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Dedicated to Professor Bogdan Choczewski, my Colleague,
on the occasion of his 70th birthday.

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STABILITY OF THE EQUATION OF HOMOMORPHISM AND COMPLETENESS OF THE UNDERLYING SPACE

Abstract. We prove that all assumptions of a Theorem of Forti and Schwaiger (cf. [4]) on the coherence of stability of the equation of homomorphism with the completeness of the space of values of all these homomorphisms, are essential. We give some generalizations of this theorem and certain examples of applications.

Keywords: functional equation, stability of equations of homomorphisms, superstability, complete vector spaces.

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1. INTRODUCTION

Let $g : G \to V$ be a homomorphism of a group $(G, +)$ into a vector space $(V, +)$, i.e.,

$$g(x + y) = g(x) + g(y), \quad x, y \in G. \tag{1}$$

It is well known that completeness is of great importance in the theory of stability of functional equations, in particular for the stability of equation (1) (cf. Hyers's result in [5]). Completeness of the space of values of a homomorphism is to some extent necessary for the stability of equation of the homomorphism. In paper [4], G.L. Forti and J. Schwaiger have proved an important result to this effect, using the following notion of stability:

Definition 1.1. *Equation* (1) *is said to be stable* (*in this paper: b-stable*) *if for every function* $f : G \to V$ *satisfying the inequality*

$$|f(x + y) - f(x) - f(y)| \le \delta, \quad x, y \in G \tag{2}$$

with some real number $\delta > 0$*, there exist a real number* $\varepsilon > 0$ *and a solution* g *of equation* (1) *such that*

$$|f(x) - g(x)| \le \varepsilon, \quad x \in G. \tag{3}$$

Theorem 1.2 ([4]). *Assume that*

1) $(G, +)$ *is an Abelian group,*
2) *there exists an element of infinite order in* G*,*
3) V *is a normed space,*
4) *equation* (1) *is stable.*

Then the space V *is complete.*

The main aim of this note is to prove that all assumptions 1)–4) are essential for Theorem 1.2 to hold true. We shall also deal with other notions of stability of equation (1) and we give some generalizations of Theorem 1.2 as well as its applications.

2. DEFINITIONS OF STABILITY

Definition 2.1. *Equation* (1) *is said to be Ulam-Hyers stable if for every* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that for every function* $f : G \to V$ *satisfying* (2) *there exists a solution* g *of* (1) *such that condition* (3) *is fulfilled.*

Definition 2.2. *Equation* (1) *is called uniformly b-stable if for every* $\delta > 0$ *there exists an* $\varepsilon > 0$ *such that for every function* $f : G \to V$ *satisfying* (2) *there exists a solution* g *of* (1) *such that condition* (3) *is fulfilled.*

In all that follows expressions "stability" or "equation is stable in any sense" mean: in the sense of one of Definitions 1.1, 2.1, or 2.2.

We shall also consider two notions of superstability.

Definition 2.3. *Equation* (1) *is called superstable if every function* $f : G \to V$ *satisfying condition* (2) *for any* $\delta > 0$ *is bounded or it is a solution of* (1).

Definition 2.4. *Let* L *and* R *by any mappings of some function space into another one. The functional equation* $L(g) = R(g)$ *is said to be uniformly superstable if for every* $\delta > 0$ *there exists an* $\varepsilon > 0$ *such that for every* $f : G \to V$ *if* $|L(f) - R(f)| \le \delta$*, then* $|f(x)| \le \varepsilon$ *for* $x \in G$ *or* $L(f) = R(f)$.

Remark 2.5. *Uniform superstability implies superstability, but not vice-versa. Indeed, it suffices to consider the following equation*

$$\frac{1}{g(x)} = 0 \quad \textit{for} \quad g : G = \{0\} \to V = \mathbb{R}.$$

Since every function from G *into* V *is bounded, this equation is superstable. However, it is not uniformly superstable (setting* $f_n(0) = n \in \mathbb{N}$ *we plainly have* $|\frac{1}{f_n(0)}| \le 1$*, but neither is* f_n *a solution of our equation, nor does there exist an* $\varepsilon > 0$ *for which* $|f_n(0)| = |n| \le \varepsilon$, $n \in \mathbb{N}$*).*

Since the equation in consideration has no solutions at all, it is also neither Ulam-Hyers stable nor b-stable (in the sense of Definitions 1.1 and 2.1 adapted to this equation).

