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FUNCTIONAL MODELS FOR NEVANLINNA FAMILIES

Abstract. The class of Nevanlinna families consists of \mathbb{R} -symmetric holomorphic multivalued functions on $\mathbb{C} \setminus \mathbb{R}$ with maximal dissipative (maximal accumulative) values on \mathbb{C}_+ (\mathbb{C}_- , respectively) and is a generalization of the class of operator-valued Nevanlinna functions. In this note Nevanlinna families are realized as Weyl families of boundary relations induced by multiplication operators with the independent variable in reproducing kernel Hilbert spaces.

Keywords: symmetric operator, selfadjoint extension, boundary relation, Weyl family, functional model, reproducing kernel Hilbert space.

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1. INTRODUCTION

Let \mathcal{H} be a Hilbert space and let M be a *Nevanlinna function* whose values are bounded linear operators in \mathcal{H} , i.e., M is holomorphic on $\mathbb{C} \setminus \mathbb{R}$, $M(\bar{\lambda}) = M(\lambda)^*$ holds for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and $\text{Im } M(\lambda)$ is a nonnegative operator for all λ in the upper open half plane \mathbb{C}_+ . If, in addition, $\text{Im } M(\lambda)$, $\lambda \in \mathbb{C}_+$, is uniformly positive, then M will be called *uniformly strict*. It is well known that a uniformly strict Nevanlinna function M can be realized as a so-called Q -function of a closed simple symmetric operator S in a Hilbert space \mathfrak{H} and a selfadjoint extension of S in \mathfrak{H} , cf. [9] and [10]. The notion of Q -function “coincides” with the modern concept of Weyl function associated with an ordinary boundary triplet for symmetric operators or relations.

Recall that an *ordinary boundary triplet* $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for a closed symmetric operator or relation S with equal defect numbers in a Hilbert space \mathfrak{H} consists of a Hilbert space \mathcal{H} and two linear mappings $\Gamma_0, \Gamma_1 : S^* \rightarrow \mathcal{H}$ such that $\Gamma := (\Gamma_0, \Gamma_1)^\top : S^* \rightarrow \mathcal{H} \times \mathcal{H}$ is surjective and the abstract Green’s identity

$$(f', g) - (f, g') = (\Gamma_1 \widehat{f}, \Gamma_0 \widehat{g}) - (\Gamma_0 \widehat{f}, \Gamma_1 \widehat{g})$$

holds for all $\widehat{f} = \{f, f'\}, \widehat{g} = \{g, g'\} \in S^*$. V.A. Derkach and M.M. Malamud [6, 7] have supplemented this notion by defining the concept of a *Weyl function* as an abstract analogon of the classical Titchmarsh-Weyl coefficient in the theory of singular Sturm-Liouville operators:

$$M(\lambda) = \Gamma(\widehat{\mathfrak{N}}_\lambda(S^*)) = \{\{\Gamma_0 \widehat{f}_\lambda, \Gamma_1 \widehat{f}_\lambda\} : \widehat{f}_\lambda \in \widehat{\mathfrak{N}}_\lambda(S^*)\}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \quad (1.1)$$

where $\widehat{\mathfrak{N}}_\lambda(S^*) = \{\{f, f'\} \in S^* : f' = \lambda f\}$. It follows that M is a uniformly strict Nevanlinna function and that, conversely, every uniformly strict Nevanlinna function can be realized as the Weyl function of a closed simple symmetric operator S in a Hilbert space \mathfrak{H} and an ordinary boundary triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for S^* .

In [5] the notions of boundary relations and associated Weyl families were introduced as generalizations of the concepts of (ordinary and generalized) boundary triplets and their Weyl functions. In contrast to ordinary boundary triplets, a boundary relation Γ for a symmetric relation S (whose defect numbers may be unequal and infinite) is in general only defined on a core \mathcal{T} of the adjoint S^* and the mapping Γ can be multivalued; see Definition 3.1. The Weyl family M associated with a boundary relation Γ is defined in a similar way as the Weyl function in (1.1) and it follows that the Weyl family is a so-called *Nevanlinna family*, i.e., M is a holomorphic family of maximal dissipative (in \mathbb{C}_+) linear relations in \mathcal{H} , cf. Definition 2.1 below. In [5] it was shown that, conversely, every Nevanlinna family can be realized as the Weyl family of a boundary relation in an abstract model space.

This note concerns an explicit *functional model* for a Nevanlinna family M . In this functional model there is a “natural” boundary relation Γ whose Weyl family is given by M . Here the model space is a *reproducing kernel Hilbert space* in which multiplication by the independent variable is a closed simple symmetric operator S which gives rise to the boundary relation. The operator of multiplication by the independent variable in the context of reproducing kernel Hilbert spaces of scalar entire functions was already considered in [3]. For uniformly strict Nevanlinna functions, the model in the present note reduces to the model constructed in [7] (see also [11]).

The purpose of the present note is to provide a brief introduction to the notions of boundary relations and corresponding Weyl families and to outline the functional models associated with them. Very recently, a similar functional model has also been obtained with different methods by Derkach in [4]. An essentially wider analysis involving connections to the reproducing kernel space models for operator-valued Schur functions and transfer functions of unitary colligations is carried by the authors in [2]. In that paper, classes of generalizations of boundary triplets corresponding to special classes of Nevanlinna families (or Nevanlinna functions) are also treated in detail.

