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GENERALIZED CHARACTERISTIC SINGULAR INTEGRAL EQUATION WITH HILBERT KERNEL

Abstract. In this paper an explicit solution of a generalized singular integral equation with a Hilbert kernel depending on indices of characteristic operators is presented.

Keywords: singular integral equation, characteristic equation, exact solution, Hilbert kernel.

Mathematics Subject Classification: 45E99.

1. INTRODUCTION

In the theory of singular integral equations [1,5–7], solutions of the following equations

$$a\,(s)\,\varphi\,(s) - \frac{b\,(s)}{2\pi}\int_0^{2\pi}\varphi\,(\sigma)\cot\frac{\sigma - s}{2}d\sigma = f\,(s)\,, \qquad s \in [0, 2\pi]\,, \tag{1}$$

$$a\,(s)\,\varphi\,(s) - \frac{1}{2\pi}\int_0^{2\pi}b\,(\sigma)\,\varphi\,(\sigma)\cot\frac{\sigma - s}{2}d\sigma = f\,(s)\,, \qquad s \in [0, 2\pi]\,, \tag{2}$$

are very well known, whenever the functions $a\,(s)$, $b\,(s)$ and the unknown function $\varphi\,(s)$ are 2π-periodic real Hölder continuous and satisfy the condition $a^2\,(s) + b^2\,(s) > 0$.

We will find explicit formulae for the solution of the following equation

$$\begin{aligned} a_0\,(s)\,\varphi\,(s) - \frac{a_1\,(s)}{2\pi}\int_0^{2\pi}b_2\,(\sigma)\,\varphi\,(\sigma)\cot\frac{\sigma - s}{2}d\sigma - \\ - \frac{b_1\,(s)}{2\pi}\int_0^{2\pi}a_2\,(\sigma)\,\varphi\,(\sigma)\cot\frac{\sigma - s}{2}d\sigma = f\,(s)\,, \quad s \in [0, 2\pi]\,, \end{aligned} \tag{3}$$

which we will call a generalized characteristic equation. In this equation coefficients $a_0(s)$, $a_1(s)$, $a_2(s)$, $b_1(s)$, $b_2(s)$, $f(s)$ are 2π-periodic real Hölder continuous functions. We look for a solution $\varphi(s)$ of (3) in the same class of functions in which the coefficients are. We assume that the coefficients satisfy the following conditions

$$a_0(s) = a_1(s)\, a_2(s) - b_1(s)\, b_2(s), \tag{4}$$

$$a_1^2(s) + b_1^2(s) > 0, \quad a_2^2(s) + b_2^2(s) > 0. \tag{5}$$

2. SOLUTION OF THE EQUATION

One can check (cf. [4]) that equation (3) can be transformed into the following system of two characteristic equations:

$$a_2(s)\,\varphi(s) - \frac{1}{2\pi}\int_0^{2\pi} \varphi(\sigma)\, b_2(\sigma) \cot\frac{\sigma - s}{2} d\sigma = \psi(s), \tag{6}$$

and

$$a_1(s)\,\psi(s) - \frac{b_1(s)}{2\pi}\int_0^{2\pi} \psi(\sigma) \cot\frac{\sigma - s}{2} d\sigma = f(s), \quad s \in [0, 2\pi], \tag{7}$$

with the condition

$$\frac{1}{2\pi}\int_0^{2\pi} b_2(\sigma)\,\varphi(\sigma)\, d\sigma = 0. \tag{8}$$

We can rewrite system of equations (6), (7) as the following vector equation

$$A(s)\,\omega(s) - \frac{B(s)}{2\pi}\int_0^{2\pi} \omega(\sigma) \cot\frac{\sigma - s}{2} d\sigma + \frac{1}{2\pi}\int_0^{2\pi} K(s,\sigma)\,\omega(\sigma) \cot\frac{\sigma - s}{2} = F(s), \tag{9}$$

where

$$A(s) = \begin{pmatrix} a_1(s), & 0 \\ -1, & a_2(s) \end{pmatrix}, \quad B(s) = \begin{pmatrix} b_1(s), & 0 \\ 0, & b_2(s) \end{pmatrix},$$
$$K(s,\sigma) = \begin{pmatrix} 0, & 0 \\ 0, & b_2(s) - b_2(\sigma) \end{pmatrix}, \quad F(s) = \begin{pmatrix} f(s) \\ 0 \end{pmatrix}, \quad \omega(s) = \begin{pmatrix} \psi(s) \\ \varphi(s) \end{pmatrix}.$$

