2$. □

Proposition 2. *Let X be a Frechet space, Y a locally convex linear space, $F : X \to \mathbb{R} \cup \{+\infty\}$ and $T : Y \to X$ a linear continuous map admitting an adjoint map T^*. Let $U \subset X$ be a convex body and $V = T^{-1}(U)$. Then for every $Tv \in \mathrm{int}U$, where $v \in V$, there is*

$$\partial(F \circ T)(v; V) = T^*(\partial F(T(v; V))$$

Proof. Let $x \in \partial F(Tv; U)$. Obviously,

$$\langle x^*, x - Tv\rangle \le F(x) - F(T(v)) \quad \text{for every } x \in U.$$

Taking $x = T(y)$ with $y \in V$, we can rewrite the last inequality in the form

$$\langle x^*, Ty - Tv\rangle \le F(T(y)) - F(T(v)) \quad \text{for every } y \in V$$

or

$$\langle T^*x^*, y - v\rangle \le (F \circ T)(y) - (F \circ T)(v) \quad \text{for every } y \in V,$$

which means that $T^*x^* \in \partial(F \circ T)(v; V)$. Thus $\partial(F \circ T)(v; V) \supset T^*(\partial F(T(v); V))$.

To prove the inverse inclusion, take $y^* \in \partial(F \circ T)(v; V)$. Clearly

$$\langle y^*, y - v\rangle \le (F \circ T)(y) - (F \circ T)(v) \quad \text{for every } y \in V.$$

Taking $x^* \in X^*$ such that $y^* = T^*x^*$, we can rewrite the last inequality in the form

$$\langle x^*, Ty - Tv\rangle \le F(T(y)) - F(T(v)) \quad \text{for every } y \in V$$

or

$$\langle x^*, x - Tv\rangle \le F(x) - F(T(v)) \quad \text{for every } x \in U,$$

which means that $y^* = T^*x^* \in T^*(\partial F(T(v; V)))$ and this completes the proof. □

Theorem 1. *Let U be a convex body in X, $F : X \mapsto \mathbb{R} \cup \{+\infty\}$ be a convex functional on U and a lower semicontinuous functional on $int\, U$ ($int\, U \subset domF$). Then for every $x_0 \in int\, U$ and every $h \in X$, the quantity*

$$D_+F(x_0; h) = \lim_{t \to 0+} \frac{F(x_0 + th) - F(x_0)}{t} \tag{2}$$

is finite and the following statements hold true:

(i) *there exists a counterbalanced (cf. [12]) convex absorbing neighborhood of zero Θ ($x_0 + \Theta \subset int\, U$) such that for every $h \in \Theta$*

$$F(x_0) - F(x_0 - h) \le D_+F(x_0; h) \le F(x_0 + h) - F(x_0); \tag{3}$$

(ii) *the functional $int\, U \times X \ni (x; h) \mapsto D_+F(x; h)$ is upper semicontinuous;*

(iii) *the functional $D_+F(x_0; \cdot) : X \mapsto \mathbb{R}$ is positively homogeneous and semiadditive for every $x_0 \in int\, U$;*

(iv) *there exist a neighborhood $O(h_0)$ and a constant $c_1 > 0$ such that for every $x_0 \in int\, U$ and every $h_0 \in X$,*

$$|D_+F(x_0; h) - D_+F(x_0; h_0)| \le c_1 d(h, h_0) \quad \text{for every } h \in O(h_0).$$

Proof. First we introduce some auxiliary statements.

Claim 1. *The functional F is locally upper bounded on $int\, U$, that is for every $x_0 \in int\, U$ there exist positive constants r and c such that $F(x) \le c$, for each $x \in B_r(x_0)$, where $B_r(x_0) = \{x \in X \mid d(x, x_0) < r\}$.*

Proof of Claim 1. For arbitrary $x_0 \in int\, U$ there exists $\varepsilon_1 > 0$ such that $B_{2\varepsilon_1}(x_0) \subset int\, U \subset domF$, hence $\overline{B_{\varepsilon_1}(x_0)} \subset B_{2\varepsilon_1}(x_0) \subset domF$. Since F is lower semicontinuous, than for each $n = 1, 2, \ldots$ the set

$$A_n = \{x \in \overline{B_{\varepsilon_1}(x_0)} \mid F(x) \le n\}$$

is closed in X and

$$\bigcup_{n=1}^{+\infty} A_n = \overline{B_{\varepsilon_1}(x_0)} \subset dom\,F.$$

Since the metric space $(\overline{B_{\varepsilon_1}(x_0)}, d)$ is complete, due to the Baire Category Theorem, there exists $n_0 \in \mathbb{N}$ such that $intA_{n_0} \neq \emptyset$ in $\overline{B_{\varepsilon_1}(x_0)}$. We now prove that $intA_{n_0} \neq \emptyset$ in X. Since $intA_{n_0} \neq \emptyset$ in $\overline{B_{\varepsilon_1}(x_0)}$, we conclude that there exist $x_1 \in intA_{n_0}$ and $\varepsilon_2 > 0$ such that the following equality holds true:

$$A_{n_0} \supset \{x \in \overline{B_{\varepsilon_1}(x_0)} \mid d(x, x_1) < \varepsilon_2\} = \overline{B_{\varepsilon_1}(x_0)} \cap B_{\varepsilon_2}(x_1) \neq \emptyset.$$

Thus the following two cases are possible:

1) $B_{\varepsilon_1}(x_0) \cap B_{\varepsilon_2}(x_1) \neq \emptyset$;
2) $B_{\varepsilon_1}(x_0) \cap B_{\varepsilon_2}(x_1) = \emptyset$, $\partial\overline{B_{\varepsilon_1}(x_0)} \cap B_{\varepsilon_2}(x_1) \neq \emptyset$.

In the first case, the set $B_{\varepsilon_1}(x_0) \cap B_{\varepsilon_2}(x_1)$ is open in topology τ; therefore, there exist $x_2 \in X$ and $\varepsilon_3 > 0$ such that $B_{\varepsilon_3}(x_2) \subset B_{\varepsilon_1}(x_0) \cap B_{\varepsilon_2}(x_1) \subset A_{n_0}$. Thus, for each $x \in B_{\varepsilon_3}(x_2)$, there is $F(x) \leq n_0$. Hence $x_2 \in intA_{n_0}$ in X.

