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NUMERICAL METHODS FOR HYPERBOLIC DIFFERENTIAL FUNCTIONAL PROBLEMS

Abstract. The paper deals with the initial boundary value problem for quasilinear first order partial differential functional systems. A general class of difference methods for the problem is constructed. Theorems on the error estimate of approximate solutions for difference functional systems are presented. The convergence results are proved by means of consistency and stability arguments. A numerical example is given.

Keywords: functional differential equations, stability and convergence.

Mathematics Subject Classification: 65M12, 35R10.

1. INTRODUCTION

For any metric spaces U and V, by $C(U,V)$ we denote the class of all continuous functions defined on U and taking values in V. We will use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components. Let $M_{k\times n}$ be the set of $k\times n$ matrices with real elements. For $x = (x_1,\ldots,x_n)\in R^n$, $p=(p_1,\ldots,p_k)\in R^k$ and $X\in M_{k\times n}$, $X=[X_{ij}]_{i=1,\ldots,k,j=1,\ldots,n}$ we put

$$\|x\| = |x_1| + \ldots + |x_n|, \quad \|p\| = \max\{\, |p_i| \colon\ 1\le i\le k\,\},$$

$$\|X\| = \max\Big\{ \sum_{j=1}^{n} |x_{ij}| \colon\ \ 1\le i\le k\Big\}.$$

Let $a>0$, $\tau_0\in R_+$, $R_+=[0,+\infty)$, $\tau=(\tau_1,\ldots,\tau_n)\in R_+^n$ and $b=(b_1,\ldots,b_n)\in R^n$ be given, where $b_i>0$ for $1\le i\le n$. Let $c=(c_1,\ldots,c_n)=b+\tau$. Define the sets

$$E=[0,a]\times(-b,b), \qquad D=[-\tau_0,0]\times[-\tau,\tau],$$

and

$$E_0=[-\tau_0,0]\times[-c,c], \qquad \partial_0E=([0,a]\times[-c,c])\setminus E, \qquad E^*=E_0\cup E\cup\partial_0E.$$

Given a function $z\colon E^* \to R^k$ and a point $(t,x) \in \overline{E}$, we consider the function $z_{(t,x)}\colon D \to R^k$ defined by

$$z_{(t,x)}(s,y) = z(t+s, x+y), \qquad (s,y) \in D.$$

The function $z_{(t,x)}$ is the restriction of z to the set $[t-\tau_0, t] \times [x-\tau, x+\tau]$ and this restriction is shifted to the set D. For a function $w \in C(D, R^k)$ we put

$$\|w\|_D = \max\{\, |w(t,x)|\colon \ (t,x) \in D\,\}.$$

Assume that

$$\varrho\colon E \times C(D,R) \to M_{k\times n}, \quad \varrho = [\,\varrho_{ij}\,]_{i=1,\ldots,k,\, j=1,\ldots,n},$$
$$f\colon E \times C(D,R) \to R^k, \qquad f = (f_1, \ldots, f_k),$$

are given functions in the variables (t,x,w). Given a function $\varphi\colon E_0 \cup \partial_0 E \to R^k$, we consider the quasilinear differential functional system

$$\partial_t\, z_i(t,x) = \sum_{j=1}^{n} \varrho_{ij}(t,x,z_{(t,x)})\, \partial_{x_j} z_i(t,x) + f_i(t,x,z_{(t,x)}), \quad i = 1,\ldots,k, \tag{1}$$

with the initial boundary condition

$$z(t,x) = \varphi(t,x) \qquad \text{for} \quad (t,x) \in E_0 \cup \partial_0 E. \tag{2}$$

We consider classical solution of the above problem.

A number of papers concerning difference methods for nonlinear first order differential or functional differential equations have been published in recent years [1,5,8,10,13]. Nonlinear equations and finite systems of equations with initial conditions and mixed problems have been studied in these papers. It is easy to construct Euler's type explicit or implicit difference method (for a nonlinear problem) which satisfies the consistency conditions on all sufficiently regular solutions of a differential or differential functional equations. The main task of these research is to find a finite difference approximation which is stable. The method of difference inequalities and simple theorems on recurrent inequalities are used in the investigation of the stability of nonlinear difference-functional equations generated by initial or mixed problems.

It is easy to see that convergence results of the papers cited above are not applicable to quasilinear systems (1) with initial boundary condition (2). Until now there have been no results on the numerical approximations of classical solutions of problem (1), (2). The aim of the paper is to construct a general class of difference methods for (1), (2). We prove a theorem on the error estimates of approximate solutions for quasilinear functional difference equations of the Volterra type with unknown function in several variables. By an approximate solution, we mean a function satisfying (8), (9). In Theorem (1) we give an estimate of the difference between the exact and approximate solution of (6), (7). We will assume that the functions f_h and ϱ_h in (5) satisfy nonlinear estimates of the Perron type with respect to functional variables.

Then the error of an approximate solution is estimated by a solution of an initial problem for a nonlinear difference equation. We apply this general idea to the investigation of the stability of difference functional system generated by (1), (2). The functions f_h and ϱ_h are superpositions of f and ϱ with suitable interpolating operators. It is an essential fact in our consideration that we have assumed nonlinear estimates of the Perron type for given functions with respect to the functional variables. These assumptions imply the uniqueness of a classical solution of problem (1), (2). In the paper, we use these general ideas for difference equations which were introduced in [2, 11, 12].