By the general theory of systems of singular integral equations [3,6,10], particularly with a Hilbert kernel [8,9], the index κ of system (9) equals the index of the linear conjugate problem of the form

$$\Phi^+(t) = G(s)\,\Phi^-(s) + i\,(A(s) - iB(s))^{-1} F(s), \tag{10}$$

where

$$\Phi(z)=\begin{pmatrix}\Phi_1(z)\\ \Phi_2(z)\end{pmatrix},$$

$$\Phi_1(z)=\frac{1}{4\pi}\int_L \psi(\sigma)\frac{\tau+z}{\tau-z}\frac{d\tau}{\tau},\quad \Phi_2(z)=\frac{1}{4\pi}\int_L \varphi(\sigma)\frac{\tau+z}{\tau-z}\frac{d\tau}{\tau},$$

$$G(s)=(A(s)-iB(s))^{-1}(A(s)+iB(s))=$$
$$=\begin{pmatrix}\frac{a_1(s)+ib_1(s)}{a_1(s)-ib_1(s)}, & 0\\ \frac{2ib_1(s)}{(a_1(s)-ib_1(s))(a_2(s)-ib_2(s))}, & \frac{a_2(s)+ib_2(s)}{a_2(s)-ib_2(s)}\end{pmatrix},$$

i.e.,

$$\kappa=\operatorname{Ind}\det G(s)=2\kappa_1+2\kappa_2,$$

where

$$\kappa_1=\operatorname{Ind}(a_1(s)+ib_1(s)),\quad \kappa_2=\operatorname{Ind}(a_2(s)+ib_2(s)),$$

κ_1, κ_2 are indices [3,6] of characteristic equations (6) and (7). Moreover, the component indices [10] of system (9) equal the component indices of problem (10). Some complicated transformations are required to find the indices [2]. In our case it makes no sense, since we can solve (3) in a simple way. We find $\psi(s)$ from (7), and then we find the unknown solution $\varphi(s)$ of (3) from (6).

First, let us consider the case of positive indices of characteristic equations (6) and (7), i.e., $\kappa_1>0$, $\kappa_2>0$. Using the formula given in [7], a solution of (7) takes the form

$$\psi(s)=\frac{a_1(s)}{a_1^2(s)+b_1^2(s)}f(s)+\frac{b_1(s)}{a_1^2(s)+b_1^2(s)}\frac{Z_1(s)}{2\pi}\int_0^{2\pi}\frac{f(\sigma)}{Z_1(\sigma)}\cot\frac{\sigma-s}{2}d\sigma+$$
$$+\frac{b_1(s)Z_1(s)}{a_1^2(s)+b_1^2(s)}\left(\gamma_0+\ldots+\gamma_{\kappa_1}t^{\kappa_1}+\overline{\gamma_0}+\ldots+\overline{\gamma_{\kappa_1}}\frac{1}{t^{\kappa_1}}\right),\quad t=e^{is},\tag{11}$$

where $\gamma_k=\alpha_k^{(1)}+i\beta_k^{(1)}$, $k=0,\ldots,\kappa_1$, are arbitrary complex constants, and

$$\alpha_{\kappa_1}^{(1)}\cos\alpha_1+\beta_{\kappa_1}^{(1)}\sin\alpha_1=0.\tag{12}$$