Remark 2.6. *Let G and V satisfy the assumptions of Theorem 1.2. Then Definitions 1.1, 2.1 and 2.2 are equivalent (cf. [8]). Generally they are not equivalent if V is a metric space ([7], see also Section 4 below). The superstability implies the b-stability for a normed space V but not conversely (the equation (1) for $g : (\mathbb{R}, +) \to (\mathbb{R}, +)$ is b-stable and it is not superstable). Hence Theorem 1.2 is also true if "stable" in 4) is meant in the sense of Definitions 2.1 and 2.2.*

Remark 2.7. *If the equation of homomorphism (1) is superstable, then it is also uniformly superstable. Indeed, to see this, assume that the function $f : G \to V$ satisfies (2). Then f either is bounded or is a homomorphism, which is also bounded; otherwise the function $f + a$, where $a \in V$ and $|a| = \delta$, satisfies (2), it is unbounded and is not the solution of (1). Thus, f satisfying (2) is always bounded. Since $|f(2x) - 2f(x)| \le \delta$ for $x \in G$, then $|f(x)| \le \delta$ for $x \in G$ by Lemma 7.2 below. The equation (1) is also uniformly superstable ($\varepsilon = \delta$).*

3. COMMUTATIVITY OF THE GROUP G

J. Lawrence has proved that any torsion-free group (also a non-commutative one) can be embedded in a group for which the equation (1), where V is a normed space, is b-stable (cf. [3, p. 149]). For this reason, assumption that G is Abelian is essential for Theorem 1.2 to hold true.

In the noncommutative case, we obtain the following result.

Theorem 3.1. *Let $(G, +)$ be a semigroup cancellative by 2 with the following property: there exists an element $x_0 \in G$ such that for every $n, m \in \mathbb{N}_0$, there is $2^n x_0 \neq 2^m x_0$ for $n \neq m$. Let $(V, +)$ be a semigroup with a metric ρ such that*

$$\rho(2a, 2b) = 2\rho(a, b), \quad a, b \in V.$$

If the equation

$$g(2x) = 2g(x), \quad x \in G, \tag{4}$$

where g is a function from G into V, is b-stable (in Definition 1.1 replace (1) by (4) and take $\rho(u, v)$ in place of $|u - v|$), then V is complete.

Proof. Assume that equation (4) is b-stable. Take a Cauchy sequence $\{a_n\}$ in V. Let $\{b_n\}$ be a subsequence of $\{a_n\}$ such that $\rho(b_{n+1}, b_n) \le 2^{-n}$ for every $n \in \mathbb{N}$. We define the function $f : G \to V$ by

$$\begin{aligned} f(2^n x_0) &:= 2^{n-1} b_n && \text{for} \quad n \in \mathbb{N}, \\ f(x) &:= b_1 && \text{for} \quad x \in G \setminus \{2^n x_0 : n \in \mathbb{N}\}. \end{aligned}$$

First, note that for every $n \in \mathbb{N}$ there is

$$\rho\big(f(2^{n+1}x_0), 2f(2^n x_0)\big) = 2^n \rho(b_{n+1}, b_n) \le 1$$

and consequently

$$\rho\big(f(2x), 2f(x)\big) \le 1, \quad x \in \{2^n x_0 : n \in \mathbb{N}\}. \tag{5}$$

Now, if $x \in G \setminus \{2^n x_0 : n \in \mathbb{N}\}$ and $x \neq x_0$, then $2x$ belongs to this set as well and

$$\rho\big(f(2x), 2f(x)\big) = \rho(b_1, 2b_1). \tag{6}$$

The equality remains valid for $x = x_0$ (see the definition of f). Conditions (5) and (6) lead to the conclusion that

$$\rho\big(f(2x), 2f(x)\big) \leq \delta := \max\{1, \rho(b_1, 2b_1)\}, \quad x \in G. \tag{7}$$

Owing to b-stability of equation (4), there exist a solution g of (4) and an $\varepsilon > 0$ such that $\rho(f(x), g(x)) \leq \varepsilon$ for every $x \in G$. Hence,

$$\rho\big(f(2^n x_0), 2^n g(x_0)\big) = \rho\big(f(2^n x_0), g(2^n x_0)\big) \leq \varepsilon, \quad n \in \mathbb{N},$$

which in turn yields

$$\rho\big(b_n, 2g(x_0)\big) \leq 2^{-n+1}\varepsilon, \quad n \in \mathbb{N}. \tag{8}$$

From (8) we deduce that the sequence $\{b_n\}$ tends to $2g(x_0)$. Consequently, the sequence $\{a_n\}$ is convergent and the space V is complete. □

Proposition 3.2. *If $(G, +)$ is an Abelian semigroup and $(V, +)$ is a semigroup uniquely divisible by 2 with a metric ρ such that*

$$2\rho(a, b) \leq \rho(2a, 2b), \quad a, b \in V, \tag{9}$$

then the stability of equation (4) in any sense implies stability of equation (1) in the same sense.