Differential equations with a deviated argument and integral differential problems can be obtained from (1), (2) by a specification of the given operators. Existence results are given in [6].

2. DIFFERENCE FUNCTIONAL EQUATIONS

Let $\mathbb{N}$ and $\mathbb{Z}$ be the sets of natural numbers and integers, respectively. For $x, \bar{x} \in R^n$, $x = (x_1, \ldots, x_n)$, $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$, we write $x * \bar{x} = (x_1\bar{x}_1, \ldots, x_n\bar{x}_n)$. We define a mesh on the set E^* in the following way. Suppose that $h = (h_0, h')$ where $h' = (h_1, \ldots, h_n)$ stand for steps of the mesh. Denote by Δ the set of all $h = (h_0, h')$ such that there exist $\tilde{N}_0 \in \mathbb{Z}$ and $N = (N_1, \ldots, N_n) \in \mathbb{Z}^n$ with the properties: $N_0 h_0 = \tau_0$ and $N * h' = \tau$. We assume that $\Delta \neq \emptyset$ and that there exists a sequence $\{h^{(j)}\}$, $h^{(j)} \in \Delta$ such that $\lim_{j\to\infty} h^{(j)} = 0$. For $h \in \Delta$, we put $\|h\| = h_0 + h_1 + \ldots + h_n$. We define nodal points as follows:

$$t^{(r)} = rh_0, \qquad x^{(m)} = m * h', \qquad x^{(m)} = \left(x_1^{(m_1)}, \ldots, x_n^{(m_n)}\right),$$

where $(r, m) \in \mathbb{Z}^{1+n}$. There exists $N_0 \in \mathbb{N}$ such that $N_0 h_0 \le a < (N_0 + 1)h_0$. Let

$$R_h^{1+n} = \left\{(t^{(r)}, x^{(m)}) \colon \ (r, m) \in \mathbb{Z}^{1+n} \right\}$$

and

$$D_h = D \cap R_h^{1+n}, \qquad E_h = E \cap R_h^{1+n},$$

$$\partial_0 E_h = \partial_0 E \cap R_h^{1+n}, \qquad E_{0.h} = E_0 \cap R_h^{1+n}, \qquad E_h^* = E_h \cup E_{0.h} \cup \partial_0 E_h.$$

For a function $z \colon E_h^* \to R^k$, we write $z^{(r,m)} = z(t^{(r)}, x^{(m)})$. For the above z and for a point $(t^{(r)}, x^{(m)}) \in E_h$, we define the function $z_{[r,m]} \colon D_h \to R^k$ by the formula:

$$z_{[r,m]}(s, y) = z(t^{(r)} + s, x^{(m)} + y), \qquad (s, y) \in D_h.$$

The function $z_{[r,m]}$ is the restriction of z to the set

$$\left([t^{(r)} - \tau_0, t^{(r)}] \times [x^{(m)} - \tau, x^{(m)} + \tau]\right) \cap R_h^{1+n}$$

and this restriction is shifted to the set D_h. For a function $w \colon D_h \to R^k$, we put

$$\|w\|_h = \max\{\, \|w^{(r,m)}\| \colon \ (t^{(r)}, x^{(m)}) \in D_h \,\}.$$

Let $e_j = (0,\dots,0,1,0,\dots,0) \in R^n$ with 1 standing on j-th place. For a function $z = (z_1,\dots,z_k),: E_h^* \to R^k$, we define difference operators δ_0, $\delta = (\delta_1,\dots,\delta_n)$ as follows

$$\delta_0 z_i^{(r,m)} = \frac{1}{h_0}[\, z_i^{(r+1,m)} - A z_i^{(r,m)}\,], \quad 1 \le i \le k \tag{3}$$

where

$$A z_i^{(r,m)} = \frac{1}{2n}\sum_{j=1}^{n}\big(\, z_i^{(r,m+e_j)} + z_i^{(r,m-e_j)}\,\big),$$

$$\delta_j z_i^{(r,m)} = \frac{1}{2h_j}[z_i^{(r,m+e_j)} - z_i^{(r,m-e_j)}], \quad 1 \le i \le k, \ \ 1 \le j \le n \tag{4}$$

and

$$\delta_0 z^{(r,m)} = (\delta_0 z_1^{(r,m)},\dots,\delta_0 z_k^{(r,m)}).$$

Let

$$E_h' = \big\{\, (t^{(r)}, x^{(m)}) \in E_h : \quad (t^{(r)} + h_0, x^{(m)}) \in E_h \,\big\}$$

and now by $\mathfrak{F}(D_h, R^k)$ we denote the set of all functions $w\colon D_h \to R^k$. Suppose that

$$\begin{aligned}
&\varrho_h\colon E_h' \times \mathfrak{F}(D_h,R^k) \to M_{k\times n}, && \varrho_h = [\, \varrho_{h\cdot ij}\,]_{i=1,\dots,k,\, j=1,\dots,n},\\
&f_h\colon E_h' \times \mathfrak{F}(D_h,R^k) \to R^k, && f_h = (f_{h\cdot 1},\dots,f_{h\cdot k}),\\
&\varphi_h\colon E_{0\cdot h} \cup \partial_0 E_h \to R^k, && \varphi_h = (\varphi_{h\cdot 1},\dots,\varphi_{h\cdot k}),
\end{aligned}$$

