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STRONG GEODOMINATION IN GRAPHS

Abstract. A pair x, y of vertices in a nontrivial connected graph G is said to geodominate a vertex v of G if either $v \in \{x, y\}$ or v lies in an $x - y$ geodesic of G. A set S of vertices of G is a geodominating set if every vertex of G is geodominated by some pair of vertices of S. In this paper we study strong geodomination in a graph G.

Keywords: geodomination, k-geodomination, open geodomination.

Mathematics Subject Classification: 05C12, 05C70.

1. INTRODUCTION

For two vertices x and y in a connected graph G, the distance $d(x, y)$ is the length of a shortest $x - y$ path in G. An $x - y$ path of length $d(x, y)$ is called an $x - y$ geodesic. A vertex v is said to lie in an $x - y$ geodesic P if v is an internal vertex of P. The closed interval $I[x, y]$ consists of x, y and all vertices lying in some $x - y$ geodesic of G, while for $S \subseteq V(G)$,

$$I[S] = \cup_{x,y \in S} I[x, y].$$

A set S of vertices in a graph G is a *geodetic set* if $I[S] = V(G)$, and the minimum cardinality of a geodetic set is the *geodetic number* $g(G)$. A geodetic set of cardinality $g(G)$ is called a $g(G)$-set (see [1–7]).

Geodetic concepts were studied from the point of view of domination (see [2]). Geodetic sets and the geodetic number were referred to as geodominating sets and geodomination number (see [2]); we adopt these names in this paper.

A pair x, y of vertices in a nontrivial connected graph G is said to *geodominate* a vertex v of G if either $v \in \{x, y\}$ or v lies in an $x - y$ geodesic of G. A set S of vertices of G is a geodominating set if every vertex of G is geodominated by some pair of vertices of S. A vertex of G is *link-complete* if the subgraph induced by its *neighborhood* is complete. It is easily seen that any link-complete vertex belongs to any geodominating set. For a graph G and an integer $k \geq 1$, a vertex v of G is *k-geodominated* by a pair x, y of distinct vertices in G if v is geodominated by x, y

and $d(x, y) = k$. A set S of vertices of G is a k-geodominating set of G if each vertex v in $V(G)\backslash S$ is k-geodominated by some pair of distinct vertices of S. The minimum cardinality of a k-geodominating set of G is its *k-geodomination number* $g_k(G)$. A pair x, y of vertices in G is said to *openly geodominate* a vertex v of G if $v \neq x, y$ and v is geodominated by x and y. A set S is an *open geodominating set* of G if for each vertex v, either (1) v is link-complete and $v \in S$ or (2) v is openly geodominated by some pair of vertices of S. The minimum cardinality of an open geodominating set of G is its *open geodomination number* $og(G)$. A k-geodominating set of cardinality $g_k(G)$ is called a $g_k(G)$-set of G and an open geodominating set of cardinality $og(G)$ is called an $og(G)$-set.

In this paper we introduce and study strong, open strong and k-strong geodomination in a graph G. All graphs in this paper are connected and we denote the Cartesian product of two graphs G, H by $G \times H$, and it is the graph with the vertex set $V(G) \times V(H)$ specified by putting (u, v) adjacent to $(u^{'}, v^{'})$ if and only if (1) $u = u^{'}$ and $vv^{'} \in E(H)$, or (2) $v = v^{'}$ and $uu^{'} \in E(G)$. This graph has $|V(G)|$ copies of H as rows and $|V(H)|$ copies of G as columns. In this paper, for an edge $e = \{u, v\}$ of a graph G with $\deg(u) = 1$ and $\deg(v) > 1$, we call e a pendant edge and u a pendant vertex.

2. STRONG GEODOMINATION

The concept of being "strong" is defined for numerous graph structures such as domination and vertex colorings. We study strong geodomination in graphs.

We say that a pair of vertices x, y in a connected graph G *strongly geodominates* a vertex v if one of the following holds:

1) $v \in \{x, y\}$ or
2) v lies in an x-y geodesic L of G and there is another x-y geodesic $L' \neq L$ of G of length $d(x, y)$.