Proof. Assume that equation (4) is b-stable. Take a function $f : G \to V$ such that $\rho(f(x+y), f(x)+f(y)) \leq \delta$ for every $x, y \in G$. Then, in particular, $\rho(f(2x), 2f(x)) \leq \delta$. Hence, there exist a solution g of (4) and an $\varepsilon > 0$ such that $\rho(f(x), g(x)) \leq \varepsilon$ for every $x \in G$. The latter inequality leads to

$$2^n \rho\big(2^{-n} f(2^n x), g(x)\big) \leq \rho\big(f(2^n x), g(2^n x)\big) \leq \varepsilon.$$

As a consequence, there exists a finite limit of the sequence $\{2^{-n} f(2^n x)\}$ and it is equal to $g(x)$. Since

$$2^n \rho\big(2^{-n} f(2^n x + 2^n y), 2^{-n} f(2^n x) + 2^{-n} f(2^n y)\big) \leq \rho\big(f(2^n x + 2^n y), f(2^n x) + f(2^n y)\big) \leq \delta,$$

we obtain

$$\rho\big(2^{-n} f(2^n x + 2^n y), 2^{-n} f(2^n x) + 2^{-n} f(2^n y)\big) \leq 2^{-n}\delta.$$

This in turn implies that $\rho(g(x+y), g(x)+g(y)) = 0$ for every $x, y \in G$, the function g is a solution of (1) and equation (1) is b-stable. In the case of uniform b-stability or Ulam-Hyers stability the proof is analogous to the one presented above. □

We now formulate a generalization of Theorem 1.2, saying when the stability of equation (1) implies that of equation (4), which is a counterpart of Proposition 3.2. First, we introduce necessary notation. Namely, if G is a group, then we write G^1 for the commutator subgroup of G. The symbol G/G^1 stands for the quotient group of G by G^1.

Theorem 3.3. *Let V be a normed space and let G be a group such that the quotient group G/G^1 contains an element of infinite order, i.e. there exists an element $a \in G$ with the property that for every $n \in \mathbb{N}$ the element a^n does not belong to G^1. Then the following conditions hold true:*

(i) *if the equation of homomorphism from G to V is stable in any sense, then V is complete;*
(ii) *the normed space V is complete if and only if the equation of homomorphism from G/G^1 to G is stable in any sense.*

Proof. Since every homomorphism g from G to V is identically equal to zero on G^1, thus G^1 is included in the kernel of g. Hence, stability of the equation of homomorphism from G to V implies stability of equation of homomorphism from G/G^1 to V. From this and Theorem 1.2, we derive part (i) of Theorem 3.3. Part (ii) of Theorem 3.3 is evident. □

Remark 3.4. *Let us emphasis that stability of the equation of homomorphism from G to V does not follow from stability of the equation of homomorphism from G/G^1 to V. It suffices to consider a free group G generated by two elements and to take a Banach space V. Since the group G/G^1 is Abelian, the equation of homomorphism from G/G^1 to V is stable. However, in view of results contained in* [2], *the equation of homomorphism from G to V is not stable.*

Remark 3.5. *It may happen that a group G has an element of infinite order, but simultaneously the group G/G_1 does not contain such elements. This follows from a result, due to J. Lawrence, saying that any group can be embedded into a group whose every element is a commutator.* (*The result was communicated to G.L. Forti by F. Zorzitto in his letter of May 31, 1988.*)

Remark 3.6. *In paper* [4], *G.L. Forti and J. Schwaiger noted that their result is valid for every* (*not necessarily commutative*) *group having an element α of infinite order and isomorphic to a certain subgroup of the tensor product $G_1 \otimes G_2$ of two groups G_1 and G_2, where $Z_\alpha \subset G_1 \subset \mathbb{R}$ and Z_α is the subgroup generated by α. In this case, if we take an $a \in \mathbb{Z}_\alpha \setminus \{0\}$, then due to $G_1 = \{0\} \times G_2^1$, the element $\{a\} \times G \in G/G^1$ is of infinite order.*

4. AN ELEMENT OF INFINITE ORDER IN G

It turns out that assumption 2) cannot be dropped without affecting the validity of Theorem 1.2. To see this, let us consider the equation of homomorphism from $G = \{0\}$ into a normed (not necessarily complete) space V. Such equation is stable in any sense. Indeed, if the function $f : G \to V$ satisfies

$$|f(0 + 0) - f(0) - f(0)| = |f(0)| \le \delta,$$

then the inequality $|f(0) - g(x)| \le \delta$ holds, with the function g identically equal to zero being a solution of equation (1).