are given functions. Let the operator F_h be defined by

$$\begin{aligned}
&F_h[z]^{(r,m)} = (\, F_{h\cdot 1}[z]^{(r,m)},\dots, F_{h\cdot k}[z]^{(r,m)}),\\
&F_{h\cdot i}[z]^{(r,m)} = \sum_{j=1}^{n} \varrho_{h\cdot ij}(\, t^{(r)}, x^{(m)}, z_{[r,m]})\, \delta_j z_i^{(r,m)} + f_{h\cdot i}(\, t^{(r)}, x^{(m)}, z_{[r,m]}), \quad 1 \le i \le k.
\end{aligned} \tag{5}$$

We will approximate solutions of problem (1), (2) by means of solutions of the difference equation

$$\delta_0\, z^{(r,m)} = F_h[z]^{(r,m)} \tag{6}$$

with the initial boundary condition

$$z^{(r,m)} = \varphi_h^{(r,m)} \qquad \text{on } \ E_{0\cdot h} \cup \partial_0 E_h. \tag{7}$$

There exists exactly one solution $u_h\colon E^* \to R^k$ of problem (6), (7). We need to know what is the relation between the solution u_h of (6), (7) and a function $v_h\colon E_h \to R^k$ satisfying the condition

$$\|\delta_0 v_h^{(r,m)} - F_h[v_h]^{(r,m)}\| \le \alpha(h) \quad \text{on} \quad E_h' \tag{8}$$

and

$$\|v_h^{(r,m)} - \varphi_h^{(r,m)}\| \le \alpha_0(h) \quad \text{on} \quad E_{0\cdot h} \cup \partial_0 E_h \tag{9}$$

where

$$\alpha, \alpha_0 \colon \Delta \to R_+ \quad \text{and} \quad \lim_{h\to 0} \alpha_0(h) = 0, \quad \lim_{h\to 0} \alpha(h) = 0.$$

The function v_h satisfying the above relation is considered as an approximate solution of problem (6), (7). We will need the following assumptions.

Assumption H $[\sigma]$**.** *Suppose that*

1) *the function* $\sigma \colon [0,a] \times R_+ \to R_+$ *is continuous;*
2) $\sigma(t,0) = 0$ *for* $t \in [0,a]$*;*
3) σ *is nondecreasing with respect to both variables;*
4) *for any* $\tilde{c} > 1$*, the Cauchy problem*

$$y'(t) = \tilde{c}\,\sigma(t, y(t)), \qquad y(0) = 0 \tag{10}$$

has the only solution $y(t) = 0$ *for* $t \in [0,a]$*.*

Assumption H $[\varrho_h, f_h]$**.** *Suppose that*

$$\varrho_h \colon E'_h \times \mathfrak{F}(D_h, R^k) \to M_{k\times n} \quad \text{and} \quad f_h \colon E'_h \times \mathfrak{F}(D_h, R^k) \to R^k$$

and there is a function $\sigma \colon [0,a] \times R_+ \to R_+$ *satisfying Assumption* $H[\sigma]$ *and such that*

$$\| \varrho_h(t^{(r)}, x^{(m)}, w) - \varrho_h(t^{(r)}, x^{(m)}, \bar{w})\| \le \sigma(t^{(r)}, \|w - \bar{w}\|_h),$$
$$\| f_h(t^{(r)}, x^{(m)}, w) - f_h(t^{(r)}, x^{(m)}, \bar{w})\| \le \sigma(t^{(r)}, \|w - \bar{w}\|_h)$$

on $E'_h \times \mathfrak{F}(D_h, R^k) \to R^k$*.*

Theorem 1. *Suppose that Assumption* $\mathrm{H}[\varrho_h, f_h]$ *is satisfied and*

1) $h \in \Delta$ *and*

$$\frac{1}{n} - \frac{h_0}{h_j} |\,\varrho_{h\cdot ij}(t,x,w)| \ge 0 \quad \text{on } E'_h \times \mathfrak{F}(D_h, R^k), \quad 1 \le j \le n,\ 1 \le i \le k; \tag{11}$$

2) $u_h \colon E^*_h \to R^k$ *is the solution of problem* (6), (7)*;*
3) $v_h \colon E^*_h \to R^k$ *satisfies relations* (8), (9)*;*
4) *there is* $c_0 \in R_+$ *such that*

$$\|\,\delta_j v_h^{(r,m)}\,\| \le c_0 \qquad \text{on } E_h \quad \text{for } 1 \le j \le n.$$

Under these assumptions, there is $\eta \colon \Delta \to R_+$ *such that*

$$\|\,u_h^{(r,m)} - v_h^{(r,m)}\,\| \le \eta(h) \quad \text{on } E_h \quad \text{and} \quad \lim_{h\to 0} \eta(h) = 0. \tag{12}$$