Next, from equation (6) we get

$$\varphi(s)=\frac{a_2(s)}{a_2^2(s)+b_2^2(s)}\psi(s)+\frac{Z_2(s)}{a_2^2(s)+b_2^2(s)}\frac{1}{2\pi}\int_0^{2\pi}\frac{b_2(\sigma)}{Z_2(\sigma)}\psi(\sigma)\cot\frac{\sigma-s}{2}d\sigma+$$
$$+\frac{Z_2(s)}{a_2^2(s)+b_2^2(s)}\left(q_0+\ldots+q_{\kappa_2}t^{\kappa_2}+\overline{q_0}+\ldots+\overline{q_{\kappa_2}}\frac{1}{t^{\kappa_2}}\right),\tag{13}$$

where $q_k=\alpha_k^{(2)}+i\beta_k^{(2)}$, $k=0,1,\ldots,\kappa_2$, are arbitrary complex constants, and

$$\alpha_{\kappa_2}^{(2)}\cos\alpha_2+\beta_{\kappa_2}^{(2)}\sin\alpha_2=0.\tag{14}$$

In formulae (11) and (13), there is $\alpha_k = \frac{1}{2\pi}\int_0^{2\pi} \arg\left(a_k(\sigma) + ib_k(\sigma)\right) d\sigma$, $0 \leq \arg(a_k(s) + ib_k(s)) < 2\pi$, $Z_k(s) = \left(a_k(s) - ib_k(s)\right) X_k^+(t)$ $(k = 1, 2)$, where $X_k(z)$ is a canonical function of the linear conjugation problem $X_k^+(t) = \frac{a_k(s)+ib_k(s)}{a_k(s)-ib_k(s)} X_k^-(t)$, $t = e^{is}$, $s \in [0, 2\pi]$, satisfying symmetry conditions $X_k^+(z) = \overline{X_k^-\left(\frac{1}{\overline{z}}\right)}$, $|z| < 1$, $X_k^-(z) = \overline{X_k^+\left(\frac{1}{\overline{z}}\right)}$, $|z| > 1$. Since condition (8) has hold, then substituting the right side of (13) into (8) we obtain relation $\alpha_{\kappa_2}^{(2)} \sin\alpha_2 - \beta_{\kappa_2}^{(2)} \cos\alpha_2 = 0$, and taking into account (14) we get $\alpha_{\kappa_2}^{(2)} = \beta_{\kappa_2}^{(2)} = 0$.

Substituting the right side of (11) into (13), we arrive at

$$
\begin{aligned}
\varphi(s) = & \frac{a_2(s)}{a_2^2(s) + b_2^2(s)} \cdot \\
& \cdot \left\{ \frac{a_1(s) f(s)}{a_1^2(s) + b_1^2(s)} + \frac{b_1(s)}{a_1^2(s) + b_1^2(s)} \frac{Z_1(s)}{2\pi} \int_0^{2\pi} \frac{f(\sigma)}{Z_1(\sigma)} \cot\frac{\sigma - s}{2} d\sigma \right\} + \\
& + \frac{Z_2(s)}{a_2^2(s) + b_2^2(s)} \frac{1}{2\pi} \int_0^{2\pi} \frac{b_2(\sigma)}{Z_2(\sigma)} \cdot \\
& \cdot \left\{ \frac{a_1(\sigma) f(\sigma)}{a_1^2(\sigma) + b_1^2(\sigma)} + \frac{b_1(\sigma) Z_1(\sigma)}{a_1^2(\sigma) + b_1^2(\sigma)} \frac{1}{2\pi} \int_0^{2\pi} \frac{f(\sigma_1)}{Z_1(\sigma_1)} \cot\frac{\sigma_1 - \sigma}{2} d\sigma_1 \right\} \cot\frac{\sigma - s}{2} d\sigma + \\
& + \frac{a_2(s)}{a_2^2(s) + b_2^2(s)} \frac{b_1(s) Z_1(s)}{a_1^2(s) + b_1^2(s)} \left(\gamma_0 + \ldots + \gamma_{\kappa_1} t^{\kappa_1} + \overline{\gamma_0} + \ldots + \overline{\gamma_{\kappa_1}} \frac{1}{t^{\kappa_1}} \right) + \\
& + \frac{Z_2(s)}{a_2^2(s) + b_2^2(s)} \frac{1}{2\pi} \int_0^{2\pi} \frac{b_2(\sigma)}{Z_2(\sigma)} \frac{b_1(\sigma) Z_1(\sigma)}{a_1^2(\sigma) + b_1^2(\sigma)} \cdot \\
& \cdot \left(\gamma_0 + \ldots + \gamma_{\kappa_1} \tau^{\kappa_1} + \overline{\gamma_0} + \ldots + \overline{\gamma_{\kappa_1}} \frac{1}{\tau^{\kappa_1}} \right) \cot\frac{\sigma - s}{2} d\sigma + \\
& + \frac{Z_2(s)}{a_2^2(s) + b_2^2(s)} \left(q_0 + \ldots + q_{\kappa_2 - 1} t^{\kappa_2 - 1} + \overline{q_0} + \ldots + \overline{q_{\kappa_2 - 1}} \frac{1}{t^{\kappa_2 - 1}} \right).
\end{aligned}
\tag{15}
$$