In the second case, for an arbitrary $x \in \partial\overline{B_{\varepsilon_1}(x_0)} \cap B_{\varepsilon_2}(x_1)$, there exists $\{x_n\}_{n\geq 1} \subset B_{\varepsilon_1}(x_0)$ such that $x_n \to x$ as $n \to +\infty$. Since $x \in B_{\varepsilon_2}(x_1)$, then there exists N such that for each $n \geq N$, $x_n \in B_{\varepsilon_2}(x_1)$. Therefore, $x_n \in B_{\varepsilon_1}(x_0) \cap B_{\varepsilon_2}(x_1) \neq \emptyset$, and we may proceed further as in the first case. Thus $intA_{n_0} \neq \emptyset$ in X.

Now we show that the functional F is upper bounded in some neighborhood of x_0. Let $x_2 \neq x_0$, $y = x_2 + \frac{x_0 - x_2}{1-\lambda}$, where $\lambda = \frac{\varepsilon_1/d(x_2,x_0)}{1+\varepsilon_1/d(x_2,x_0)}$. Therefore,

$$y = x_0 + \frac{\varepsilon_1}{d(x_2, x_0)}(x_0 - x_2),$$

$$d(y, x_0) = d\left(\frac{\varepsilon_1}{d(x_0, x_2)}(x_0 - x_2), 0\right) < \frac{\varepsilon_1}{d(x_0, x_2)} d(x_0, x_2) = \varepsilon_1,$$

that is $y \in B_{\varepsilon_1}(x_0) \subset domF$. For an arbitrary $x \in B_{\lambda\varepsilon_3}(x_0)$, we consider $z = (x + \lambda x_2 - x_0)/\lambda = (x - (1-\lambda)y)/\lambda$. Since $0 < \lambda < 1$, we conclude that

$$d(z, x_2) = d\left(x_2 + \frac{x - x_0}{\lambda}, x_2\right) = d\left(\frac{x - x_0}{\lambda}, 0\right) < \frac{1}{\lambda} d(x, x_0) < \frac{\lambda\varepsilon_3}{\lambda} = \varepsilon_3,$$

hence $z \in B_{\varepsilon_3}(x_2)$, and $F(z) \leq n_0$. Due to convexity of F, there is

$$F(x) = F(\lambda z + (1-\lambda)y) \leq \lambda F(z) + (1-\lambda)F(y) \leq n_0 + (1-\lambda)F(y).$$

From this we conclude that F is upper bounded in the neighborhood $B_r(x_0)$ with $r = \lambda\varepsilon_3$ and $c = n_0 + (1-\lambda)F(y)$. □

Claim 2. *The functional F is locally Lipschitzean on $int\,U$, i.e., for every $x_0 \in int\,U$ there exist $r_1 > 0$ and $c_1 > 0$ such that*

$$|F(x) - F(y)| \leq c_1 d(x, y) \quad \textit{for all } \ x, y \in B_{r_1}(x_0).$$

Proof of Claim 2. The local upper boundedness of the functional F on $int\, U$ follows from Claim 1. Therefore, for every $x_0 \in int\, U$ there exist $r > 0$ and $c > 0$ such that $F(x) \le c$ for every $x \in B_r(x_0)$.

For an arbitrary $x \in B_r(x_0)$ $(x \neq x_0)$ and $t = \frac{d(x,x_0)}{r+d(x,x_0)}$, we put

$$y = \frac{x_0 + (t-1)x}{t} = x_0 + \frac{1-t}{t}(x_0 - x),$$

where $t \in (0,1)$. Then

$$d(y, x_0) = d\left(\frac{1-t}{t}(x_0 - x), 0\right) = d\left(\frac{r}{d(x,x_0)}(x_0 - x), 0\right) < \frac{r}{d(x,x_0)} d(x, x_0) = r,$$

i.e., $F(y) \le c$. Due to convexity of F,

$$F(x_0) = F(ty + (1-t)x) \le tF(y) + (1-t)F(x) \le tc + (1-t)F(x),$$

or $(1-t)F(x_0) \le t(c - F(x_0)) + (1-t)F(x)$. Hence

$$F(x_0) - F(x) \le \frac{t}{1-t}(c - F(x_0)) = \frac{(c - F(x_0))}{r} d(x, x_0). \tag{4}$$

Now let $z = \frac{x-(1-\tau)x_0}{\tau} = x_0 + \frac{x-x_0}{\tau}$, where $\tau = \frac{d(x,x_0)}{r} \in (0,1)$. Then

$$d(z, x_0) = d\left(\frac{x - x_0}{\tau}, 0\right) < \frac{1}{\tau} d(x, x_0) = r,$$

i.e., $F(z) \le c$, and since F is convex, we obtain

$$F(x) = F(\tau z + (1-\tau)x_0) \le \tau F(z) + (1-\tau)F(x_0) \le \tau c + (1-\tau)F(x_0)$$

or

$$F(x) - F(x_0) \le \tau(c - F(x_0)) = \frac{c - F(x_0)}{r} d(x, x_0). \tag{5}$$

Relations (4) and (5) imply the following estimate

$$|F(x) - F(x_0)| \le \frac{c - F(x_0)}{r} d(x, x_0). \tag{6}$$

Now we show that the Lipschitz condition holds true for F on $B_{\varepsilon_1}(x_0)$ with $\varepsilon_1 = r/3$. Hence in view of (6), for all $x_1, x_2 \in B_{3\varepsilon_1}(x_0)$ $F(x_1) \le c$, $F(x_2) \le c$. If $x_1 \in B_{\varepsilon_1}(x_0)$, then $B_{2\varepsilon_1}(x_1) \subset B_r(x_0)$, that is $x_1 \in int\, U$. Therefore, from (6) we obtain

$$|F(x) - F(x_1)| \le \frac{c - F(x_1)}{2\varepsilon_1} d(x, x_1) \quad \text{for every} \quad x \in B_{2\varepsilon_1}(x_1). \tag{7}$$