Proof. Let

$$\Gamma_h \colon E'_h \to R^k, \qquad \Gamma_h = (\Gamma_{h\cdot 1}, \dots, \Gamma_{h\cdot k}),$$
$$\Gamma_{0\cdot h} \colon E_{0\cdot h} \cup \partial_0 E_h \to R^k, \qquad \Gamma_{0\cdot h} = (\Gamma_{0\cdot h1}, \dots, \Gamma_{0\cdot hk})$$

be functions defined by the relations

$$\delta_0\, v_h^{(r,m)} = F_h[v_h]^{(r,m)} + \Gamma_h^{(r,m)} \qquad \text{on } E_h' \tag{13}$$

and

$$v_h^{(r,m)} = \varphi_h^{(r,m)} + \Gamma_{0\cdot h}^{(r,m)} \qquad \text{on } E_{0\cdot h} \cup \partial_0 E_h. \tag{14}$$

Then

$$\begin{aligned} &\|\Gamma_h^{(r,m)}\| \le \alpha(h) && \text{on } E_h',\\ &\|\Gamma_{0\cdot h}^{(r,m)}\| \le \alpha_0(h) && \text{on } E_{0\cdot h} \cup \partial_0 E_h \end{aligned} \tag{15}$$

and

$$\lim_{h\to 0} \alpha_0(h) = 0, \qquad \lim_{h\to 0} \alpha(h) = 0. \tag{16}$$

The function $w_h = u_h - v_h$, $w_h = (w_{h\cdot 1}, \ldots, w_{h\cdot k})$, satisfies the difference functional system

$$\begin{aligned} \delta_0\, w_{h\cdot i}^{(r,m)} \;=\; & \sum_{j=1}^{n} \varrho_{h\cdot ij}\big(\, t^{(r)}, x^{(m)}, (u_h)_{[r,m]}\,\big)\, \delta_j w_{h\cdot i}^{(r,m)} + \\ & + \sum_{j=1}^{n} \Big[\, \varrho_{h\cdot ij}\big(\, t^{(r)}, x^{(m)}, (u_h)_{[r,m]}\,\big) - \varrho_{h\cdot ij}\big(\, t^{(r)}, x^{(m)}, (v_h)_{[r,m]}\,\big)\,\Big] \delta_j v_{h\cdot i}^{(r,m)} + \\ & + f_{h\cdot i}\big(\, t^{(r)}, x^{(m)}, (u_h)_{[r,m]}\,\big) - f_{h\cdot i}\big(\, t^{(r)}, x^{(m)}, (v_h)_{[r,m]}\,\big) - \\ & - \Gamma_{h\cdot i}^{(r,m)}, \qquad 1 \le i \le k. \end{aligned} \tag{17}$$

Write

$$P^{(r,m)}[z] = (\, t^{(r)}, x^{(m)}, z_{[r,m]}\,) \tag{18}$$

and $\Lambda_h = (\Lambda_{h\cdot 1}, \ldots, \Lambda_{h\cdot k})$ where

$$\begin{aligned} \Lambda_{h\cdot i}^{(r,m)} = & \sum_{j=1}^{n} \Big[\varrho_{h\cdot ij}(\, P^{(r,m)}[u_h]\,) - \varrho_{h\cdot ij}(\, P^{(r,m)}[v_h]\,) \Big]\, \delta_j v_{h\cdot i}^{(r,m)} + \\ & + f_{h\cdot i}(\, P^{(r,m)}[u_h]\,) - f_{h\cdot i}(\, P^{(r,m)}[v_h]\,) - \Gamma_{h\cdot i}^{(r,m)}. \end{aligned} \tag{19}$$

From (17), it follows that the function w_h satisfies the recursive equations

$$\begin{aligned} w_{h\cdot i}^{(r+1,m)} = & \frac{1}{2} \sum_{j=1}^{n} w_{h\cdot i}^{(r,m+e_j)} \Big[\frac{1}{n} + \frac{h_0}{h_j} \varrho_{h\cdot ij}(\, P^{(r,m)}[u_h]\,) \Big] + \\ & + \frac{1}{2} \sum_{j=1}^{n} w_{h\cdot i}^{(r,m-e_j)} \Big[\frac{1}{n} - \frac{h_0}{h_j} \varrho_{h\cdot ij}(\, P^{(r,m)}[u_h]\,) \Big] + h_0 \Lambda_{h\cdot i}^{(r,m)}. \end{aligned} \tag{20}$$

Write

$$\begin{aligned} \omega_h^{(r)} &= \omega_h(t^{(r)}) = \\ &= \max\big\{\, \|w_h^{(j,m)}\| \colon\;\; (t^{(j)}, x^{(m)}) \in E_h^* \cap \big([-\tau_0, t^{(r)}] \times R^n \big) \,\big\}, \quad 0 \le r \le N_0. \end{aligned} \tag{21}$$

The term Λ_h can be estimated as follows

$$\|\Lambda_h^{(r,m)}\| \le \sigma(t^{(r)}, \omega_h^{(r)})(1+c_0) + \alpha(h) \qquad \text{on } E'_h. \tag{22}$$