2. PRELIMINARIES

2.1. LINEAR RELATIONS IN HILBERT SPACES

Let \mathfrak{H} be a Hilbert space with scalar product (\cdot, \cdot) . In this note (closed) linear relations in \mathfrak{H} , that is (closed) linear subspaces of the Cartesian product $\mathfrak{H} \times \mathfrak{H}$ are studied. The

elements of a linear relation T will be denoted by $\widehat{f} = \{f, f'\} \in T, f, f' \in \mathfrak{H}$. For a linear relation T in \mathfrak{H} , the symbols $\text{dom } T, \ker T, \text{ran } T$, and $\text{mul } T$ stand for the domain, kernel, range, and the multi-valued part, respectively. The inverse relation is defined by $T^{-1} = \{\{f', f\} : \{f, f'\} \in T\}$. Closed linear operators in \mathfrak{H} will be identified with closed linear relations via their graphs. The linear space of everywhere defined bounded linear operators in \mathfrak{H} will be denoted by $\mathbf{B}(\mathfrak{H})$.

Let S be a linear relation in \mathfrak{H} . The adjoint S^* of S is a closed linear relation in \mathfrak{H} defined by

$$S^* := \{ \{g, g'\} : (f', g) = (f, g') \text{ for all } \{f, f'\} \in S \}.$$

A linear relation S is called *symmetric (selfadjoint)* if $S \subset S^*$ ($S = S^*$, respectively). For a closed symmetric relation S in the Hilbert space \mathfrak{H} the defect space corresponding to $\lambda \in \mathbb{C}$ will be denoted by $\widehat{\mathfrak{N}}_\lambda(S^*), \widehat{\mathfrak{N}}_\lambda(S^*) = \{\{f, f'\} \in S^* : f' = \lambda f\}$, so that $\text{dom } \widehat{\mathfrak{N}}_\lambda(S^*) = \ker(S^* - \lambda)$, which will be denoted by $\mathfrak{N}_\lambda(S^*)$.

For a fixed $\mu \in \mathbb{C}_+$ the *Cayley transform* $C_\mu(T)$ of a linear relation T in \mathfrak{H} is defined by

$$C_\mu(T) := \{ \{f' - \mu f, f' - \bar{\mu} f\} : \{f, f'\} \in T \}.$$

Clearly, $C_\mu(T)$ maps $\text{ran}(T - \mu)$ onto $\text{ran}(T - \bar{\mu})$. Furthermore, $C_\mu(T)$ is an isometric (unitary) operator if and only if T is a symmetric (selfadjoint, respectively) relation. The *inverse Cayley transform* of a linear relation V is given by

$$C_\mu^{(-1)}(V) := \{ \{h' - h, \mu h' - \bar{\mu} h\} : \{h, h'\} \in V \}. \tag{2.1}$$

If V is a unitary operator, then the selfadjoint relation $A := C_\mu^{(-1)}(V)$ in (2.1) is given by

$$A = \{ \{(V - I)h, (\mu V - \bar{\mu})h\} : h \in \mathfrak{H} \}.$$

2.2. NEVANLINNA FAMILIES, NEVANLINNA PAIRS, AND ASSOCIATED REPRODUCING KERNEL HILBERT SPACES

Nevanlinna families are a natural generalization of the class of Nevanlinna functions.

Definition 2.1. A family $M = \{M(\lambda) : \lambda \in \mathbb{C} \setminus \mathbb{R}\}$ of linear relations in \mathcal{H} is said to be a *Nevanlinna family* in \mathcal{H} if:

- (i) $\text{Im}(f', f) \geq 0$ for all $\{f, f'\} \in M(\lambda), \lambda \in \mathbb{C}_+$, and $\text{Im}(f', f) \leq 0$ for all $\{f, f'\} \in M(\lambda), \lambda \in \mathbb{C}_-$;
- (ii) $M(\bar{\lambda}) = M(\lambda)^*$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$;
- (iii) for some, and hence for all, $\nu \in \mathbb{C}_+ (\nu \in \mathbb{C}_-)$, the operator function $\lambda \mapsto (M(\lambda) + \nu)^{-1}$ is $\mathbf{B}(\mathcal{H})$ -valued and holomorphic on $\mathbb{C}_+ (\mathbb{C}_-, \text{ respectively})$.

The multivalued part $\text{mul } M(\lambda)$ of a Nevanlinna family M does not depend on $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and M can be decomposed into the direct orthogonal sum of a Nevanlinna family M_0 of densely defined operators in $\mathcal{H}_0 := \mathcal{H} \ominus \text{mul } M(\lambda)$ and the selfadjoint relation $M_\infty = \{0\} \times \text{mul } M(\lambda)$,

$$M(\lambda) = M_0(\lambda) \oplus M_\infty, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

In particular, M is an operator function if and only if $\text{dom } M(\lambda)$ is dense for some, and hence for all, $\lambda \in \mathbb{C} \setminus \mathbb{R}$. A Nevanlinna family $M = \{M(\lambda) : \lambda \in \mathbb{C} \setminus \mathbb{R}\}$ in \mathcal{H} is said to be *uniformly strict* if $M(\lambda) \hat{+} M(\lambda)^* = \mathcal{H}^2$ for some, and hence for all, $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Here $\hat{+}$ denotes the componentwise direct sum of linear subspaces in \mathcal{H}^2 . It turns out that a uniformly strict Nevanlinna family M is automatically a $\mathbf{B}(\mathcal{H})$ -valued Nevanlinna function with $0 \in \rho(\text{Im } M(\lambda))$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$, cf. [5].