We now establish a theorem which is a generalization of the final remark appearing at the end of note [4] (see p. 220 therein). The justification of that remark in [4] is incomplete, because the reference to [2, Proposition 1] is incorrect. More precisely, a vector space Y used in [4] is normed, but Y in [2] is assumed to be a Banach space (it is denoted by B therein).

Theorem 4.1. *Let $(G, +)$ be a group which has no element of infinite order. Assume that $(Y, +)$ is a groupoid uniquely divisible by 2 with the element 0 such that $2 \cdot 0 = 0$ and with a metric ρ such that condition (9) holds. Then for every function $f : G \to Y$ the inequality*

$$\rho\big(f(2x), 2f(x)\big) \le \delta, \quad x, y \in G,$$

implies

$$\rho\big(f(x), 0\big) \le \delta, \quad x \in G.$$

Moreover, the equation of homomorphism from G to Y is stable in any sense.

Proof. By induction, for every $n \in \mathbb{N}$ and $a \in Y$ we obtain $2^n \rho(2^{-n}a, 0) \le \rho(a, 0)$, and consequently, $\rho(2^{-n}a, 0) \le 2^{-n}\rho(a, 0)$. Hence, $\rho(2^{-n}a, 0) \to 0$ $(n \to +\infty)$, which means that $2^{-n}a \to 0$ $(n \to +\infty)$ for every $a \in Y$. The set $\{2^n x : n \in \mathbb{N}\}$ is finite for every $x \in G$, thus $2^{-n} f(2^n x) \to 0$ for $n \to +\infty$. Moreover, by induction, $\rho(2^{-n} f(2^n x), f(x)) \le (1 - 2^{-n})\delta$. Hence, $\rho(f(x), 0) \le \delta$ for every $x \in G$. Since the function $g(x) \equiv 0$ is a homomorphism from G to Y, the equation of homomorphism is stable. □

5. THE METRIC SPACE V

As Example 5.1 below shows, Theorem 1.2 (with the completeness as the sequential one) fails to be true if V is assumed to be a topological space only (see the problem in p. 150 in [6]). More precisely, the pertinent implication does not hold true:

> if G is as in Theorem 1.2, V is a metric vector space and equation (1) is uniformly b-stable or superstable, then V is complete. (10)

Example 5.1. *Let G be an arbitrary Abelian group, containing an element of infinite order and let V be a metric vector space which is bounded and not complete (for instance one can take $V = (\mathbb{R}, +)$ with $\rho(a, b) = |\arctan a - \arctan b|$). Then equation of homomorphism (1) is uniformly b-stable and superstable (the function f is bounded for every $f : G \to V$) and V is not complete.*

For $(G, +) = (V, +) = (\mathbb{R}, +)$ and $\rho(a, b) = |\arctan a - \arctan b|$, where $a, b \in V$, equation (1) is not Ulam-Hyers stable. Indeed, assume that it is. For $\varepsilon = \frac{\pi}{4}$ there exists a real number $\delta > 0$ such that for every function $f : G \to V$ satisfying the condition

$$|\arctan f(x + y) - \arctan\big(f(x) + f(y)\big)| \le \delta, \quad x, y \in G,$$

there exists a solution g of equation (1) such that

$$|\arctan f(x) - \arctan g(x)| \le \varepsilon, \quad x \in G.$$

One can assume that $\delta < 1/2$, *in order to define a constant function* $f : G \to V$ *by the formula*

$$f(x) = \frac{1 + (1 - 4\delta^2)^{\frac{1}{2}}}{2\delta} =: a, \quad x \in G.$$

Then $a > 1$ *and for a real number* θ *such that* $a \le \theta \le 2a$ *we obtain*

$$|\arctan f(x+y) - \arctan\big(f(x) + f(y)\big)| = \arctan 2a - \arctan a = \\ = \frac{a}{1+\theta^2} \le \frac{a}{1+a^2} = \delta, \quad x, y \in G.$$