From (9), (20) and (22), we conclude that the function ω_h satisfies the recursive inequality

$$\omega_h^{(r+1)} \le \omega_h^{(r)} + \tilde{c}\, h_0\, \sigma(t^{(r)}, \omega_h^{(r)}) + h_0 \alpha(h), \qquad 0 \le r \le N_0 - 1, \tag{23}$$

with $\tilde{c} = (1 + c_0)$ and

$$\omega_h^{(0)} \le \alpha_0(h). \tag{24}$$

Consider the differential equation

$$\eta'(t) = \tilde{c}\, \sigma(t, \eta(t)) + \alpha(h) \tag{25}$$

with the initial condition

$$\eta(0) = \alpha_0(h) \tag{26}$$

and its solution η_h. From (16) and Assumption H$[\varrho]$, it follows that

$$\lim_{h \to 0} \eta_h(\cdot) = 0.$$

Then, because η_h is a convex function:

$$\eta_h^{(r+1)} \ge \eta_h^{(r)} + h_0\, \tilde{c}\, \sigma(t, \eta_h^{(r)}) + h_0 \alpha(h).$$

Using induction we prove that

$$\omega_h^{(r)} \le \eta_h^{(r)}, \qquad 0 \le r \le N_0.$$

This gives (12) with $\eta(h) = \eta_h(a)$ and Theorem 1 is proved. □

Now we consider difference functional problem (6), (7) where F_h is given by (5) and the difference operators δ_0, $\delta = (\delta_1, \ldots, \delta_n)$ are calculated in the following way:

$$\delta_0 z_i^{(r,m)} = \frac{1}{h_0}[z_i^{(r+1,m)} - z_i^{(r,m)}], \tag{27}$$

$$\delta_j z_i^{(r,m)} = \frac{1}{h_j}[z_i^{(r,m+e_j)} - z_i^{(r,m)}] \qquad \text{if} \quad \varrho_{h\cdot ij}(\, t^{(r)}, x^{(m)}, z_{[r,m]}) \ge 0, \tag{28}$$

$$\delta_j z_i^{(r,m)} = \frac{1}{h_j}[z_i^{(r,m)} - z_i^{(r,m-e_j)}] \qquad \text{if} \quad \varrho_{h\cdot ij}(\, t^{(r)}, x^{(m)}, z_{[r,m]}) < 0, \tag{29}$$

where $1 \le i \le k$.
It is easily seen that problem (6), (7) with difference operators defined by (27)–(29) has exactly one solution $u_h \colon E_h^* \to R^k$.

Now we give an estimate of the difference between the exact and approximate solution of the above problem.

Theorem 2. *Suppose that Assumption* $\mathrm{H}[\varrho_h, f_h]$ *is satisfied and*

1) $h \in \Delta$ *and*

$$1 - h_0 \sum_{j=1}^{n} \frac{1}{h_j} |\, \varrho_{h\cdot ij}(t,x,w)| \geq 0 \qquad on\ E'_h \times \mathfrak{F}(D_h, R^k),\ 1 \leq i \leq k; \tag{30}$$

2) $u_h : E^*_h \to R^k$ *is the solution of the problem* (6), (7) *with* δ_0 *and* δ *given by* (27)–(29);
3) $v_h : E^*_h \to R^k$ *satisfies relations* (8), (9);
4) *there is* $c_0 \in R_+$ *such that*

$$\| \, \delta_j v_h^{(r,m)} \| \leq c_0 \qquad on\ E_h,\ 1 \leq j \leq n.$$

Under these assumptions, there is $\eta \colon \Delta \to R_+$ *such that*

$$\| \, u_h^{(r,m)} - v_h^{(r,m)} \| \leq \eta(h) \quad on\ E_h \quad and \quad \lim_{h \to 0} \eta(h) = 0. \tag{31}$$

Proof. Let $\Gamma_h : E'_h \to R$ and $\Gamma_{0\cdot h} : E_{0\cdot h} \cup \partial_0 E_h \to R$ be the functions defined by (13) and (14) with δ_0 and δ given by (28), (29). Then estimate (15) is satisfied and the function $w_h = u_h - v_h$ satisfies the difference functional system

$$\begin{aligned} w_{h\cdot i}^{(r+1,m)} = w_{h\cdot i}^{(r,m)} &+ h_0 \sum_{j=1}^{n} \varrho_{h\cdot ij}(\, P^{(r,m)}[u_h]\,)\, \delta_j w_{h\cdot i}^{(r,m)} + \\ &+ h_0 \sum_{j=1}^{n} \big[\, \varrho_{h\cdot ij}(\, P^{(r,m)}[u_h]\,) - \varrho_{h\cdot ij}(\, P^{(r,m)}[v_h]\,)\,\big]\, \delta_j v_{h\cdot i}^{(r,m)} + \\ &+ h_0 \big[\, f_{h\cdot i}(\, P^{(r,m)}[u_h]\,) - f_{h\cdot i}(\, P^{(r,m)}[v_h]\,)\,\big] - h_0 \Gamma_{h\cdot i}^{(r,m)}, \end{aligned}$$

where $(t^{(r)}, x^{(m)}) \in E'_h$, $1 \leq i \leq k$ and $P^{(r,m)}[z]$ is given by (18). Write

$$\begin{aligned} I_{i\cdot +}^{(r,m)} &= \{\, j \colon \ 1 \leq j \leq n, \quad \varrho_{h\cdot ij}(\, P^{(r,m)}[u_h]\,) \geq 0\,\}, \\ I_{i\cdot -}^{(r,m)} &= \{1, \ldots, n\} \setminus I_{i\cdot +}^{(r,m)} \end{aligned}$$