Definition 2.2. A pair $\{\Phi, \Psi\}$ of $\mathbf{B}(\mathcal{H})$ -valued functions is said to be a *Nevanlinna pair* in \mathcal{H} if:

- (i) $(\text{Im } \lambda) \text{Im}(\Psi(\lambda)\Phi(\lambda)^*) \geq 0$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$;
- (ii) $\Psi(\lambda)\Phi(\bar{\lambda})^* = \Phi(\lambda)\Psi(\bar{\lambda})^*$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$;
- (iii) $(\Psi(\lambda) + \nu\Phi(\lambda))^{-1} \in \mathbf{B}(\mathcal{H})$ for all $\lambda \in \mathbb{C}_{\pm}$, $\nu \in \mathbb{C}_{\pm}$.

Let $\{\Phi, \Psi\}$ be a Nevanlinna pair. Then

$$M(\lambda) := \{\{f, f'\} : \Phi(\lambda)f + \Psi(\lambda)f' = 0\} = \{\{\Psi(\bar{\lambda})^*g, -\Phi(\bar{\lambda})^*g\} : g \in \mathcal{H}\} \quad (2.2)$$

is a Nevanlinna family in \mathcal{H} . Conversely, if M is a Nevanlinna family in \mathcal{H} and $\mu \in \mathbb{C}_+$, then the pair $\{A, B\}$ defined by

$$A(\lambda) := \begin{cases} (M(\lambda) + \mu)^{-1}, & \lambda \in \mathbb{C}_+; \\ (M(\lambda) + \bar{\mu})^{-1}, & \lambda \in \mathbb{C}_-; \end{cases} \quad B(\lambda) := \begin{cases} I - \mu(M(\lambda) + \mu)^{-1}, & \lambda \in \mathbb{C}_+; \\ I - \bar{\mu}(M(\lambda) + \bar{\mu})^{-1}, & \lambda \in \mathbb{C}_-, \end{cases}$$

is a Nevanlinna pair in \mathcal{H} which is *symmetric*, that is,

$$A(\lambda) = A(\bar{\lambda})^* \quad \text{and} \quad B(\lambda) = B(\bar{\lambda})^*, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Moreover, the pair $\{\Phi, \Psi\}$ defined by $\Phi(\lambda) := -B(\lambda)$ and $\Psi(\lambda) := A(\lambda)$, is a Nevanlinna pair such that (2.2) holds. In other words

$$M(\lambda) = \{\{A(\lambda)g, B(\lambda)g\} : g \in \mathcal{H}\}.$$

In the special case of the Nevanlinna family M being a $\mathbf{B}(\mathcal{H})$ -valued Nevanlinna function it is clear that $\{I, M\}$ is a symmetric Nevanlinna pair.

Let $\{A, B\}$ be a Nevanlinna pair in \mathcal{H} . The corresponding *Nevanlinna kernel* on $(\mathbb{C}_+ \cup \mathbb{C}_-) \times (\mathbb{C}_+ \cup \mathbb{C}_-)$ is defined by

$$\mathsf{K}_{A,B}(\xi, \lambda) := \frac{B(\lambda)A(\xi)^* - A(\lambda)B(\xi)^*}{\lambda - \bar{\xi}}, \quad \lambda, \xi \in \mathbb{C}_+ \cup \mathbb{C}_-, \quad \xi \neq \bar{\lambda};$$

and $\mathsf{K}_{A,B}(\bar{\lambda}, \lambda) = 0$. The kernel $\mathsf{K}_{A,B}(\cdot, \cdot)$ is nonnegative, Hermitian, and holomorphic. The corresponding *reproducing kernel Hilbert space* will be denoted by $\mathfrak{H}(A, B)$. It consists of vector functions which are holomorphic on $\mathbb{C} \setminus \mathbb{R}$, see, e.g. [1]. If M is a $\mathbf{B}(\mathcal{H})$ -valued Nevanlinna function, the notation $\mathfrak{H}(M)$ will be used instead of $\mathfrak{H}(I, M)$.

3. BOUNDARY RELATIONS AND WEYL FAMILIES

3.1. GENERAL DEFINITIONS AND BASIC PROPERTIES

The concepts of boundary relations and associated Weyl families were introduced in [5] as a generalization of the notions of (generalized) boundary triplets and their Weyl functions. Below, the definitions and some elementary properties of boundary relations and Weyl families are briefly recalled.

Definition 3.1. Let S be a closed symmetric relation in a Hilbert space \mathfrak{H} and let \mathcal{H} be an auxiliary Hilbert space. A linear relation $\Gamma \subset \mathfrak{H}^2 \times \mathcal{H}^2$ is called a *boundary relation* for S^* if:

- (i) $\mathcal{T} := \text{dom } \Gamma$ is dense in S^* and the abstract Green's identity

$$(g', l)_{\mathfrak{H}} - (g, l')_{\mathfrak{H}} = (k', m)_{\mathcal{H}} - (k, m')_{\mathcal{H}}$$

holds for every $\{\widehat{g}, \widehat{k}\}, \{\widehat{l}, \widehat{m}\} \in \Gamma$;

- (ii) if $\{\widehat{l}, \widehat{m}\} \in \mathfrak{H}^2 \times \mathcal{H}^2$ satisfies the abstract Green's identity for every $\{\widehat{g}, \widehat{k}\} \in \Gamma$, then $\{\widehat{l}, \widehat{m}\} \in \Gamma$;

where $\widehat{g} = \{g, g'\}, \widehat{l} = \{l, l'\} \in \mathfrak{H}^2$, and $\widehat{k} = \{k, k'\}, \widehat{m} = \{m, m'\} \in \mathcal{H}^2$.