In particular, inequality (7) is valid for an arbitrary element of $B_{\varepsilon_1}(x_0)$. Further, due to (6)

$$-F(x_1) \le (c - F(x_0)) + |F(x_1) - F(x_0)| \le$$
$$\le (c - F(x_0)) + \frac{c - F(x_0)}{r} d(x_1, x_0) < 2(c - F(x_0)).$$

From the last relation, using (7), we finally obtain

$$|F(x_2) - F(x_1)| \le \frac{c - F(x_0)}{\varepsilon_1} d(x_2, x_1) \quad \text{for all } \; x_1, x_2 \in B_{\varepsilon_1}(x_0),$$

i.e., $c_1 = \frac{c - F(x_0)}{\varepsilon_1}$, $r_1 = \varepsilon_1$. □

Now we continue to prove Theorem 1. Let $x_0 \in int\, U$ and $B_r(x_0) = x_0 + B_r(0)$. Then due to Claims 1 and 2 the upper boundness and the Lipschitz condition for F on $B_r(x_0)$ follow. We recall that, unlike in the case of a Banach space, $B_r(0)$ is not absolutely convex, but at the same time there exists a convex absorbing counterbalanced set $\Theta = \Theta(x_0)$ in a basis of topology τ, such that $\Theta \subset B_r(0)$. Then $F(x) \le c$, for every $x \in x_0 + \Theta$,

$$|F(x_1) - F(x_2)| \le c_1 d(x_1, x_2) \quad \text{for all } x_1, x_2 \in x_0 + \Theta. \tag{8}$$

For each $u \in X$ there exists $t = t(u) > 0$ such that $t^{-1}u \in \Theta$ (if $u \in \Theta$, then we take $t = 1$). So for each $\tau \in (0, t^{-1}]$ the element $\tau u \in \Theta$, as $t\Theta \subset \frac{1}{\tau}\Theta$. Further, due to convexity of F, for every $\tau_1, \tau_2 \in \mathbb{R}$ such that $0 < \tau_1 \le \tau_2 \le t^{-1}$, there follows:

$$\begin{aligned} F(x_0 + \tau_1 u) - F(x_0) &= F\left(x_0\left(1 - \frac{\tau_1}{\tau_2}\right) + (x_0 + \tau_2 u)\frac{\tau_1}{\tau_2}\right) - F(x_0) \le \\ &\le \left(1 - \frac{\tau_1}{\tau_2}\right) F(x_0) + \frac{\tau_1}{\tau_2} F(x_0 + \tau_2 u) - F(x_0) = \\ &= \frac{\tau_1}{\tau_2}(F(x_0 + \tau_2 u) - F(x_0)). \end{aligned}$$

Hence the function $\tau \mapsto \frac{F(x_0 + \tau u) - F(x_0)}{\tau}$ monotonely decreases as $\tau \to 0+$.

For each $u \in \Theta$, the quantity $D_+F(x_0; u)$ is finite. In fact, $\alpha u \in \Theta$ for every α such that $|\alpha| \le 1$, therefore

$$\begin{aligned} D_+F(x_0; u) = \inf_{\tau > 0} \frac{F(x_0 + \tau u) - F(x_0)}{\tau} &\le \\ \le F(x_0 + u) - F(x_0) < +\infty \text{ as } x_0 + u \in B_r(x_0) \subset int\, U \subset dom\, F. \end{aligned}$$

On the other hand, for every $\tau \in (0, 1)$ $\;x_0 = \frac{1}{1+\tau}(x_0 + \tau u) + \frac{\tau}{1+\tau}(x_0 - u)$, i.e., $-u \in \Theta$, and moreover, for each $\tau \in (0, 1)$,

$$-\infty < F(x_0) - F(x_0 - u) \le \frac{F(x_0 + \tau u) - F(x_0)}{\tau}.$$

Thus, for each $u \in \Theta$,

$$-\infty < F(x_0) - F(x_0 - u) \le D_+F(x_0; u) \le F(x_0 + u) - F(x_0) < +\infty,$$

i.e., $D_+F(x_0; u) \in \mathbb{R}$ for every $u \in \Theta$. The validity of (3) follows from these facts.

From (2) we immediately obtain

$$D_+F(x_0; \alpha u) = \alpha D_+F(x_0; u) \quad \text{for all } \; \alpha > 0 \text{ and } u \in X, \tag{9}$$

and since the set Θ is absorbing, then for each $u \in X$ there is $\alpha > 0$ such that $\alpha u \in \Theta$. Then from (9) we obtain

$$D_+F(x_0; u) \in \mathbb{R} \quad \text{for all } x_0 \in int\, U \text{ and } u \in X.$$

Now taking $t > 0$, we consider the function

$$int\, U \times \Theta \ni (x, h) \mapsto F_t(x; h) = \frac{F(x + th) - F(x)}{t}. \tag{10}$$

Claim 3. *For each pair $(x_0; h_0) \in int\, U \times X$ there exists $l > 0$ such that for each $t \in (0, l)$ the function $F_t(\cdot\,;\cdot)$ is continuous at the point $(x_0; h_0)$.*

Proof of Claim 3. Let $h_0 \in X$ be arbitrary, $x_0 \in int\, U$ (the set $\Theta = \Theta(x_0)$ is defined above), then there exists $t_0 > 0$ such that $h_0 \in t_0\Theta$. For $t \in (0, l)$, putting $l = \min(\frac{1}{2t_0}, 1)$, we consider the function $F_t(x; h)$ in a neighborhood of the point (x_0, h_0) :

$$\begin{aligned} |F_t(x; h) - F_t(x_0; h_0)| = \frac{1}{t}\Big|\big[F(x_0 + th_0 + (x - x_0) + t(h - h_0)) - \\ - F(x_0 + th_0)\big] + \big[F(x_0) - F(x)\big]\Big| \end{aligned} \tag{11}$$

If we take $x \in x_0 + \frac{1}{4}\Theta$, $h \in h_0 + \frac{1}{4}\Theta$, then

$$x_0 + th_0 \in x_0 + \frac{1}{2}\Theta \subset x_0 + \Theta, \quad t(h - h_0) \in \frac{1}{4}\Theta,$$

$$(x - x_0) + t(h - h_0) \in \frac{1}{2}\Theta, \quad x_0 + th_0 + (x - x_0) + t(h - h_0) \in x_0 + \Theta.$$