We call a set S of vertices of G a *strong geodominating set* if every vertex of G is strongly geodominated by some pair of vertices of S. The minimum cardinality of a strong geodominating set is the *strong geodomintion number* $g_s(G)$.

We call a $g_s(G)$-set a strong geodominating set of size $g_s(G)$. By definition, the inequality $g_s(G) \geq g(G)$ is obvious. For complete graphs, there also holds $g_s(K_n) = g(K_n) = n$. On the other hand, for a tree T with n vertices and l leaves, it is well known that $g(T) = l$. But T has no proper strong geodominating set, so the following is true.

Observation 1. *Let T be a tree with n vertices and l leaves. Then $g_s(T) = n$.*

In particular, for any positive integer n, $g_s(P_n) = n, g_s(K_{1,n}) = n + 1$. In what follows, we obtain the strong geodomination number for some families of graphs.

Proposition 2. 1) $g_s(C_n) = \begin{cases} 2, & n \text{ even}, \\ n, & n \text{ odd}, \end{cases}$

2) *If* $\min\{m, n\} \geq 2$, *then* $g_s(P_m \times P_n) = 2$,

3) $g_s(K_m \times K_n) = \max\{m, n\}$,

4) $g_s(K_2 \times C_n) = \begin{cases} 2, & n \text{ even}, \\ 3, & n \text{ odd}, \end{cases}$

5) $g_s(K_m \times C_{2n}) = m$,

6) *If* $\min\{m, n\} \geq 2$, *then* $g_s(P_m \times P_n) = 2$,

7) $g_s(K_m \times K_n) = \max\{m, n\}$,

8) $g_s(K_2 \times C_n) = g_s(C_n) = \begin{cases} 2, & n \text{ even}, \\ 3, & n \text{ odd}, \end{cases}$

9) $g_s(K_m \times C_{2n}) = m$.

Proof. We only prove 1). The other statements are similarly verified. Let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$. First, let n be even, then $\{v_1, v_{\frac{n}{2}+1}\}$ is a strong geodominating set and the result follows from $g_s(C_n) \geq g(C_n) = 2$.

Now let n be odd. For any two vertices v_i, v_j there exists exactly one $v_i - v_j$ geodesic, so $g_s(C_n) = n$. □

In the general case, we have the following proposition.

Proposition 3. $g_s(K_2 \times G) \leq 2g_s(G) - 2$ *and this bound is sharp.*

Proof. Let $V(K_2 \times G) = \{(1, v_1), (1, v_2), \ldots, (1, v_n), (2, v_1), (2, v_2), \ldots, (2, v_n)\}$ where $(1, v_i)$ is adjacent to $(2, v_i)$ for $i = 1, 2, \ldots, n$ and $V(G) = \{v_1, v_2, \ldots, v_n\}$. Let S be a $g_s(G)$-set, and let $S = \{v_{k_1}, v_{k_2}, \ldots, v_{k_l}\}$. Then

$$\{(1, v_{k_1}), (1, v_{k_2}), \ldots, (1, v_{k_l - 1})\} \cup \{(2, v_{k_2}), \ldots, (2, v_{k_l})\}$$

is a strong geodominating set for $K_2 \times G$.

By part 1 of Proposition 2, the bound for $g_s(G) = 2$ is sharp. Now let K_n^2 denote the multigraph of order n for which every two vertices of K_n^2 are joined by two edges. Let G' be obtained from K_n^2 by adding a pendant vertex to each of the vertices of K_n^2 and let G be the graph obtained from G' by subdividing any non-pendant edge of G'. Then $g_s(G) = n$ and $g_s(K_2 \times G) = 2n - 2$. □

In the next theorem, we prove the existence of a connected graph G of order b and strong geodomination number a for any two positive integers a, b with $b \geq a + 2$.

Theorem 4. *For any two positive integers* a, b *with* $b \geq a+2$, *there exists a connected graph* G *with* $|V(G)| = b, g_s(G) = a$.