Let S be a closed symmetric relation in \mathfrak{H} . If $\Gamma \subset \mathfrak{H}^2 \times \mathcal{H}^2$ is a boundary relation for S^* , then Γ is necessarily closed and $S = \ker \Gamma$ holds. Moreover, it is not difficult to see that $\Gamma \subset \mathfrak{H}^2 \times \mathcal{H}^2$ with $\ker \Gamma = S$ is a boundary relation for S^* if and only if

$$\widetilde{A} := \left\{ \left\{ \begin{pmatrix} g \\ k \end{pmatrix} \begin{pmatrix} g' \\ -k' \end{pmatrix} \right\} : \left\{ \begin{pmatrix} g \\ g' \end{pmatrix} \begin{pmatrix} k \\ k' \end{pmatrix} \right\} \in \Gamma \right\} \subset (\mathfrak{H} \oplus \mathcal{H}) \times (\mathfrak{H} \oplus \mathcal{H}) \quad (3.1)$$

is a selfadjoint relation in $\mathfrak{H} \oplus \mathcal{H}$. Therefore, every selfadjoint relation \widetilde{A} in $\mathfrak{H} \oplus \mathcal{H}$ with $S = \widetilde{A} \cap \mathfrak{H}^2$ yields a boundary relation for S^* and vice versa, and hence for a given S there always exists a Hilbert space \mathcal{H} and a boundary relation $\Gamma \subset \mathfrak{H}^2 \times \mathcal{H}^2$ for S^* . Note that Γ is not unique.

Let $\mathfrak{N}_\lambda(\mathcal{T}) = \ker(\mathcal{T} - \lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, be the eigenspace and set

$$\widehat{\mathfrak{N}}_\lambda(\mathcal{T}) = \{ \{g_\lambda, \lambda g_\lambda\} : g_\lambda \in \mathfrak{N}_\lambda(\mathcal{T}) \}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Definition 3.2. The *Weyl family* $M = \{M(\lambda) : \lambda \in \mathbb{C} \setminus \mathbb{R}\}$ of S corresponding to the boundary relation $\Gamma \subset \mathfrak{H}^2 \times \mathcal{H}^2$ is defined by

$$M(\lambda) := \Gamma(\widehat{\mathfrak{N}}_\lambda(\mathcal{T})) = \{ \widehat{k} : \{\widehat{g}_\lambda, \widehat{k}\} \in \Gamma, \widehat{g}_\lambda = \{g_\lambda, \lambda g_\lambda\} \in \widehat{\mathfrak{N}}_\lambda(\mathcal{T}) \}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

If the values $M(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, are operators, then M is called a *Weyl function*.

It is easy to see that the Weyl family M and the selfadjoint relation \widetilde{A} in (3.1) are connected via

$$P_{\mathcal{H}}(\widetilde{A} - \lambda)^{-1} \upharpoonright_{\mathcal{H}} = -(M(\lambda) + \lambda)^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \quad (3.2)$$

This also implies that M is a Nevanlinna family in \mathcal{H} , cf. [5]. Conversely every Nevanlinna family can be interpreted as the Weyl family of a boundary relation, cf. [5, Theorem 3.9].

Boundary relations extend the concepts of ordinary and generalized boundary triplets. More precisely, if S is a closed symmetric relation in \mathfrak{H} and $\Gamma \subset \mathfrak{H}^2 \times \mathcal{H}^2$ is a boundary relation for S^* such that $\text{ran } \Gamma = \mathcal{H}^2$, or, equivalently, $\text{dom } \Gamma = \mathcal{T} = S^*$, then $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is said to be an *ordinary boundary triplet* for S^* , see [6, 7] and [8]. In this case the Weyl family M is a uniformly strict $\mathbf{B}(\mathcal{H})$ -valued Nevanlinna function, see [5, Proposition 5.3].

3.2. BOUNDARY RELATIONS, UNITARY COLLIGATIONS, WEYL FAMILIES, AND TRANSFER FUNCTIONS

Let \mathfrak{H} and \mathcal{H} be Hilbert spaces, and let \tilde{A} be a selfadjoint relation in $\mathfrak{H} \oplus \mathcal{H}$. Then, according to (3.1), \tilde{A} gives rise to a boundary relation Γ for the adjoint of the closed symmetric relation $S := \tilde{A} \cap \mathfrak{H}^2$ in \mathfrak{H} . On the other hand, for a fixed $\mu \in \mathbb{C}_+$, the selfadjoint relation \tilde{A} in $\mathfrak{H} \oplus \mathcal{H}$ is the inverse Cayley transform $C_\mu^{(-1)}(U)$ of a unitary operator, more precisely, of a so-called unitary colligation $U \in \mathbf{B}(\mathfrak{H} \oplus \mathcal{H})$,

$$U = \begin{pmatrix} T & F \\ G & H \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathcal{H} \end{pmatrix} \rightarrow \begin{pmatrix} \mathfrak{H} \\ \mathcal{H} \end{pmatrix}, \quad (3.3)$$

that is, \tilde{A} admits the representation

$$\tilde{A} = \left\{ \left\{ \left((T - I)h + Ff \right), \left(\mu T - \bar{\mu} \right)h + \mu Ff \right\}, \left\{ \left(Gh + (H - I)f \right), \mu Gh + (\mu H - \bar{\mu})f \right\} : h \in \mathfrak{H}, f \in \mathcal{H} \right\}, \quad (3.4)$$

with $T \in \mathbf{B}(\mathfrak{H})$, $F \in \mathbf{B}(\mathcal{H}, \mathfrak{H})$, $G \in \mathbf{B}(\mathfrak{H}, \mathcal{H})$, and $H \in \mathbf{B}(\mathcal{H})$ having the properties

$$\begin{aligned} T^*T + G^*G &= I, & F^*T + H^*G &= 0, & F^*F + H^*H &= I, \\ TT^* + FF^* &= I, & GT^* + HF^* &= 0, & GG^* + HH^* &= I. \end{aligned} \quad (3.5)$$