From (11) using (8), we derive

$$|F_t(x; h) - F_t(x_0; h_0)| \leq \frac{c_1}{t}\left(d(x + th, x_0 + th_0) + d(x, x_0)\right) \to 0$$

as $x \to x_0$, $h \to h_0$. □

Claim 3 implies the upper semicontinuity of the map

$$int\, U \times X \ni (x; h) \mapsto D_+F(x; h) = \inf_{t>0} F_t(x; h) = \inf_{t \in (0,l)} F_t(x; h),$$

since it is a "pointwise infimum" of continuous functions. The positive homogeneity of $D_+F(x_0; \cdot)$ is obvious. Now we show that this map is semiadditive. Indeed, for all $v_1, v_2 \in X$

$$\begin{aligned} D_+F(x_0; v_1 + v_2) &= \inf_{t>0} \frac{F(x_0 + t(v_1 + v_2)) - F(x_0)}{t} = \\ &= \lim_{t\to 0+} \frac{2F(\frac{x_0 + tv_1}{2} + \frac{x_0 + tv_2}{2}) - 2F(x_0)}{t} \leq \\ &\leq \lim_{t\to 0+} \frac{F(x_0 + tv_1) - F(x_0)}{t} + \lim_{t\to 0+} \frac{F(x_0 + tv_2) - F(x_0)}{t} = \\ &= D_+F(x_0; v_1) + D_+F(x_0; v_2). \end{aligned}$$

In order to complete the proof it suffices to show that the map $D_+F(x_0;\cdot)$ satisfies (iv). From semiadditivity it follows that

$$|D_+F(x_0;h) - D_+F(x_0;h_0)| \le \max\{D_+F(x_0;h-h_0), D_+F(x_0;h_0-h)\} \le c_1 d(h,h_0)$$

for any $h \in h_0 + \frac{1}{4}\Theta$. This completes the proof of Theorem 1. □

Definition 1. *We call a set $B \subset X^*$ bounded in the $\sigma(X^*;X)$ topology (*-bounded), if $\sup_{y\in B} |\langle y,x\rangle_X| < +\infty$ for each $x \in X$.*

It is obvious that each bounded set in X^* is *-bounded.

Definition 2. *A multivalued map $A : X \rightrightarrows X^*$ is called:*

a) **-bounded, if for any bounded set B in X the image $A(B)$ is *- bounded in X^*;*
b) **-upper semicontinuous, if for any set B open in the $\sigma(X^*,X)$ topology the set $A_M^{-1}(B) = \{x \in X \mid A(x) \subset B\}$ is open in X ;*
c) *upper hemicontinuous, if the function*

$$X \ni x \mapsto [A(x),y]_+ = \sup_{d\in A(x)} \langle d,y\rangle_X$$

is upper semicontinuous for each $y \in X$.

Let us note that c) follows from b).

Theorem 2. *Let U be a convex body and $int\, U \subset \mathrm{dom}F$, where $F : X \to \overline{\mathbb{R}}$ is a convex functional on U and a semicontinuous function on $int\, U$. Then:*

i) *$\partial F(x;U)$ is a nonempty convex compact set for every $x \in int\, U$ in the $\sigma(X^*;X)$ topology;*
ii) *$\partial F(\cdot;U) : U \rightrightarrows X^*$ is a monotone map (on U);*
iii) *the map $int\, U \ni x \mapsto \partial\varphi(x;U) \subset X^*$ is *-upper semicontinuous (on $intU$) and*

$$[\partial\varphi(x_0;U),h]_+ = D_+\varphi(x_0;h) \quad \text{for all } h \in X \text{ and } x_0 \in int\, U. \qquad (12)$$

Proof. First we prove condition ii). Let $x_1,x_2 \in U$ and $\xi_i \in \partial F(x_i;U)$, $i = 1,2$. Then

$$F(x_2) - F(x_1) \ge \langle\xi_1, x_2 - x_1\rangle_X,\ F(x_1) - F(x_2) \ge \langle\xi_2, x_1 - x_2\rangle_X.$$

Adding the first inequality to the second, we obtain

$$\langle\xi_1 - \xi_2, x_1 - x_2\rangle_X \ge 0,$$

or

$$[\partial F(x_1;U), x_1 - x_2]_- \ge [\partial F(x_1;U), x_1 - x_2]_+ \quad \text{for all } x_1,x_2 \in U.$$

The last relation proves the monotonicity on U. Convexity and weak star closure are obvious. Let us prove nonemptiness. Let us set an arbitrary $x,h \in int\, U$ and consider the real convex function $\varphi(t) = F(x + t(h - x))$ defined on $[0, 1]$. So there exist $\varphi(t-)$, $\varphi(t+)$ such that

$$\varphi(t-) \leq \varphi(t+) = \lim_{t \to +0} \frac{\varphi(t) - \varphi(0)}{t} = D_+\varphi(x; x - h),$$

or

$$D_-\varphi(x; x - h) \leq D_+\varphi(x; x - h),$$

where $D_-\varphi(x; v) = -D_+\varphi(x; -v)$. From Theorem 1 it follows that

$$\frac{\varphi(\alpha) - \varphi(0)}{\alpha} \leq \varphi(1) - \varphi(0) \quad \text{for every} \quad \alpha \in (0, 1)$$

or

$$D_+F(x; h - x) \leq F(h) - F(x) \quad \text{for all} \quad x, h \in int\, U. \tag{13}$$

Claim 4. *For arbitrary $x \in int\, U$ there exists $\xi(x) \in X^*$ such that*

$$D_-F(x; h) \leq \langle \xi(x), h \rangle_X \leq D_+F(x; h) \quad \text{for every } h \in X.$$