Proof. Let G' be a graph obtained from $K_{1,a-1}$ by subdividing an edge xy to xwy. For each $i = 1, 2, \ldots, b - a - 1$, we add an ear xw_iy to obtain a graph G. Then $|V(G)| = b, g_s(G) = a$. □

A geodominating set S is *essential* if for every two vertices u, v in S, there exists a vertex $w \neq u, v$ of G that lies in a $u - v$ geodesic but in no $x - y$ geodesic for $x, y \in S$ and $\{x, y\} \neq \{u, v\}$ (see [1, 2]). A geodominating set S is k-*essential* if there is a subset $S' \subseteq S$ with $|S'| = k$ such that for every two vertices u, v in S', there exists a vertex $w \neq u, v$ of G that lies in a $u - v$ geodesic but in no $x - y$ geodesic for $x, y \in S$ and $\{x, y\} \neq \{u, v\}$.

Corollary 5. *If G has a k-essential $g_s(G)$-set, then* $g_s(K_m \times G) \leq \left\lfloor \frac{m}{2} \right\rfloor (k-2) + \left\lceil \frac{m}{2} \right\rceil g_s(G)$.

Proof. First we prove that $g_s(K_2 \times G) \leq k + g_s(G) - 2$. Let

$$V(K_2 \times G) = \{(1, v_1), (1, v_2), \ldots, (1, v_n), (2, v_1), (2, v_2), \ldots, (2, v_n)\}$$

where $(1, v_i)$ is adjacent to $(2, v_i)$ for $i = 1, 2, \ldots, n$ and $V(G) = \{v_1, v_2, \ldots, v_n\}$. Let $S = \{v_{l_1}, v_{l_2} \ldots, v_{l_t}\}$ be a k-essential $g_s(G)$-set and, without loss of generality, suppose that $S' = \{v_{l_1}, v_{l_2} \ldots, v_{l_k}\} \subseteq S$ be such that for every two vertices u, v in S', there exists a vertex $w \neq u, v$ of G that lies in a $u - v$ geodesic but in no $x - y$ geodesic for $x, y \in S$ and $\{x, y\} \neq \{u, v\}$. Then

$$\{(1, v_{l_1}), (1, v_{l_2}), \ldots, (1, v_{l_k - 1})\} \cup \{(2, v_{l_2}), \ldots, (2, v_{l_k})\} \cup \{(2, v_{l_k + 1}), \ldots, (2, v_{l_t})\}$$

is a strong geodominating set for $K_2 \times G$. Now consider $K_m \times G$. We know that $K_m \times G$ contains $\left\lfloor \frac{m}{2} \right\rfloor$ disjoint copies of $K_2 \times G$. So we consider a strong geodominating set described above for each copy of $K_2 \times G$ together with a $g_s(G)$-set corresponding to the last copy of G if m is odd, to obtain a strong geodominating set for $K_m \times G$ of size $\left\lfloor \frac{m}{2} \right\rfloor (k-2) + \left\lceil \frac{m}{2} \right\rceil g_s(G)$. □

In what follows, we show that the strong geodomination number is affected by adding a pendant vertex.

Proposition 6. *Let G' be a graph obtained from G by adding a pendant vertex. Then* $g_s(G) \leq g_s(G') \leq 1 + g_s(G)$.

Proof. Let G' be obtained from G by adding the pendant edge uv with $v \notin V(G), u \in V(G)$. Let S be a $g_s(G)$-set.

If $u \in S$, then $(S \backslash \{u\}) \cup \{v\}$ is a strong geodominating set for G', and if $u \notin S$, then $S \cup \{v\}$ is a strong geodominating set for G'. So $g_s(G') \leq 1 + g_s(G)$.

Now let S' be a $g_s(G')$-set. Then $v \in S'$ and so $(S' \backslash \{v\}) \cup \{u\}$ is a strong geodominating set for G, hence $g_s(G) \leq g_s(G')$. □

Let k be a positive integer. A set S of vertices in a connected graph G is k-*uniform* if the distance between every two vertices of S is the same fixed number k (see [1,2]). It can be seen that if a graph G has a proper essential k-uniform strong geodominating set, then $|V(G)| \geq (g_s(G) - 1)(k + 1) + 1$.