Hence the following theorem holds.

Theorem 1. *Let the functions appearing in equation* (3), *i.e.,* $a_0(s)$, $a_1(s)$, $a_2(s)$, $b_1(s)$, $b_2(s)$, $f(s)$, *be* 2π*-periodic real Hölder continuous functions, and let conditions* (4) *and* (5) *be satisfied. If* $\kappa_1 > 0$, $\kappa_2 > 0$, *then the* 2π*-periodic real Hölder continuous solution* $\varphi(s)$ *of equation* (3), *satisfying condition* (8) *is given by formula* (15), *the right side of which includes* $2\kappa_1 + 2\kappa_2 - 1$ *arbitrary real constants.*

Let us now consider the case $\kappa_2 < 0 < \kappa_1$. Then equation (6) to be solvable, the following conditions must be satisfied [6, 7]

$$\int_0^{2\pi} \frac{b_2(\sigma)}{Z_2(\sigma)} \psi(\sigma) \cos k\sigma d\sigma = 0, \quad k = 0, 1, \ldots, |\kappa_2| - 1, \tag{16}$$

$$\int_0^{2\pi} \frac{b_2(\sigma)}{Z_2(\sigma)} \psi(\sigma) \sin k\sigma d\sigma = 0, \quad k = 1, \ldots, |\kappa_2| - 1, \tag{17}$$

$$\int_0^{2\pi} \frac{b_2(\sigma)}{Z_2(\sigma)} \psi(\sigma) \sin(|\kappa|\sigma - \alpha_2) d\sigma = 0, \tag{18}$$

and $q_0 = \ldots = q_{\kappa_2} = 0$. Substituting (11) into (16), (17), (18) and into the condition of solvability (8), we get the following system of equations

$$\left\{ \begin{aligned} & 2\alpha_0^{(1)} L_0(\cos k\sigma) + \sum_{\substack{j=-\kappa_1 \\ j\neq 0}}^{\kappa_1} \alpha_j^{(1)} L_j(\cos k\sigma) + i \sum_{\substack{j=-\kappa_1 \\ j\neq 0}}^{\kappa_1} \beta_j^{(1)} \operatorname{sgn}(j) L_j(\cos k\sigma) = \\ & \quad = R(\cos k\sigma), \quad k = 0, \ldots, |\kappa_2| - 1; \\ & 2\alpha_0^{(1)} L_0(\sin k\sigma) + \sum_{\substack{j=-\kappa_1 \\ j\neq 0}}^{\kappa_1} \alpha_j^{(1)} L_j(\sin k\sigma) + i \sum_{\substack{j=-\kappa_1 \\ j\neq 0}}^{\kappa_1} \beta_j^{(1)} \operatorname{sgn}(j) L_j(\sin k\sigma) = \\ & \quad = R(\sin k\sigma), \quad k = 1, \ldots, |\kappa_2| - 1; \\ & 2\alpha_0^{(1)} L_0(\sin(|\kappa_2|\sigma - \alpha_2)) + \sum_{\substack{j=-\kappa_1 \\ j\neq 0}}^{\kappa_1} \alpha_j^{(1)} L_j(\sin(|\kappa_2|\sigma - \alpha_2)) + \\ & \quad + i \sum_{\substack{j=-\kappa_1 \\ j\neq 0}}^{\kappa_1} \beta_j^{(1)} \operatorname{sgn}(j) L_j(\sin(|\kappa_2|\sigma - \alpha_2)) = R(\sin(|\kappa_2|\sigma - \alpha_2)); \\ & 2\alpha_0^{(1)} L_0(\cos(|\kappa_2|\sigma - \alpha_2)) + \sum_{\substack{j=-\kappa_1 \\ j\neq 0}}^{\kappa_1} \alpha_j^{(1)} L_j(\cos(|\kappa_2|\sigma - \alpha_2)) + \\ & \quad + i \sum_{\substack{j=-\kappa_1 \\ j\neq 0}}^{\kappa_1} \beta_j^{(1)} \operatorname{sgn}(j) L_j(\cos(|\kappa_2|\sigma - \alpha_2)) = R(\cos(|\kappa_2|\sigma - \alpha_2)), \end{aligned} \right. \tag{19}$$