Proof of Claim 4. Let us fix $h_0 \in X$ and consider the one-dimensional subspace $X_0 = \{\alpha h_0 \mid \alpha \in \mathbb{R}\}$. Let us choose an element $\xi \in X^*$ satisfying the following condition

$$\langle \xi, \alpha h_0 \rangle_X = D_+F(x, \alpha h_0), \quad \alpha \geq 0$$

(We remark that since x is an interior point of U, then due to Theorem 1 for every $h \in X$ there exists $D_+F(x; h)$). It is possible to choose ξ in such a way, since $X \ni h \mapsto D_+F(x; h)$ is a positively homogeneous functional. Further, taking into account the semiadditivity of $X \ni h \mapsto D_+F(x; h)$, we obtain

$$0 = D_+F(x; h - h) \leq D_+F(x; h) + D_+F(x; -h)$$

or

$$-D_+F(x; h) \leq D_+F(x; -h). \tag{14}$$

Then for $\alpha < 0$, from (14), the following relation follows:

$$\begin{aligned} \langle \xi, \alpha h_0 \rangle_X &= \alpha D_+F(x; h_0) = -|\alpha| D_+F(x; h_0) \leq \\ &\leq |\alpha| D_+F(x; -h_0) = D_+F(x; -|\alpha| h_0) = D_+F(x; \alpha h_0). \end{aligned}$$

Since $\langle \xi, v \rangle_X \leq D_+F(x; v)$ for each $v \in X_0$ and $X \ni h \mapsto D_+F(x; h)$ is a continuous positively homogeneous semiadditive functional, then according to the Hahn-Banach Theorem there exists $\zeta \in X^*$ such that $\langle \zeta, h \rangle_X \leq D_+F(x; h)$ for each $h \in X$ and $\langle \zeta, h_0 \rangle_X = \langle \xi, h_0 \rangle_X$. Hence we obtain $\langle \zeta, -h \rangle_X \leq D_+F(x; -h)$ and

$$\langle \zeta, h \rangle_X = -\langle \zeta, -h \rangle_X \geq -D_+F(x; -h) = D_-F(x; h) \quad \text{for every } h \in X.$$

The last relation proves the required inequality. □

Claim 4 and inequality (13) guarantee the existence of $\xi(x) \in X^*$ such that

$$\langle \xi(x), h - x\rangle_X \leq D_+F(x; h - x) \leq F(h) - F(x) \quad \text{for every} \ \ h \in U,$$

i.e., $\xi(x) \in \partial F(x, U)$, and hereby the nonemptiness of $\partial F(x, U)$ is proved.

Claim 5. *For every $x_0 \in int\, U$, the following inequality holds true:*

$$\partial\varphi(x_0; U) = \{p \in X^* \mid \langle p, h\rangle_X \leq D_+\varphi(x_0; h) \ \ \textit{for every} \ \ h \in X\}.$$

Proof of Claim 5. Let $p \in \partial F(x_0; U)$. Then there exists an open convex set V containing zero such that $x_0 + V \subset int\, U$ and

$$\langle p, h\rangle_X \leq F(x_0 + h) - F(x_0) \quad \text{for every} \ \ h \in V.$$

Hence,

$$\langle p, h\rangle_X \leq \frac{F(x_0 + th) - F(x_0)}{t} \quad \text{for every} \ \ t \in (0, 1).$$

Due to Theorem 1,

$$\langle p, h\rangle_X \leq \inf_{t>0} \frac{F(x_0 + th) - F(x_0)}{t} = D_+F(x_0; h) \quad \text{for every} \ \ h \in V.$$

Since the set V is absorbing and functions

$$X \ni h \mapsto D_+F(x; h), \quad X \ni h \mapsto \langle p, h\rangle_X$$

are positively homogeneous, then

$$\langle p, h\rangle_X \leq D_+F(x_0; h) \quad \text{for every} \ \ h \in X.$$

On the other hand, let for every $h \in X$ the relation $\langle p, h\rangle_X \leq D_+F(x_0; h)$ hold true. Due to Theorem 1, there follows the existence of a counterbalanced convex absorbing neighborhood of zero Θ ($x_0 + \Theta \subset int\, U$) such that

$$D_+F(x_0; v) \leq F(x_0 + v) - F(x_0) \quad \text{for every} \ \ v \in \Theta.$$

Let us fix an arbitrary $h \in U \cap domF$. Then there is $\alpha \in (0, 1)$ such that $\alpha(h - x_0) \in \Theta$. Therefore,

$$\begin{aligned} \alpha \cdot \langle p, h - x_0\rangle_X = \langle p, \alpha(h - x_0)\rangle_X \leq D_+F(x_0; \alpha(h - x_0)) \leq F(x_0 + \alpha(h - x_0)) - \\ - F(x_0) \leq \alpha F(h) + (1 - \alpha)F(x_0) - F(x_0) = \alpha(F(h) - F(x_0)). \end{aligned}$$

Hence we obtain that $\langle p, h - x_0\rangle_X \leq F(h) - F(x_0)$ for each $h \in U \cap domF$, and for this reason $\langle p, h - x_0\rangle_X \leq F(h) - F(x_0)$ for each $h \in U$. Hence $p \in \partial F(x_0; U)$. □

By Claim 5, it immediately follows that

$$[\partial F(x_0; U), h]_+ \leq D_+F(x_0; h) \quad \text{for every} \ \ h \in X,$$

that is, due to Claim 5,

$$\{p \in X^* \mid \langle p, h - x_0 \rangle_X \le [\partial F(x_0;U), h - x_0]_+ \ \text{ for every } \ h \in X\} \subset \\ \subset \{p \in X^* \mid \langle p, h \rangle_X \le D_+F(x_0; h - x_0) \ \text{ for every } \ h \in X\} = \partial F(x_0;U).$$

On the other hand, every element $p \in \partial F(x_0;U)$ satisfies the condition

$$\langle p, h \rangle_X \le [\partial F(x_0;U), h]_+ \ \text{ for every } h \in X,$$

which proves the inverse inclusion. Therefore, equality (12) holds.