The *transfer function* Θ of the unitary colligation U is defined by

$$\Theta(z) := H + zG(I - zT)^{-1}F, \quad z \in \mathbb{D} = \{w \in \mathbb{C} : |w| < 1\}, \quad (3.6)$$

and $\Theta(z) := \Theta(1/\bar{z})^*$ for $z \in \mathbb{D}^* = \{w \in \mathbb{C} \cup \{\infty\} : |w| > 1\}$. It is clear that Θ is a $\mathbf{B}(\mathcal{H})$ -valued holomorphic function on $\mathbb{D} \cup \mathbb{D}^*$ and a straightforward computation using (3.5) shows that

$$I - \Theta(z)\Theta(z)^* = G(I - zT)^{-1}(1 - z\bar{z})(I - \bar{z}T^*)^{-1}G^* \geq 0, \quad z \in \mathbb{D},$$

and therefore, $\|\Theta(z)\| \leq 1$, $z \in \mathbb{D} \cup \mathbb{D}^*$, i.e., Θ is a $\mathbf{B}(\mathcal{H})$ -valued *Schur function*.

In the next theorem the closed symmetric relation S , the domain \mathcal{T} of the boundary relation Γ corresponding to \tilde{A} in (3.4), and the Weyl family M associated to Γ are specified.

Theorem 3.3. *Let \tilde{A} be a selfadjoint relation in $\mathfrak{H} \oplus \mathcal{H}$ of the form (3.4). Then the following statements hold:*

- (i) $S := \{(T - I)h, (\mu T - \bar{\mu})h\} : h \in \ker G\}$ is a closed symmetric relation in \mathfrak{H} and the linear relation

$$\mathcal{T} := \{(T - I)h + Ff, (\mu T - \bar{\mu})h + \mu Ff\} : h \in \mathfrak{H}, f \in \mathcal{H}$$

is dense in S^* , so that $S^* = \overline{\mathcal{T}}$, and $S = \mathcal{T}^*$.

- (ii) The defect space $\mathfrak{N}_\lambda(\mathcal{T}) = \ker(\mathcal{T} - \lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, is given by

$$\mathfrak{N}_\lambda(\mathcal{T}) = \begin{cases} \text{span} \{(1 - zT)^{-1}Ff : f \in \mathcal{H}\}, & \lambda \in \mathbb{C}_+, \\ \text{span} \{(1 - z^{-1}T^*)^{-1}G^*f : f \in \mathcal{H}\}, & \lambda \in \mathbb{C}_-, \end{cases}$$

where $z = (\lambda - \mu)/(\lambda - \bar{\mu})$.

- (iii) A boundary relation Γ for S^* is defined by

$$\Gamma := \left\{ \left(\begin{pmatrix} (T - I)h + Ff \\ (\mu T - \bar{\mu})h + \mu Ff \end{pmatrix}, \begin{pmatrix} Gh + (H - I)f \\ -\mu Gh - (\mu H - \bar{\mu})f \end{pmatrix} \right) : h \in \mathfrak{H}, f \in \mathcal{H} \right\}.$$

- (iv) If Θ is the transfer function of the unitary colligation $U = C_\mu(\tilde{A})$ in (3.3), then the Weyl family M corresponding to the boundary relation Γ is

$$M(\lambda) = \begin{cases} \{(I - \Theta(z))f, (\mu\Theta(z) - \bar{\mu})f\} : f \in \mathcal{H}\}, & \lambda \in \mathbb{C}_+, \\ \{(I - \Theta(z))f, (\bar{\mu}\Theta(z) - \mu)f\} : f \in \mathcal{H}\}, & \lambda \in \mathbb{C}_-. \end{cases}$$

Proof. (i) It is not difficult to see that S and \mathcal{T} coincide with $\tilde{A} \cap \mathfrak{H}^2$ and $\widehat{P}_{\mathfrak{H}} \tilde{A}$, respectively, where $\widehat{P}_{\mathfrak{H}}$ is the projection of the entries in \tilde{A} onto \mathfrak{H}^2 . Now [5, Proposition 2.12] implies (i).

- (ii) An element $(T - I)h + Ff$ belongs to $\ker(\mathcal{T} - \lambda)$ if and only if

$$(\mu T - \bar{\mu})h + \mu Ff = \lambda(T - I)h + \lambda Ff.$$

For $\lambda \in \mathbb{C}_+$, this is equivalent to $h = z(I - zT)^{-1}Ff$ and hence $\mathfrak{N}_\lambda(\mathcal{T})$ is of the asserted form for $\lambda \in \mathbb{C}_+$. Making use of the unitarity of U in (3.3), the relation \mathcal{T} may equivalently be written in the form

$$\mathcal{T} = \{(I - T^*)h - G^*f, (\mu - \bar{\mu}T^*)h - \bar{\mu}G^*f\} : h \in \mathfrak{H}, f \in \mathcal{H}$$

and a similar argument as above implies $\mathfrak{N}_\lambda(\mathcal{T}) = \text{span} \{(1 - z^{-1}T^*)^{-1}G^*f : f \in \mathcal{H}\}$ for $\lambda \in \mathbb{C}_-$.

- (iii) Clearly $\ker \Gamma = S$, where S is as in part (i). Now the fact that Γ is a boundary relation for S^* follows immediately from the selfadjointness of \tilde{A} .