Theorem 7. *For each integer $k \geq 2$, there exists such a connected graph G with $g_s(G) = k$ which contains a uniform, essential, minimum strong geodominating set.*

Proof. Let $K_k^{(k-1)}$ denote the multigraph of order k for which every two vertices of $K_k^{(k-1)}$ are joined by $k-1$ edges and let $G_k = S(K_k^{(k-1)})$ be the subdivision graph of $K_k^{(k-1)}$. It was shown that $V(K_k^{(k-1)})$ is a uniform, essential, minimum geodetic set for G_k and $g(G_k) = k$ (see [1]). It is now easy to see that $g_s(G_k) = g_k(G_k) = g(G_k) = k$ and $V(K_k^{(k-1)})$ is a uniform, essential, minimum strong geodominating set for G_k. □

3. OPEN STRONG AND k-STRONG GEODOMINATION

Here we study open strong geodomination and k-strong geodomination in a graph G. We first introduce the following definitions.

We say that a pair x, y of vertices in a graph G *open strongly geodominates* a vertex v of G if $v \neq x, y$ and v is strongly geodominated by x, y. We call a set S an *open strong geodominating* set of G if for each vertex v, either (1) v is link-complete and $v \in S$ or (2) v is open strongly geodominated by some pair of vertices of S. The minimum cardinality of an open strong geodominating set of G is its open-strong geodomination number $og_s(G)$.

We also call an $og_s(G)$-set an open strong geodominating set of cardinality $og_s(G)$. For a graph G of order $n \geq 2$, by the definitions, there is $og_s(G) \geq g_s(G)$ and $2 \leq og_s(G) \leq n$.

For a graph G and an integer $k \geq 1$, we say that a vertex v of G is *k-stongly geodominated* by a pair x, y of distinct vertices in G if v is strongly geodominated by x, y and $d(x, y) = k$. A set S of vertices of G is a *k-strong geodominating set* of G if each vertex v in $V(G)\backslash S$ is k-strongly geodominated by some pair of distinct vertices of S. The minimum cardinality of a k-strong geodominating set of G is its *k-stong geodomination number* $g_{ks}(G)$.

We call a $g_{ks}(G)$-set a k-strong geodominating set of cardinality $g_{ks}(G)$. By definition, any k-strong geodominating set is both a k-geodominating set and a strong geodominating set.

Let $k \geq 1$. A set S of vertices of G is an *open k-strong geodominating set* of G if for each vertex v, either (1) v is link-complete and $v \in S$ or (2) v is open strongly geodominated by some pair x, y of vertices of S with $d(x, y) = k$. The minimum cardinality of an open k-strong geodominating set of G is its *open k-strong geodomination number* $og_{ks}(G)$.

Now we are ready to investigate the open-strong geodomination numbers as well as open k-strong geodomination numbers for some families of graphs. The proofs are straightforward and we omit them.

— If T is a tree with n vertices and l leaves, then $og_s(T) = n$.

As a result, $og_s(P_n) = n$ and $og_s(K_{1,n}) = n + 1$.

— $og_s(C_n) = \begin{cases} 4, & n \text{ even}, \\ n, & n \text{ odd}, \end{cases} \quad og_s(K_n) = g_s(K_n) = g(K_n) = n.$

— $og_s(P_m \times P_n) = 4, \quad og_s(K_n \times K_n) = \begin{cases} 2n, & n \text{ even}, \\ 2n-1, & n \text{ odd}. \end{cases}$

— If a graph G has a proper open strong geodominating set, then $|V(G)| \geq 4$.

— If a graph G has $k \geq 2$ link-complete vertices and $g_s(G) = k$, then $og_s(G) = k$.

— If G is a graph with no link-complete vertices, then $og_s(G \times K_2) \leq 2og_s(G)$.

— If G is a graph with exactly one link-complete vertex, then $og_s(G \times K_2) \leq 2og_s(G) - 1$.

— If G is a graph with at least two link-complete vertices, then $og_s(G \times K_2) \leq 2og_s(G) - 2$.
— $g_{1s}(G) = og_{1s}(G) = |V(G)|$.
— If $k > diam(G)$, then $g_{ks}(G) = og_{ks}(G) = |V(G)|$.
— $og_{ks}(G) \geq g_{ks}(G) \geq g_s(G)$, $2 \leq g_{ks}(G) \leq |V(G)|$ and $g_{ks}(G) \leq og_{ks}(G) \leq 3(og_{ks}(G))$.