Further, due to (12) and Theorem 1, $\partial F(\cdot;U)$ is upper hemicontinuous on $int\, U$. Moreover, the boundedness of $\partial F(x_0;U)$ follows from the estimate

$$[\partial F(x_0;U), h]_+ = D_+F(x_0, h) \le c_1 d(h, 0) \ \text{ for every } h \in \Theta,$$

where Θ is absorbing. So, by virtue of the Banach-Alaoglu Theorem (cf. [10]), $\partial F(x_0;U)$ is a compact set in the $\sigma(X^*, X)$ topology. Under these conditions, upper hemicontinuity of the map $\partial F(\cdot;U)$ and the Castaing Theorem (cf. [2]) imply *-upper semicontinuity of $\partial F(\cdot;U)$ on $int\, U$. This completes the proof of Theorem 2. □

Theorem 3. *Let $F_1, F_2 : X \to \overline{\mathbb{R}}$ and $U = U_1 \cap U_2$, where $int\, U \ne \emptyset$, U_1, U_2 are convex sets and*

$$\partial F_1(x_1;U_1) \ne \emptyset, \ \partial F_2(x_2;U_2) \ne \emptyset \quad \textit{for all } \ x_1 \in U_1, x_2 \in U_2.$$

Then $\partial F(x;U) \ne \emptyset$ for every $x \in U$, where $F = F_1 + F_2$, and

$$\partial F(x;U) = \partial F_1(x;U) + \partial F_2(x;U) \quad \textit{for every } \ x \in intU.$$

Proof. Suppose that $x \in U$. It is clear that

$$\partial F(x;U) \supset \partial F_1(x;U) + \partial F_2(x;U) \supset \partial F_1(x;U_1) + \partial F_2(x;U_2) \ne \emptyset.$$

In order to complete the proof, it is necessary to show that for every $x \in int\, U$ and for every $h \in X$ the following equality is fulfilled:

$$D_+F(x;h) = D_+F_1(x;h) + D_+F_2(x;h). \tag{15}$$

Indeed, since functions F, F_1, F_2 satisfy assumptions of Proposition 1, then all conditions of Theorem 2 hold true for them as well. Thus, due to equality (12) and [11, Proposition 1],

$$[\partial F(x;U), h]_+ = D_+F(x;h) = D_+F_1(x;h) + D_+F_2(x;h) = \\ = [\partial F_1(x;U), h]_+ + [\partial F_2(x;U), h]_+ = \\ = [\partial F_1(x;U) + \partial F_2(x;U), h]_+ \ \text{ for all } \ x \in int\, U \text{ and } h \in X.$$

Hence

$$\partial F(x;U) = \partial F_1(x;U) + \partial F_2(x;U) \quad \text{for every } \ x \in int\, U.$$

Now we prove (15). For functions $F,\ F_1,\ F_2$, due to Proposition 1, Theorem 1 holds true. Consequently, for all $x \in int\, U$ and $h \in X$, we obtain

$$\begin{aligned} D_+F(x;h) &= \lim_{t\to 0+} \frac{F(x+th)-F(x)}{t} = \\ &= \lim_{t\to 0+} \frac{F_1(x+th)-F_1(x)+F_2(x+th)-F_2(x)}{t} = \\ &= \lim_{t\to 0+} \frac{F_1(x+th)-F_1(x)}{t} + \lim_{t\to 0+} \frac{F_2(x+th)-F_2(x)}{t} = \\ &= D_+F_1(x;h)+D_+F_2(x;h). \end{aligned}$$

This completes the proof of Theorem 3. □

Definition 3. *Suppose that U is a convex body. The functional $F : X \mapsto \mathbb{R} \cup \{+\infty\}$ ($int\, U \subset dom F$) is said to be upper bounded on $int\, U$ if for every bounded set $B \subset int\, U$ the image $F(B)$ is upper bounded in $\mathbb{R}$.*

The following result is new even in the case of X being a Banach space.

Theorem 4. *Let $F : X \mapsto \mathbb{R}$ be a convex lower semicontinuous functional. Then the following statements are equivalent:*

a) *F is an upper bounded functional on X ;*
b) *a multivalued map $\partial F(\cdot) = \partial F(\cdot; X)$ is *-bounded on X.*

Proof. The following statements are true.

Claim 6. *If B is a bounded set in X and C is a *-bounded set in X^*, then the quantity $\sup_{x\in B}\sup_{p\in C} |\langle p, x\rangle_X|$ is finite.*

Proof of Claim 6. Let $\rho(x) = \sup_{p\in C} |\langle p, x\rangle_X|$. *-boundedness of C implies that the given functional is well defined on X. We remark that $\rho(-x) = \rho(x)$ for $x \in X$. Moreover, ρ is convex positively homogeneous and lower semicontinuous as the supremum of convex positively homogeneous continuous functionals. Hence, due to Claim 2, ρ is continuous on X, i.e., ρ is a continuous seminorm on X. By Theorem V.23 in [12], the boundedness of B in X implies that $\sup_{x\in B}\sup_{p\in C} |\langle p, x\rangle_X| = \sup_{x\in B} \rho(x) < +\infty$. □

Definition 4. *Let X be a separable locally convex topological space, $U \subset X$ be an unbounded convex body. Then the functional $F : U \mapsto \mathbb{R} \cup \{+\infty\}$ is called coercive on U if $F(x) \to +\infty$ as $\rho(x) \to +\infty$, $x \in U$, where ρ an arbitrary continuous seminorm on X.*

Claim 7. *Let $B \subset X$ be a nonempty set satisfying one of the two conditions:*

(i) *B is bounded,*
(ii) *F is coercive on B.*

Then $\inf_{x\in B} F(x) > -\infty$.

Proof of Claim 7. For some integer n, we consider the following set:

$$A_n = \{x \in B \mid F(x) \leq n\} \neq \emptyset.$$

The boundedness of A_n follows from the boundedness of B or coercivity of F. Indeed, if the set A_n is unbounded, then there exists a continuous seminorm ρ and a sequence $\{x_n\}_{n\geq 1} \subset B$ such that $\rho(x_n) \to +\infty$. Thus we obtain $F(x_n) \to +\infty$, and this fact contradicts the construction of A_n. Therefore, taking into account Theorem 2 and Claim 6 with $C = \{p\}$, $p \in \partial F(\bar{0})$, we deduce that $\inf_{x\in B} F(x) \geq F(\bar{0}) - \sup_{x\in B} |\langle p, x\rangle| > -\infty$. This completes the proof. □

We continue with the proof of Theorem 4. Let the set B be bounded in X. First we assume that the multivalued map $\partial F(\cdot)$ is *-bounded on X. Then, by definition of a subdifferential,

$$F(x_0) - F(x) \geq \langle p_x, x_0 - x\rangle_X \quad \text{for all } x \in B \text{ and } p_x \in \partial F(x).$$