- (iv) The definition of the Weyl family and the transfer function in (3.6) together with the form of the boundary relation Γ in (iii) lead to

$$M(\lambda) = \{(I - \Theta(z))f, (\mu\Theta(z) - \bar{\mu})f\} : f \in \mathcal{H} \quad \text{for } \lambda \in \mathbb{C}_+.$$

For $\lambda \in \mathbb{C}_-$, the statement follows from the symmetry property $M(\bar{\lambda}) = M(\lambda)^*$. \square

If S , \mathcal{T} , and Γ are as in Theorem 3.3 then the relations Γ_0 and Γ_1 are given by

$$\Gamma_0 = \left\{ \left\{ \left(\begin{array}{l} (T - I)h + Ff \\ (\mu T - \bar{\mu})h + \mu Ff \end{array} \right), Gh + (H - I)f \right\} : h \in \mathfrak{H}, f \in \mathcal{H} \right\}$$

and

$$\Gamma_1 = \left\{ \left\{ \left(\begin{array}{l} (T - I)h + Ff \\ (\mu T - \bar{\mu})h + \mu Ff \end{array} \right), -\mu Gh - (\mu H - \bar{\mu})f \right\} : h \in \mathfrak{H}, f \in \mathcal{H} \right\},$$

respectively, and their kernels are symmetric extensions of S given by

$$\begin{aligned} A_0 &= \{ \{ (T - I)h + Ff, (\mu T - \bar{\mu})h + \mu Ff \} : Gh + (H - I)f = 0 \}, \\ A_1 &= \{ \{ (T - I)h + Ff, (\mu T - \bar{\mu})h + \mu Ff \} : \mu Gh + (\mu H - \bar{\mu})f = 0 \}. \end{aligned}$$

The special case of the Weyl family M in Theorem 3.3 being a uniformly strict Nevanlinna function may be characterized in terms of the operators H , F , and G .

Proposition 3.4. *Let $S \subset \mathcal{T}$ and Γ be as in Theorem 3.3 and let M be the Weyl family corresponding to Γ . Then M is uniformly strict if and only if $I - HH^* = GG^*$ is uniformly positive or, equivalently, $I - H^*H = F^*F$ is uniformly positive. In this case $M(\lambda) \in \mathbf{B}(\mathcal{H})$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and $\text{Im } M(\lambda)$ is uniformly positive (uniformly negative) for all $\lambda \in \mathbb{C}_+$ ($\lambda \in \mathbb{C}_-$, respectively). Furthermore, $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is an ordinary boundary triplet for S^* .*

4. A REPRODUCING KERNEL HILBERT SPACE MODEL FOR NEVANLINNA FAMILIES

The following theorem states that each Nevanlinna family may be realized as the Weyl family of a boundary relation associated with the operator of multiplication by the independent variable in the reproducing kernel Hilbert space $\mathfrak{H}(A, B)$; see also [2], [4].

Theorem 4.1. *Let M be a Nevanlinna family in \mathcal{H} , let $\{A, B\}$ be a symmetric Nevanlinna pair such that $M(\lambda) = \{ \{ A(\lambda)g, B(\lambda)g \} : g \in \mathcal{H} \}$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and let $\mathfrak{H}(A, B)$ be the corresponding reproducing kernel Hilbert space. Then:*

- (i) *the multiplication by the independent variable in $\mathfrak{H}(A, B)$, that is,*

$$S = \{ \{ \varphi, \psi \} \in \mathfrak{H}(A, B)^2 : \psi(\lambda) = \lambda \varphi(\lambda) \},$$

defines a closed simple symmetric operator in $\mathfrak{H}(A, B)$;

- (ii) *the linear relation*

$$\mathcal{T} = \{ \{ \varphi, \psi \} \in \mathfrak{H}(A, B)^2 : \psi(\lambda) - \lambda \varphi(\lambda) = A(\lambda)c_1 + B(\lambda)c_2, c_1, c_2 \in \mathcal{H} \}$$

is dense in S^ ;*

(iii) the linear relation

$$\Gamma = \left\{ \left\{ \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, \begin{pmatrix} c_2 \\ -c_1 \end{pmatrix} \right\} : \{\varphi, \psi\} \in \mathcal{T} \right\}$$

is a boundary relation for S^* whose Weyl family is M .

Proof. To the Nevanlinna family $M(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, associate the Schur function

$$\Theta(z) = \begin{cases} I - (\mu - \bar{\mu})(M(\lambda) + \mu)^{-1}, & \lambda \in \mathbb{C}_+, \\ I - (\bar{\mu} - \mu)(M(\lambda) + \bar{\mu})^{-1}, & \lambda \in \mathbb{C}_-, \end{cases} \quad z = \frac{\lambda - \mu}{\lambda - \bar{\mu}}, \quad (4.1)$$

and define the corresponding Schur kernel $S_\Theta(w, z)$ on $(\mathbb{D} \cup \mathbb{D}^*) \times (\mathbb{D} \cup \mathbb{D}^*)$ by

$$S_\Theta(w, z) := \begin{cases} \frac{1 - \Theta(z)\Theta(w)^*}{1 - z\bar{w}}, & w \in \mathbb{D}, z \in \mathbb{D}, \\ \frac{\Theta(z) - \Theta(w)^*}{1/z - \bar{w}}, & w \in \mathbb{D}, z \in \mathbb{D}^*, \\ \frac{\Theta(z) - \Theta(w)^*}{z - 1/\bar{w}}, & w \in \mathbb{D}^*, z \in \mathbb{D}, \\ \frac{1 - \Theta(z)\Theta(w)^*}{1 - 1/(z\bar{w})}, & w \in \mathbb{D}^*, z \in \mathbb{D}^*. \end{cases} \quad (4.2)$$