Therefore, there exists a solution g *of equation* (1) *such that*

$$|\arctan f(x) - \arctan g(x)| \le \varepsilon, \quad x \in G. \tag{11}$$

Substituting $x = 0$ *into* (11), *we easily deduce that* $\arctan a \le \varepsilon = \frac{\pi}{4}$. *It follows that* $a \le 1$, *which is a contradiction.*

The above argument reveals that Definitions 1.1 and 2.2 are not equivalent (the uniform b-stability does not imply that in the sense of Ulam-Hyers).

The following question remains open:

Question. Does implication (10) hold true when equation (1) is stable in the Ulam-Hyers sense?

Remark 5.2. *For a function* $g : \mathbb{R} \to \mathbb{R}_+$ (*where* $\mathbb{R}_+ := \{x \in \mathbb{R} : x > 0\}$) *and the metric* $\rho(x, y) = |x - y|$ *defined on* $\mathbb{R}_+$, *the equation* $g(x + y) = g(x)g(y)$ *is uniformly* b*-stable and superstable* (*see* [1], *the first paper on superstability*), *while the space* $\mathbb{R}_+$ *is unbounded and non-complete. This equation is not Ulam-Hyers stable* (*cf.* [8]).

6. NECESSITY OF THE ASSUMPTIONS OF THEOREM 1.2

We are in a position to establish the result announced in the Introduction.

Theorem 6.1. *Each of assumptions 1)–4) of Theorem 1.2 is essential.*

Proof. From the introductory parts of Sections 2–4 it is seen that assumptions 1), 2) and 3) are essential indeed. Assumption 4) is evidently essential, because there exist normed spaces which are not complete. □

Remark 6.2. *If for a normed space* V *there exists a group* G *such that all assumptions of Theorem 1.2 are satisfied, then* V *is a Banach space. The converse implication is also true. It suffices to put* $(G, +) = (V, +)$ (*torsion free*) *and apply Theorem of Hyers* (*cf.* [5]).

7. APPLICATIONS OF THEOREM 1.2

It is possible to prove the completeness of a normed space V by choosing a group G such that the assumptions of Theorem 1.2 are fulfilled.

Example 7.1. *Let $(V, |\cdot|)$ be a Banach space and let S be an arbitrary nonempty set. Denote by V^S the set of all functions $f : S \to V$ such that $\sup_{s\in S} |f_s| < \infty$ (where $f_s = f(s)$). The set V^S equipped with the standard addition and multiplication by scalars and with the norm $\|f\| = \sup_{s\in S} |f_s|$ is a complete space. This follows from the fact that the equation of homomorphism from $(G, +) = (V, +)$ to V^S is stable in any sense.*

Indeed, if $f : G \to V^S$, then

$$\|f(x + y) - f(x) - f(y)\| = \sup_{s\in S} |f_s(x + y) - f_s(x) - f_s(y)| \le \delta, \quad x, y \in G,$$

implies that

$$|f_s(x + y) - f_s(x) - f_s(y)| \le \delta, \quad x, y \in G,\ s \in S.$$

By Theorem of Hyers (cf. [5]), there exists a homomorphism $g_s : G \to V$ such that

$$|f_s(x) - g_s(x)| \le \delta, \quad x \in G,\ s \in S.$$

Then the function $g = \{g_s\} : G \to V^S$ is also a homomorphism and we obtain

$$\|f(x) - g(x)\| = \sup_{s\in S} |f_s(x) - g_s(x)| \le \delta, \quad x \in G.$$

In particular, S can be equal to $\mathbb{N}$ or to $\{1, \dots, m\}$ for $m \in \mathbb{N}$. The same proof may be applied to the space of the convergent sequences (or sequences convergent to zero) and to the space of functions continuous on a compact set S.