where

$$R(g(\sigma)) = -\frac{1}{2\pi} \int_0^{2\pi} \frac{b_2(\sigma)}{Z_2(\sigma)} \cdot$$

$$\cdot \left\{ A_1(\sigma) f(\sigma) + B_1(\sigma) Z_1(\sigma) \frac{1}{2\pi} \int_0^{2\pi} \frac{f(\sigma_1)}{Z_1(\sigma_1)} \cot\frac{\sigma_1 - \sigma}{2} d\sigma_1 \right\} g(\sigma) d\sigma,$$

$$L_j(g(\sigma)) = \frac{1}{2\pi} \int_0^{2\pi} \frac{b_2(\sigma)}{Z_2(\sigma)} B_1(\sigma) Z_2(\sigma) \tau^j g(\sigma) d\sigma, \qquad j = -\kappa_1, \ldots, 0, \ldots, \kappa_1,$$

$$A_1(s) = \frac{a_1(s)}{a_1^2(s) + b_1^2(s)}, \quad B_1(s) = \frac{b_1(s)}{a_1^2(s) + b_1^2(s)}.$$

System of equations (19) includes $2\kappa_1$ unknowns, as the unknowns $\alpha_{\kappa_1}^{(1)}$ and $\beta_{\kappa_1}^{(1)}$ are connected trough condition (12). In this case the right side of (15) includes $2\kappa_1 - r$ arbitrary real constants, where r is the rank of the matrix of system (19). Since system (19) is a system of $2|\kappa_2|+1$ equations with $2\kappa_1$ unknowns, then it is necessary and sufficient to assume that $|\kappa_2| < \kappa_1$. Hence we get the following

Theorem 2. *Let the conditions of Theorem 1 be satisfied and let* $\kappa_2 < 0 < \kappa_1$, $|\kappa_2| < \kappa_1$. *Then the solution of equation* (3) *in the considered class of functions is given by formula* (15), *the right side of which includes* $2\kappa_1 - r$ *arbitrary real constants, where* r *is the rank of the matrix of system* (19).

The case of $|\kappa_2| \geq \kappa_1$ needs additional considerations.

If $\kappa_1 < 0$, $\kappa_2 < 0$, then the following equations need to be added to conditions (16)–(18):

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{f(\sigma)}{Z_1(\sigma)} \cos k\sigma d\sigma = 0, \quad k = 0, 1, \ldots, |\kappa_1| - 1, \tag{20}$$

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{f(\sigma)}{Z_1(\sigma)} \sin k\sigma d\sigma = 0, \quad k = 1, 2, \ldots, |\kappa_1| - 1, \tag{21}$$

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{f(\sigma)}{Z_1(\sigma)} \sin(|\kappa|\sigma - \alpha_1) d\sigma = 0 \tag{22}$$

and it is necessary to assume $\gamma_0 = \ldots = \gamma_{\kappa_1} = q_0 = \ldots = q_{\kappa_2} = 0$. System (19) takes the form

$$\begin{cases} R(\cos k\sigma) = 0, \quad k = 0, \ldots |\kappa_2| - 1, \\ R(\sin k\sigma) = 0, \quad k = 1, \ldots |\kappa_2| - 1, \\ R(\sin(|\kappa_2|\sigma - \alpha_2)) = 0, \\ R(\cos(|\kappa_2|\sigma - \alpha_2)) = 0. \end{cases} \tag{23}$$