The kernel $S_\Theta(\cdot, \cdot)$ is hermitian, holomorphic, and nonnegative, see [12]. Let $\mathfrak{S}(\Theta)$ be the reproducing kernel Hilbert space associated with the Schur kernel in (4.2). It follows from [1, Theorem 2.3.1] that

$$U = \begin{pmatrix} T & F \\ G & H \end{pmatrix} : \begin{pmatrix} \mathfrak{S}(\Theta) \\ \mathcal{H} \end{pmatrix} \rightarrow \begin{pmatrix} \mathfrak{S}(\Theta) \\ \mathcal{H} \end{pmatrix},$$

where T , F , G , and H are defined by

$$\begin{aligned} (Th)(z) &:= \begin{cases} \frac{1}{z}(h(z) - h(0)), & z \in \mathbb{D}, \\ \frac{1}{z}h(z) - \Theta(z)h(0), & z \in \mathbb{D}^*, \end{cases} \\ (Ff)(z) &:= \begin{cases} \frac{1}{z}(\Theta(z) - \Theta(0))f, & z \in \mathbb{D}, \\ (I - \Theta(z)\Theta(0))f, & z \in \mathbb{D}^*, \end{cases} \\ Gh &:= h(0), \\ Hf &:= \Theta(0)f, \end{aligned}$$

is a unitary colligation such that the transfer function of U coincides with $\Theta(z)$. The linear relations in Theorem 3.3 (i) in the Hilbert space $\mathfrak{S}(\Theta)$ will now be denoted by $S_{\mathfrak{S}(\Theta)}$ and $\mathcal{T}_{\mathfrak{S}(\Theta)}$ instead of S and \mathcal{T} , respectively. Denote by $\Gamma_{\mathfrak{S}(\Theta)}$ the boundary relation for $S_{\mathfrak{S}(\Theta)}^*$ in Theorem 3.3 (iii).

Observe that M can be recovered from (4.1) by

$$M(\lambda) = \begin{cases} \{ \{(I - \Theta(z))f, (\mu\Theta(z) - \bar{\mu})f\} : f \in \mathcal{H} \}, & \lambda \in \mathbb{C}_+, \\ \{ \{(I - \Theta(z))f, (\bar{\mu}\Theta(z) - \mu)f\} : f \in \mathcal{H} \}, & \lambda \in \mathbb{C}_-, \end{cases}$$

and that this representation of M induces the Nevanlinna pair $\{A(\lambda), B(\lambda)\}$,

$$A(\lambda) := I - \Theta(z), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \quad B(\lambda) := \begin{cases} \mu\Theta(z) - \bar{\mu}, & \lambda \in \mathbb{C}_+, \\ \bar{\mu}\Theta(z) - \mu, & \lambda \in \mathbb{C}_-. \end{cases} \quad (4.3)$$

To prove the theorem it is sufficient to consider the reproducing kernel Hilbert space $\mathfrak{H}(A, B)$ associated with this special Nevanlinna pair in (4.3). Then the Schur kernel $S_\Theta(w, z)$ and the Nevanlinna kernel $K_{A,B}(\xi, \lambda)$ are connected via

$$S_\Theta(w, z) = r(\lambda)K_{A,B}(\xi, \lambda)r(\xi)^*, \quad \lambda, \xi \in \mathbb{C}_+ \cup \mathbb{C}_-, \quad \xi \neq \bar{\lambda},$$

where $z = (\lambda - \mu)/(\lambda - \bar{\mu})$, $w = (\xi - \mu)/(\xi - \bar{\mu})$, and

$$r(\lambda) = \begin{cases} (\lambda - \bar{\mu})/(\mu - \bar{\mu}), & \lambda \in \mathbb{C}_+, \\ (\lambda - \mu)/(\bar{\mu} - \mu), & \lambda \in \mathbb{C}_-. \end{cases}$$

The multiplication by r is a unitary mapping from the reproducing kernel Hilbert space $\mathfrak{H}(A, B)$ onto the reproducing kernel Hilbert space $\mathfrak{S}(\Theta)$. Furthermore, straightforward calculations show that S and \mathcal{T} in (i) and (ii) are connected to $S_{\mathfrak{S}(\Theta)}$ and $\mathcal{T}_{\mathfrak{S}(\Theta)}$ via

$$rS = S_{\mathfrak{S}(\Theta)} \quad \text{and} \quad r\mathcal{T} = \mathcal{T}_{\mathfrak{S}(\Theta)},$$

respectively. Now, the fact that $\Gamma_{\mathfrak{S}(\Theta)}$ is a boundary relation for $S_{\mathfrak{S}(\Theta)}^*$ yields that Γ in (iii) is a boundary relation for S^* .

It remains to show that the Weyl family $M_\Gamma(\lambda)$ corresponding to Γ coincides with the Nevanlinna pair $\{A(\lambda), B(\lambda)\}$ in (4.3). Let $w \in \mathbb{C} \setminus \mathbb{R}$ and assume that $\{\varphi, \psi\} \in \hat{\mathfrak{N}}_w(\mathcal{T})$. Then $\psi(\lambda) = w\varphi(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$. In view of (ii), this means that

$$(w - \lambda)\varphi(\lambda) = A(\lambda)c_1 + B(\lambda)c_2, \quad c_1, c_2 \in \mathcal{H}.$$