Lemma 7.2. *Assume that $(G, +)$ is a groupoid. Let $(Y, +)$ be a groupoid equipped with a metric ρ satisfying condition (9), and with the element 0 such that $2\cdot 0 = 0$. If there exists a real number $\delta > 0$ such that for every $x \in G$ there is $\rho(f(2x), 2f(x)) \le \delta$, then $\rho(f(x), 0) \le \delta$ for every $x \in G$ or the function f is unbounded.*

Proof. Assume that $\rho(f(x), 0) > \delta$ for some $x \in G$. Then there exists an $\alpha > 0$ which satisfies the equality $\rho(f(x), 0) = \delta + \alpha$. The latter, when combined with the fact that

$$\delta \ge \rho\big(f(2x), 2f(x)\big) \ge \rho\big(2f(x), 0\big) - \rho\big(f(2x), 0\big),$$

gives

$$\rho\big(f(2x), 0\big) + \delta \ge \rho\big(2f(x), 0\big) = \rho\big(2f(x), 2\cdot 0\big) \ge 2\rho\big(f(x), 0\big) = 2\delta + 2\alpha.$$

Consequently, $\rho(f(2x), 0) \ge \delta + 2\alpha$. By induction, we deduce that $\rho(f(2^n x), 0) \ge \delta + 2^n\alpha$ for every $n \in \mathbb{N}$. Thus, the function f is unbounded. □

Example 7.3. *Let $(V, |\cdot|)$ be a Banach space. Denote by $V^{\mathbb{N}}$ the normed space of all sequences $x := \{x_n\}_{n=1}^{+\infty}$ such that $\sum_{n=1}^{+\infty} |x_n| < +\infty$ with the standard addition and multiplication by scalars and with the norm given by $\|x\| = \sum_{n=1}^{+\infty} |x_n|$. Since the equation of homomorphism from $(G,+) = (V,+)$ to $V^{\mathbb{N}}$ is stable in any sense, the space $V^{\mathbb{N}}$ is complete.*

To prove the required stability, take $f : G \to V^{\mathbb{N}}$ such that

$$\|f(x+y) - f(x) - f(y)\| = \sum_{n=1}^{+\infty} |f_n(x+y) - f_n(x) - f_n(y)| \le \delta, \quad x, y \in G,$$

where $f = \{f_n\}_{n=1}^{+\infty}$ and $f_n : G \to V$. Thus

$$|f_n(x+y) - f_n(x) - f_n(y)| \le \delta, \quad x, y \in G, \, n \in \mathbb{N}.$$

By Theorem of Hyers ([5]), for every $n \in \mathbb{N}$ there exists a homomorphism $g_n : G \to V$ such that $|f_n(x) - g_n(x)| \le \delta$ for every $x \in G$. Plainly, the function $g = \{g_n\}_{n=1}^{+\infty} : G \to V^{\mathbb{N}}$ is also a homomorphism. Moreover, for an arbitrary $m \in \mathbb{N}$ there is

$$\sum_{n=1}^{m} |f_n(x) - g_n(x)| \le m\delta, \quad x \in G,$$

as well as

$$\sum_{n=1}^{+\infty} |f_n(2x) - 2f_n(x)| \le \delta, \quad x \in G.$$

Hence, setting $h_n := f_n - g_n$ for $n \in \mathbb{N}$, we deduce that

$$\sum_{n=1}^{+\infty} |h_n(2x) - 2h_n(x)| \le \delta, \quad x \in G.$$

Fix $m \in \mathbb{N}$. Let us consider the normed space V^m of all sequences $x = (x_1, \dots, x_m)$, where $x_n \in V$ for $n = 1, \dots, m$, equipped with the standard addition and multiplication by scalars and with the norm given by $\|x\|_m = \sum_{n=1}^{m} |x_n|$. It turns out that for the function

$$H : G \ni x \to \big(h_1(x), \dots, h_m(x)\big) \in \mathbb{N}^m$$

and for every $x \in G$, the following inequalities hold true:

$$\|H(2x) - 2H(x)\|_m = \sum_{n=1}^{m} |h_n(2x) - 2h_n(x)| \le \delta, \quad \|H(x)\|_m \le m\delta.$$

Therefore, the function H is bounded. In view of Lemma 7.2, there is $\|H(x)\|_m \le \delta$. The latter inequality yields $\sum_{n=1}^{m} |h_n(x)| \le \delta$ for every $m \in \mathbb{N}$. As a result, we obtain

$$\sum_{n=1}^{+\infty} |h_n(x)| = \sum_{n=1}^{+\infty} |f_n(x) - g_n(x)| \le \delta, \quad x \in G.$$

This proof implies that the space V^m is complete, too.

An analogous proof works for the normed space consisting of all sequences $x := \{x_n\}_{n=1}^{+\infty}$ forming a Banach space V with

$$\|x\| = |x_1| + \sum_{n=1}^{+\infty} |x_n - x_{n+1}| < \infty.$$

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