and suppose that Λ_h is defined by (19). Then we have

$$\begin{aligned} w_{h\cdot i}^{(r+1,m)} = h_0\, \Lambda_h^{(r,m)} &+ \\ &+ w_{h\cdot i}^{(r,m)} \Big[1 - h_0 \sum_{j \in I_{i\cdot +}^{(r,m)}} \frac{1}{h_j} \varrho_{h\cdot ij}(\, P^{(r,m)}[u_h]\,) + \\ &\qquad + h_0 \sum_{j \in I_{i\cdot -}^{(r,m)}} \frac{1}{h_j} \varrho_{h\cdot ij}(\, P^{(r,m)}[u_h]\,)\Big] + \\ &+ h_0 \sum_{j \in I_{i\cdot +}^{(r,m)}} \frac{1}{h_j} \varrho_{h\cdot ij}(\, P^{(r,m)}[u_h]\,) w_{h\cdot i}^{(r,m+e_j)} - \\ &- h_0 \sum_{j \in I_{i\cdot -}^{(r,m)}} \frac{1}{h_j} \varrho_{h\cdot ij}(\, P^{(r,m)}[u_h]\,) w_{h\cdot i}^{(r,m-e_j)}, \qquad (t^{(r)}, x^{(m)}) \in E'_h. \end{aligned}$$

From (15), (22), (30), it follows that the function ω_h defined by (21) satisfies recursive inequality (23) and initial estimate (24) holds. Then we get the estimate

$$\omega_h{}^{(r)} \le \eta_h(a)$$

where η_h is the solution of (25), (26). This gives (31) with $\eta(h) = \eta_h(a)$ and Theorem 2 is proved. □

Remark 1. *The stability of difference equations generated by hyperbolic systems of conservation laws is strictly connected with the Courant-Friedrichs-Levy* (CFL) *condition* ([3], Chapter III)*. Inequalities* (11) *and* (30) *can be considered as the CFL conditions for system* (6) *with difference operators given by* (3)*,* (4) *and* (27)–(29)*, respectively.*

3. DIFFERENCE METHODS FOR MIXED PROBLEM

We will need the following operator $T_h\colon \mathfrak{F}(D_h, R^k) \to C(D, R^k)$. Let

$$S_+ = \{\, \xi = (\xi_1, \dots, \xi_n)\colon \quad \xi_j \in \{0,1\}, \quad \text{for } 0 \le j \le n \,\}.$$

Suppose that $w \in \mathfrak{F}(D_h, R^k)$. For every $(t,x) \in D$, there is $(t^{(r)}, x^{(m)}) \in D_h$ such that $(\,t^{(r+1)}, x^{(m+1)}) \in D_h$, where $m+1 = (m_1+1, \dots, m_n+1)$ and $t^{(r)} \le t \le t^{(r+1)}$, $x^{(m)} \le x \le x^{(m+1)}$. Then we put

$$\begin{aligned}\big(T_h\, w\big)(t,x) = {} & \frac{t - t^{(r)}}{h_0} \sum_{\xi \in S_+} w^{(r+1, m+\xi)} \Big(\frac{x - x^{(m)}}{h'}\Big)^{\xi} \Big(1 - \frac{x - x^{(m)}}{h'}\Big)^{1-\xi} + \\ & + \Big(1 - \frac{t - t^{(r)}}{h_0}\Big) \sum_{\xi \in S_+} w^{(r, m+\xi)} \Big(\frac{x - x^{(m)}}{h'}\Big)^{\xi} \Big(1 - \frac{x - x^{(m)}}{h'}\Big)^{1-\xi},\end{aligned}$$

where

$$\Big(\frac{x - x^{(m)}}{h'}\Big)^{\xi} = \prod_{j=1}^{n} \Big(\frac{x_j - x_j^{(m_j)}}{h_j}\Big)^{\xi_j},$$

$$\Big(1 - \frac{x - x^{(m)}}{h'}\Big)^{1-\xi} = \prod_{j=1}^{n} \Big(1 - \frac{x_j - x_j^{(m_j)}}{h_j}\Big)^{1-\xi_j},$$

and we put $0^0 = 1$ in the above formulas.

Lemma 3. *Suppose that the function $w\colon D \to R^k$ is of class C^2 and denote by w_h the restriction of w to the set D_h. Let $\tilde{C} \in R_+$ be such a constant that*

$$\|\, \partial_{tt} w(t,x)\|,\ \|\, \partial_{t x_j} w(t,x)\|,\ \|\, \partial_{x_j x_l} w(t,x)\| \le \tilde{C} \quad \text{on } D$$

where $j, l = 1, \dots, n$. Then

$$\|\, T_h w_h - w \,\|_D \le \tilde{C} \Big[\, h_0^2 + 2h_0 \sum_{j=1}^{n} h_j + \sum_{j,l=1}^{n} h_j h_l \,\Big].$$

The proof of lemma (3) is given in [6, Chapter 5].