Whence for all $x \in B$ and $p_x \in \partial F(x)$, we obtain

$$F(x) \leq F(x_0) + \langle p_x, x - x_0\rangle_X \leq |F(x_0)| + \sup_{p\in\partial F(B)} |\langle p, x - x_0\rangle_X| \leq$$
$$\leq |F(x_0)| + \sup_{x\in x_0+B}\ \sup_{p\in\partial F(B)} |\langle p, x\rangle_X|.$$

Claim 6 and the fact that $x_0 + B$ is the bounded set in X yield

$$\sup_{x\in x_0+B}\ \sup_{p\in\partial F(B)} |\langle p, x\rangle_X| < +\infty.$$

Moreover, let the functional F be upper bounded. Then, due to Theorem 2, for every $u \in X$ there is

$$\sup_{p\in\partial F(B)} |\langle p, u\rangle_X| = \sup_{x\in B}\ \sup_{p\in\partial F(x)} \langle p, u\rangle_X = \sup_{x\in B}[\partial F(x), u]_+ = \sup_{x\in B} D_+F(x; u).$$

Further, from Theorem 1 we infer that

$$\sup_{x\in B} D_+F(x; u) \leq \sup_{x\in B}(F(x + u) - F(x)) \leq \sup_{x\in B+u} F(x) - \inf_{x\in B} F(x) =: I.$$

Since B, $B+u$ are bounded sets in X, then (due to Claim 7 and the definition of an upper bounded functional) the quantity I is finite. Consequently, $\sup_{p\in\partial F(B;U)}\langle p, u\rangle_X < +\infty$ for every $u \in X$. Hence, the set $\partial F(B)$ is *-bounded. □

Remark 1. *For an arbitrary multivalued map $A : Y \subset X \rightrightarrows X^*$, coA and $\overline{co}A$ stand for multivalued maps defined as follows: $coA(y) := co(A(y))$, $\overline{co}A(y) := \overline{co}(A(y))$ for every $y \in Y$.*

Remark 2. *Claim 7 holds true if X is reflexive, but $F : X \to \mathbb{R} \cup \{+\infty\}$ is weakly lower semicontinuous.*

Corollary 1. *Let $\varphi_1, \varphi_2 : X \mapsto \mathbb{R}$ be lower semicontinuous convex functionals upper bounded on X. Then $\partial\varphi_1 + \partial\varphi_2 : X \rightrightarrows X^*$ is a *-bounded *-upper semicontinuous map with compact values in the $\sigma(X^*, X)$ topology.*

Proof. The map $G = \partial\varphi_1 + \partial\varphi_2$ is upper hemicontinuous, since it is the sum of upper hemicontinuous maps. Also, $\partial\varphi_i = \overline{co}\partial\varphi_i$ $(i = 1, 2)$. Now we prove that $\overline{co}G = G$. As $coG = G$, i.e., $\overline{co}G \supset \partial\varphi_1 + \partial\varphi_2 = G$, it remains to prove the inverse inclusion. Let $u \in \overline{co}G(y)$, then there exists a net $\{u_\alpha\} \in G(y)$ such that $u_\alpha \to u$ in X^*, and $u_\alpha = u_\alpha^{'} + u_\alpha^{''}$, where $u_\alpha^{'} \in \partial\varphi_1(y)$, $u_\alpha^{''} \in \partial\varphi_2(y)$. Since $\partial\varphi_1(y)$, $\partial\varphi_2(y)$ are compact sets in $\sigma(X^*, X)$-topology, we deduce that $u = u' + u''$, $u' \in \partial\varphi_1(y)$, $u'' \in \partial\varphi_2(y)$, i.e., $\overline{co}G(y) \subset G(y)$.

Thus, G satisfies all conditions of the Castaing Theorem, whence *-upper semicontinuity of the map $\partial\varphi_1 + \partial\varphi_2$ follows. The *-boundedness of the map $\partial\varphi_1 + \partial\varphi_2$ follows from a similar statement for $\partial\varphi_1$ and $\partial\varphi_2$.

For an arbitrary bounded set B, images $\partial\varphi_1(B)$ and $\partial\varphi_2(B)$ are *-bounded in X^*. Then

$$\begin{aligned}\sup_{g\in\partial\varphi_1(B)+\partial\varphi_2(B)} |\langle g, x\rangle_X| &= \sup_{g_1\in\partial\varphi_1(B)} \sup_{g_2\in\partial\varphi_2(B)} |\langle g_1 + g_2, x\rangle_X| \le \\ &\le \sup_{g_1\in\partial\varphi_1(B)} |\langle g_1, x\rangle_X| + \sup_{g_2\in\partial\varphi_2(B)} |\langle g_2, x\rangle_X| < \\ &< +\infty \qquad \text{for every } x \in X,\end{aligned}$$

i.e., $\partial\varphi_1 + \partial\varphi_2$ is a *-bounded set in X^*. □

Let us define

$$\varphi(y) = \varphi_1(y) + \varphi_2(y) \ - \langle f, y\rangle_X, \tag{16}$$

where U is a nonempty convex set, $f \in X^*$, $\varphi_1 : X \mapsto \mathbb{R} \cup \{+\infty\}$ is a convex upper semicontinuous functional on X $(int\ dom\varphi_1 \neq \emptyset)$, $\varphi_2 : X \mapsto \mathbb{R} \cup \{+\infty\}$ is a convex functional on U and $dom\varphi_1 \subset dom\varphi_2$.

The following results are true.