And for $k \geq 2$:

— $g_{ks}(P_n) = g_{ks}(K_n) = og_{ks}(P_n) = og_{ks}(K_n) = n$.
— $g_{ks}(C_n) = \begin{cases} 2, & k = \frac{n}{2}, \\ n, & \text{otherwise}, \end{cases}$ $og_{ks}(C_n) = \begin{cases} 4, & k = \frac{n}{2}, \\ n, & \text{otherwise}. \end{cases}$
— $g_{ks}(K_{m,n}) = og_{ks}(K_{m,n}) = \begin{cases} 4, & k = 2, \\ m + n, & \text{otherwise}. \end{cases}$
— $g_{ks}(K_n \times K_n) = \begin{cases} n, & k = 2, \\ n^2, & \text{otherwise}, \end{cases}$

$$og_{ks}(K_n \times K_n) = \begin{cases} 2n, & k = 2, n \text{ even}, \\ 2n - 1, & k = 2, n \text{ odd}, \\ n^2, & \text{otherwise}. \end{cases}$$

— If G' is a graph obtained from G by adding a pendant vertex, then $g_{ks}(G') \leq 1 + g_{ks}(G)$.

The next result guarantees the existence of a connected graph G of order b and k-strong geodomination number a for any three positive integers a, b, k with $b \geq (a-1)(k+1) + 1$, $a \geq 2$ and $k \geq 1$.

Proposition 8. *For three positive integers a, b, k with $b \geq (a-1)(k+1) + 1$, $a \geq 2$ and $k \geq 1$, there exists a connected graph G with $|V(G)| = b$ and $g_{ks}(G) = a$.*

Proof. Let $P_{(a-1)k+1}$ be the path with vertices $v_1, v_2, \ldots, v_{(a-1)k+1}$. We obtain a graph G' by adding an ear $v_{ki+1}w_iv_{ki+3}$ for each $i = 0, 1, 2, \ldots, a-2$. Now for each $j = 1, 2, \ldots, b - |V(G')|$, we add an ear $v_1u_jv_2$ to obtain a graph G. Then it is easily seen that $|V(G)| = b$ and $g_{ks}(G) = a$. □

In the next theorem we determine whether $g_s(G) = g_{ks}(G)$ in a graph G with $g_s(G) = 2$.

Theorem 9. *Let G be a connected graph of order $n \geq 3$, with $diam(G) \geq 3$ and $g_s(G) = 2$ and let $k \geq 1$ be an integer. Then $g_s(G) = g_{ks}(G)$ if and only if $k = diam(G)$.*

Proof. Let $d = diam(G)$ and let $S = \{x, y\}$ be a $g_s(G)$-set. Then x and y are antipodal, that is $d(x, y) = d$. So S is a k-strong geodominating set. Hence $g_{ks}(G) \leq g_s(G)$, which implies the equality. Now let $g_s(G) = g_{ks}(G) = 2$. Since $n \geq 3$, it follows that $2 \leq k \leq d$. Assume that $k < d$. Since $g_s(G) = 2$, every $g_s(G)$-set contains two antipodal vertices and since $g_{ks}(G) = 2$, it follows that $d < d$, a contradiction. □

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REFERENCES

[1] G. Chartrand, F. Harary, P. Zhang, *Geodetic sets in graphs,* Discussiones Mathematicae Graph Theory **20** (2000), 129–138.

[2] G. Chartrand, F. Harary, H.C. Swart, P. Zhang, *Geodomination in graphs,* Bull. ICA **31** (2001), 51–59.

[3] D.A. Mojdeh, N. Jafari Rad, *Connected geodomination in graphs,* Journal of Discrete Mathematical Science and Cryptography **9** (2006), 177–186.

[4] D.A. Mojdeh, N. Jafari Rad, *k-perfect geodominating sets in graphs,* Opuscula Mathematica **27** (2007), 51–57.

[5] R. Muntean, P. Zhang, *k-geodomination in graphs,* Ars Combinatoria **63** (2002), 33–47.

[6] R. Muntean, P. Zhang, *On geodomination in graphs,* Congressus Numerantium **143** (2000), 161–174.

[7] D.B. West, *Introduction to Graph Theory,* 2nd edition, Prentice-Hall, USA, 2001.

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