In this case we get the following

Theorem 3. *Let the conditions of Theorem 1 be satisfied and let* $\kappa_1 < 0$, $\kappa_2 < 0$. *Then for solvability of equation* (3) *it is necessary that the function* $f(s)$ *satisfies* $2|\kappa_1| + 2|\kappa_2| + 1$ *conditions* (20)–(23). *A solution of equation* (3) *in the considered class of functions is given by formula* (15), *with* $\gamma_0 = \ldots = \gamma_{\kappa_1} = q_0 = \ldots = q_{\kappa_2 - 1} = 0$.

Let us now consider the case of $\kappa_1 = 0 < \kappa_2$. If $\cos\alpha_1 \neq 0$, then

$$\alpha_0^{(1)} = \frac{\tan\alpha_1}{4\pi} \int_0^{2\pi} \frac{f(\sigma)}{Z_1(\sigma)} d\sigma, \tag{24}$$

(cf. [7]), but if $\cos\alpha_1 = 0$, then the following condition needs to be satisfied

$$\int_0^{2\pi} \frac{f(\sigma)}{Z_1(\sigma)} d\sigma = 0. \tag{25}$$

In this case the solution of (3) is given by (15), with $\gamma_1 = \ldots = \gamma_{\kappa_1} = 0$, and condition (14) must hold.

Now we consider the case of $\kappa_2 < 0 = \kappa_1$. Here we repeat the previous considerations to find γ_0. Moreover, conditions (19) must hold with $\alpha_j^{(1)} = \beta_j^{(1)} = 0$, for $j > 0$.

Let us consider the case of $\kappa_1 = \kappa_2 = 0$. We find the real part of the constant γ_0 as in the previous case; moreover in solution (15) we assume $\gamma_1 = \ldots = \gamma_{\kappa_1} = q_0 = q_1 = \ldots = q_{\kappa_2 - 1} = 0$. From the condition of solvability (8) we get

$$\int_0^{2\pi} \frac{b_2(\sigma)\,\psi(\sigma)}{Z_2(\sigma)} d\sigma = 0, \tag{26}$$

where $\psi(s)$ is given by formula (11).

In the case of $\kappa_2 = 0 < \kappa_1$, it is necessary to assume that condition (26) is satisfied. We also assume $q_0 = q_1 = \ldots = q_{\kappa_2 - 1} = 0$ and (12).

The last case we consider is that of $\kappa_1 < 0 = \kappa_2$. It is necessary to require that conditions (20)–(22) are satisfied, and it is enough to repeat the considerations for the previous two cases, when $\kappa_2 = 0$.

Example. Let us consider the equation

$$\begin{aligned} \cos s\,\varphi(s) + \frac{\cos 2s}{2\pi} \int_0^{2\pi} \sin\sigma\,\varphi(\sigma) \cot\frac{\sigma - s}{2} d\sigma - \\ - \frac{\sin 2s}{2\pi} \int_0^{2\pi} \cos\sigma\,\varphi(\sigma) \cot\frac{\sigma - s}{2} d\sigma = \cos s, \quad s \in [0, 2\pi]. \end{aligned} \tag{27}$$

In this case,

$$\kappa_1 = \operatorname{Ind}(\cos 2s + i \sin 2s) = 2, \quad \kappa_2 = \operatorname{Ind}(\cos s - i \sin s) = -1.$$

The system of algebraic equations corresponding to system (19) has the form

$$\begin{cases} -\frac{1}{2}\alpha_1^{(1)} = 0, \\ -\frac{1}{2}\beta_2^{(1)} = 0, \\ -\frac{1}{2}\alpha_0^{(1)} = 0, \end{cases} \tag{28}$$

and its rank r is equal to 3. By Theorem 2, a solution of the equation (27) is given by the followng formula

$$\varphi(\sigma) = \cos s \cos 3s - \frac{1}{2}\cos 4s + \frac{1}{2}\cos 2s + C\left(2\cos s \sin s \sin 2s + \frac{1}{2}\cos 4s - \cos 2s\right),$$

where C is an arbitrary real constant.

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