Theorem 5. *Under the above assumptions, the following conditions are equivalent:*

1) $x_0 \in int\ dom\, \varphi_1 \cap U$, $\ \varphi(x_0) = \inf\limits_{x\in U} \varphi(x)$;

2) $x_0 \in int\ dom\, \varphi_1 \cap U$, $\ [\partial\varphi_1(x_0; U), x - x_0]_+ +$

$$+\varphi_2(x) - \varphi_2(x_0) \ge \langle f, x - x_0\rangle_X \quad \textit{for every } \ x \in U. \tag{17}$$

Proof. First we prove that 1) ⇒ 2). Let a point $x_0 \in int\ dom\, \varphi_1 \cap U$ satisfy condition 1). Then for all $x \in U$ and all $t \in [0, 1]$ there is

$$\begin{aligned}\varphi(x_0) &= \varphi_1(x_0) + \varphi_2(x_0) - \langle f, x_0\rangle_X \le \\ &\le \varphi_1(x_0 + t(x - x_0)) + \varphi_2(x_0 + t(x - x_0)) - \langle f, x_0 + t(x - x_0)\rangle_X \le \\ &\le \varphi_1(x_0 + t(x - x_0)) + t\varphi_2(x) + (1 - t)\varphi_2(x_0) - t\langle f, x - x_0\rangle_X.\end{aligned}$$

Hence,

$$\frac{\varphi_1(x_0 + t(x - x_0)) - \varphi_1(x_0)}{t} + \varphi_2(x) - \varphi_2(x_0) \ge \langle f, x - x_0\rangle_X$$

or, passing to a limit as $t \to +0$,

$$D_+\varphi_1(x_0; x - x_0) + \varphi_2(x) - \varphi_2(x_0) \geq \langle f, x - x_0 \rangle_X.$$

Then, due to relation (12), we arrive at inequality (17).

To prove the inverse implication, assume that inequality (17) holds. By the definition of $\partial\varphi_1(x_0; U)$, we obtain

$$\begin{aligned} \varphi_1(x) - \varphi_1(x_0) + \varphi_2(x) - \varphi_2(x_0) &\geq [\partial\varphi_1(x_0; U), x - x_0]_+ + \varphi_2(x) - \varphi_2(x_0) \geq \\ &\geq \langle f, x - x_0 \rangle_X \quad \text{for every } x \in U \end{aligned}$$

i.e., $\varphi(x) \geq \varphi(x_0)$, which is equivalent to 1). This completes the proof of Theorem 5. □

Remark 3. *In the literature, inequality* (17) *is called a variational inequality with a multivalued map. In Banach spaces, such maps are being actively studied.*

Theorem 6. *Let X be a reflexive space and the functional φ be of the form* (16). *Let it be coercive and satisfy all conditions of Theorem 5. Let $U \subset dom\,\varphi = X$ be a closed convex set. If the functional φ_2 is lower semicontinuous on U, then variational inequality* (17) *has at least one solution $x_0 \in X$.*

Proof. The following statement is true. (It represents a generalization of the Weierstrass Theorem onto the case of Frechet spaces.)

Claim 8. *Let X be a reflexive Frechet space, $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ weakly lower semicontinuous functional, $B \subset dom\varphi$ a closed convex set. Moreover, suppose that one of the following conditions holds:*

a) *set B is bounded in X;*
b) *the functional φ is coercive on B.*
Then functional φ is lower bounded on B and reaches its exact lower bound d, and the set

$$K = \{x \in B | \varphi(x) = d\}$$

is weakly compact in X.

Proof of Claim 8. Due to Claim 7 and Remark 2, the functional φ is lower bounded. Therefore, there exists a net $\{x_\alpha\}_\alpha \subset B$ such that

$$\lim_\alpha \varphi(x_\alpha) = d = \inf_{x \in B} \varphi(x) < +\infty.$$

The set $\{x_\alpha\}_\alpha$ is bounded in X due to either the boundedness B or coercivity of φ. Hence, in virtue of the Banach-Alaoglu Theorem, there exists a subnet (which we also denote by $\{x_\alpha\}_\alpha$) such that $x_\alpha \to x_0$ in $\sigma(X; X^*)$-topology of the space X, and $x_0 \in B$, because the set B is closed in $\sigma(X; X^*)$–topology.

Hence, due to the lower semicontinuity of the functional φ in $\sigma(X; X^*)$-topology, we obtain

$$\varphi(x_0) \leq \underline{\lim}_\alpha \varphi(x_\alpha) = \lim_\alpha \varphi(x_\alpha) = d,$$

i.e., $x_0 \in K$.

Finally, let $\{x_\alpha\}_\alpha \subset K$ be an arbitrary net. By the construction, the set K is bounded. Consequently, we may assume that $x_\alpha \to x_0$ in $\sigma(X;X^*)$–topology. So, $\varphi(x_0) \leq \underline{\lim}_{\alpha} \varphi(x_\alpha) = d$, whence $x_0 \in K$. Claim 8 is proved. □

In our case, $U \subset X = dom\varphi$ and it satisfies the conditions of Claim 8; therefore, the problem $\varphi(x) \to \inf$, $x \in U$ has a solution $x_0 \in X$. In order to complete the proof it remains to use Theorem 5. This completes the proof of Thorem 6. □

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REFERENCES

[1] B.N. Pshenichniy, *Convex analysis and extremum problems,* Moscow: Nauka (1980), 317 [in Russian].

[2] J.-P. Aubin, I. Ekeland, *Applied nonlinear analysis,* A Valley-Interscience Publication (1988), 512.

[3] V.F. Dem'yanov, L.V. Vasiliev, *Non-differentiated optimization,* Moscow: Nauka (1981), 384 [in Russian].

[4] J.L. Lions, *Quaelques methodes de resolution des problemes aux limites non lineaires,* Paris: DUNOD GAUTHIER-VILLARS (1969), 587.

[5] M.Z. Zgurovsky, V.S. Mel'nik, *Nonlinear analysis and control of infinite-dimentional systems,* Kiev: Naukova dumka (1999), 630 [in Russian].

[6] J.-P. Aubin, H. Frankowska, *Set-valued analysis,* Birkhauser (1990), 461.

[7] A.A. Chikrii, *Conflict controlled processes,* Cluver academic publishers (1997), 424.

[8] A.D. Ioffe, V.M. Tichomirov *The extremal problems theory,* Moscow: Nauka (1974).

[9] M.Z. Zgurovsky, V.S. Mel'nik, A.N. Novikov, *Applied methods of the analysis and control of nonlinear processes and fields,* Kiev: Naukova dumka (2004), 590 [in Russian].

[10] W. Rudin, *Functional Analysis,* Mc Graw Hill, New York, 1973, 442.

[11] V.S. Mel'nik, *About critical points of some classes multivalued maps,* J. Cybernetics and the system analysis, **2** (1997), 87–98.

[12] M. Reed, B. Simon, *Methods of modern mathematical physics,* ACAD. PRESS INC. **1** (1980), 400.

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