In particular, $A(w)c_1 + B(w)c_2 = 0$. Now the form of Γ in (iii) implies that

$$M_\Gamma(w) \subset \{ \{c_2, -c_1\} : A(w)c_1 + B(w)c_2 = 0 \}.$$

By the symmetry property $A(\bar{w})^* = A(w)$, $B(\bar{w})^* = B(w)$ of the Nevanlinna pair $\{A(\lambda), B(\lambda)\}$ in (4.3), one has

$$M(w) = \{ \{A(w)g, B(w)g\} : g \in \mathcal{H} \} = \{ \{c_2, -c_1\} : A(w)c_1 + B(w)c_2 = 0 \}.$$

Therefore, $M_\Gamma(w) \subset M(w)$ holds. Since both families are Nevanlinna families, their maximality property implies the claim $M_\Gamma(w) = M(w)$, $w \in \mathbb{C} \setminus \mathbb{R}$. \square

If the Nevanlinna family M is a $\mathbf{B}(\mathcal{H})$ -valued Nevanlinna function, then the Nevanlinna pair $\{I, M\}$ can be chosen and the reproducing kernel Hilbert space is $\mathfrak{H}(M)$. It is left to the reader to formulate a corollary in this special situation. Furthermore, it is well known that uniformly strict $\mathbf{B}(\mathcal{H})$ -valued Nevanlinna functions may be realized as Weyl functions of ordinary boundary triplets, cf. [6, 7] and [5]. For these classes of Nevanlinna functions, Theorem 4.1 yields the following result, see [7, Proposition 5.3].

Corollary 4.2. *Let M be a uniformly strict $\mathbf{B}(\mathcal{H})$ -valued Nevanlinna function. Then $S = \{\{\varphi, \psi\} \in \mathfrak{H}(M)^2 : \psi(\lambda) = \lambda\varphi(\lambda)\}$ is a closed simple symmetric operator in $\mathfrak{H}(M)$, the linear relation*

$$\mathcal{T} = \{ \{\varphi, \psi\} \in \mathfrak{H}(M)^2 : \psi(\lambda) - \lambda\varphi(\lambda) = c_1 + M(\lambda)c_2, \ c_1, c_2 \in \mathcal{H} \}$$

coincides with S^* , $\mathcal{T} = S^*$, and $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$, where

$$\Gamma_0\{\varphi, \psi\} = c_2 \quad \text{and} \quad \Gamma_1\{\varphi, \psi\} = -c_1, \quad \{\varphi, \psi\} \in S^*,$$

is an ordinary boundary triplet for S^* .

The following example formally illustrates Theorem 4.1 and Corollary 4.2 for the uniformly strict scalar Nevanlinna function $M(\lambda) = \tan a\lambda$, where $a > 0$.

Example 4.3. Let $(-a, a)$, $a > 0$, be a finite interval and consider in $L^2((-a, a))$ the densely defined maximal first order differential operator (in graph notation)

$$T_{\max} = \{ \{f, if'\} : f \text{ absolutely continuous, } f, f' \in L^2((-a, a)) \}$$

generated by $i d/dx$. The corresponding minimal operator

$$T_{\min} = \{ \{f, if'\} \in T_{\max} : f(-a) = f(a) = 0 \}$$

is a closed symmetric operator with defect numbers $(1, 1)$ and $T_{\min}^* = T_{\max}$.

Let \mathcal{L}_a be the reproducing kernel Hilbert space generated by the kernel

$$K(\xi, \lambda) = \frac{\sin a\lambda \cos a\bar{\xi} - \cos a\lambda \sin a\bar{\xi}}{\lambda - \bar{\xi}} = \frac{\sin a(\lambda - \bar{\xi})}{\lambda - \bar{\xi}}, \tag{4.4}$$

$\lambda, \bar{\xi} \in \mathbb{C} \setminus \mathbb{R}$. The space \mathcal{L}_a consists of all those entire functions of type $\leq a$ which are square integrable on \mathbb{R} , cf. [3]. By the Paley-Wiener theorem, the mapping

$$f \mapsto \varphi(\lambda) = \int_{-a}^a e^{i\lambda t} f(t) dt, \quad \lambda \in \mathbb{C}, \tag{4.5}$$

provides an isometric isomorphism from $L^2((-a, a))$ onto \mathcal{L}_a .

Note that the images of T_{\min} and T_{\max} under the Fourier transform (4.5) are given (in graph notation) by

$$S = \{ \{\varphi, \psi\} \in \mathcal{L}_a^2 : \psi(\lambda) = \lambda\varphi(\lambda) \},$$

$$S^* = \{ \{\varphi, \psi\} \in \mathcal{L}_a^2 : \psi(\lambda) - \lambda\varphi(\lambda) = c_1 \cos a\lambda + c_2 \sin a\lambda, \ c_1, c_2 \in \mathbb{C} \}.$$

In terms of the preimage f of φ , the values c_1 and c_2 are given by

$$c_1 = i(f(a) - f(-a)) \quad \text{and} \quad c_2 = -(f(a) + f(-a)).$$

An interpretation of c_1 and c_2 in S^* is that they represent boundary values of the element $\{\varphi, \psi\} \in S^*$.

Note that the Nevanlinna pair $\{\cos a\lambda, \sin a\lambda\}$ is equivalent to the Nevanlinna function $\tan a\lambda$. The Nevanlinna kernel in (4.4) transforms accordingly into the Nevanlinna kernel

$$N(\xi, \lambda) = \frac{\tan a\lambda - \tan a\bar{\xi}}{\lambda - \bar{\xi}}, \quad \lambda, \xi \in \mathbb{C} \setminus \mathbb{R}.$$

Denote the corresponding reproducing kernel Hilbert space by \mathcal{C}_a . Then the function $\varphi(\lambda)$ belongs to \mathcal{L}_a if and only if the function $(\cos a\lambda)^{-1}\varphi(\lambda)$ belongs to \mathcal{C}_a .

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