Lemma 4. *Suppose that the function* $w=(w_1,\dots,w_k)\colon D\to R^k$ *is of class* C^1 *and* $w_h=(w_{h\cdot 1},\dots,w_{h\cdot k})$ *is the restriction of* w *to the set* D_h. *Let* c_0 *be such a constant that*

$$\|\,\partial_t w(t,x)\|\le c_0,\qquad \|\,\partial_{x_j} w(t,x)\|\le c_0\qquad \text{for } 1\le j\le n,\quad (t,x)\in D \tag{32}$$

Then

$$\|\,T_h w_h - w\,\|_D\le c_0\|h\|. \tag{33}$$

Proof. Let $(t,x)\in D$, then there is $(t^{(r)},x^{(m)})\in D_h$ such that $(t^{(r+1)},x^{(m+1)})\in D$ and $t^{(r)}\le t\le t^{(r+1)}$, $x^{(m)}\le x\le x^{(m+1)}$. It follows that there are $\theta_i,\,\tilde\theta_i\in D$, $1\le i\le k$ such that

$$\begin{aligned} w_i(t,x)-T_h w_{h\cdot i}(t,x) = w_i(t,x) &- \frac{t-t^{(r)}}{h_0}\sum_{\xi\in S_+}\Big[\,w_i(t,x)+\partial_t w_i(\theta_i)(t^{(r+1)}-t)+\\ &\qquad +\sum_{j=1}^n \partial_{x_j} w_i(\theta_i)(x_j^{(m_j+\xi_j)}-x_j)\Big]\times\\ &\times\Big(\frac{x-x^{(m)}}{h'}\Big)^{\xi}\Big(1-\frac{x-x^{(m)}}{h'}\Big)^{1-\xi}-\\ &-\Big(1-\frac{t-t^{(r)}}{h_0}\Big)\sum_{\xi\in S_+}\Big[\,w_i(t,x)-\partial_t w_i(\tilde\theta_i)(t^{(r+1)}-t)+\\ &\qquad +\sum_{j=1}^n \partial_{x_j} w_i(\tilde\theta_i)(x_j^{(m_j+\xi_j)}-x_j)\Big]\times\\ &\times\Big(\frac{x-x^{(m)}}{h'}\Big)^{\xi}\Big(1-\frac{x-x^{(m)}}{h'}\Big)^{1-\xi},\quad 1\le i\le k. \end{aligned}$$

For $x^{(m)}\le x\le x^{(m+1)}$ we have

$$\sum_{\xi\in S_+}\Big(\frac{x-x^{(m)}}{h'}\Big)^{\xi}\Big(1-\frac{x-x^{(m)}}{h'}\Big)^{1-\xi}=1.$$

Then from (32) we get (33). □

Assumption H$[\varrho,f]$**.** *Suppose that functions* $\varrho\colon E\times C(D,R^k)\to M_{k\times n}$ *and* $f\colon E\times C(D,R^k)\to R^k$ *are continuous and there is* $\sigma\colon[0,a]\times R_+\to R_+$ *satisfying Assumption* H$[\sigma]$ *and such that*

$$\|\,\varrho(t,x,w)-\varrho(t,x,\bar w)\|\le\sigma(t,\|w-\bar w\|_D),$$

$$\|\,f(t,x,w)-f(t,x,\bar w)\|\le\sigma(t,\|w-\bar w\|_D)$$

on $E\times C(D,R^k)$.

Now we consider functional differential problem (1), (2) and the difference equations

$$\delta_0 z_i^{(r,m)}=\sum_{j=1}^n \varrho_{ij}\big(\,t^{(r)},\,x^{(m)},\,T_h z_{[i,m]}\,\big)\,\delta_j z^{(r,m)}+f_i\big(\,t^{(r)},\,x^{(m)},\,T_h z_{[i,m]}\,\big) \tag{34}$$

for $1 \le i \le k$ with initial boundary condition (7). We will prove the convergence and give an error estimate for the method (34), (7), with the difference operators δ_0 and δ defined first by (3), (4) and then by (27)–(29).

3.1. δ_0, δ DEFINED BY (3), (4)

Theorem 5. *Suppose that Assumption* $\mathrm{H}[\varrho, f]$ *is satisfied and*

1) $h \in \Delta$ *and*

$$\frac{1}{n} - \frac{h_0}{h_j} | \varrho_{ij}(t,x,w)| \ge 0 \qquad \text{on } E \times C(D, R^k) \quad \text{for } 1 \le j \le n,\ 1 \le i \le k$$

and there is $M = (M_1, \ldots, M_n) \in R_+^n$ *such that* $h' \le M h_0$;

2) $u_h \colon E_h^* \to R^k$ *is a solution of problem* (34), (7);

3) $v \colon E^* \to R^k$ *is a solution of* (1), (2) *and* v_h *is the restriction of* v *to* E_h^*;

4) v *is of class* C^1 *and* $c_0 \in R_+$ *is such a constant that*

$$\| \partial_{x_j} v(t,x) \| \le c_0 \qquad \text{on } \overline{E},\ 1 \le j \le n;$$

5) *there is* $\alpha_0 \colon \Delta \to R_+$ *such that*

$$\begin{aligned} &\| \varphi_h^{(r,m)} - \varphi^{(r,m)} \| \le \alpha_0(h) \qquad \text{on } E_{0.h} \cup \partial_0 E_h, \\ &\lim_{h \to 0} \alpha_0(h) = 0. \end{aligned} \tag{35}$$

Then there is $\eta \colon \Delta \to R_+$ *such that*

$$\| u_h^{(r,m)} - v_h^{(r,m)} \| \le \eta(h) \qquad \text{on } E_h \quad \text{and} \quad \lim_{h \to 0} \eta(h) = 0. \tag{36}$$

Proof. We use Theorem 1 to prove the above assertion. Write

$$\begin{aligned} \Gamma_{h \cdot i}^{(r,m)} = \delta_0 v_{h \cdot i}^{(r,m)} - \sum_{j=1}^{n} \varrho_{ij} \big(t^{(r)}, x^{(m)}, T_h(v_h)_{[r,m]} \big) \, \delta_j v_{h \cdot i}^{(r,m)} - \\ - f_i \big(t^{(r)}, x^{(m)}, T_h(v_h)_{[r,m]} \big). \end{aligned} \tag{37}$$

We see at once that

$$\begin{aligned} \Gamma_{h \cdot i}^{(r,m)} = {} & \delta_0 v_{h \cdot i}^{(r,m)} - \partial_t v_i^{(r,m)} + \\ & + \sum_{j=1}^{n} \Big[\varrho_{ij} \big(t^{(r)}, x^{(m)}, v_{(t^{(r)}, x^{(m)})} \big) - \varrho_{ij} \big(t^{(r)}, x^{(m)}, T_h(v_h)_{[r,m]} \big) \Big] \, \delta_j v_{h \cdot i}^{(r,m)} + \\ & + \sum_{j=1}^{n} \varrho_{ij} \big(t^{(r)}, x^{(m)}, v_{(t^{(r)}, x^{(m)})} \big) \big[\partial_{x_j} v_i^{(r,m)} - \delta_j v_{h \cdot i}^{(r,m)} \big] + \\ & + f_i \big(t^{(r)}, x^{(m)}, v_{(t^{(r)}, x^{(m)})} \big) - f_i \big(t^{(r)}, x^{(m)}, T_h(v_h)_{[r,m]} \big). \end{aligned}$$

It is easily seen that there is $\alpha\colon \Delta \to R_+$ such that

$$\|\Gamma_h^{(r,m)}\| \le \alpha(h) \qquad \text{on } E'_h \quad \text{and} \quad \lim_{h\to 0} \alpha(h) = 0.$$

From Theorem 1, it follows that there exists a function $\eta\colon \Delta \to R$ satisfying (36). This completes the proof of the theorem. □

Now we give an error estimate for method (34), (7).

Lemma 6. *Suppose that*

1) *the functions* $\varrho\colon E\times C(D,R^k) \to M_{k\times n}$ *and* $f\colon E\times C(D,R^k)\to R^k$ *are continuous and there is* $L>0$ *such that*

$$\|\,\varrho(t,x,w) - \varrho(t,x,\bar w)\| \le L\|w-\bar w\|_D,$$

$$\|\,f(t,x,w) - f(t,x,\bar w)\| \le L\|w-\bar w\|_D$$

on $E\times C(D,R^k)$*;*

2) *assumptions* 1), 2), 3), 5) *of Theorem 5 are satisfied;*
3) $v\colon E^*\to R^k$ *is a solution of* (1), (2) *and* v *is of class* C^2 *on* E^**;*
4) $c_0, C, d \in R_+$ *are such constants that*

$$\|\,\partial_{tt}v(t,x)\|,\ \|\,\partial_{tx_j}v(t,x)\|,\ \|\,\partial_{x_jx_l}v(t,x)\| \le C \quad \text{on } D,\quad 1\le j,l\le n,$$

$$\|\varrho_j(t,x,v_{(t,x)})\| \le d \quad \text{on } E \qquad \text{for } 1\le j\le n$$

and

$$\|\,\partial_{x_j}v(t,x)\| \le c_0 \quad \text{on } E,\ 1\le j\le n.$$

Then

$$\|\,u_h^{(r,m)} - v_h^{(r,m)}\,\| \le \eta(h) \qquad \text{on } E_h, \tag{38}$$

where

$$\eta(h) = \alpha_0(h)e^{\,L(1+c_0)\,a} + (Ah_0 + Bh_0^2)\frac{e^{\,L(1+c_0)\,a}-1}{L(1+c_0)}. \tag{39}$$

and

$$\begin{aligned} A &= \frac12 C\Big[1+\frac1n\sum_{j=1}^n M_j^2 + d\sum_{j=1}^n M_j\Big],\\ B &= L(1+c_0)C\Big[1+2\sum_{j=1}^n M_j + \sum_{j,l=1} M_jM_l\Big]. \end{aligned} \tag{40}$$

Proof. In a general case, we have estimate (38) with $\eta(h)=\eta_h(a)$ and $\eta_h\colon[0,a]\to R_+$ is a solution of (25), (26) where $\tilde c = (1+c_0)$, $\sigma(t,p)=Lp$ and $\alpha\colon\Delta\to R_+$ is such a function that

$$\delta_0 v_h^{(r,m)} = F_h[v_h]^{(r,m)} + \Gamma_h^{(r,m)} \qquad \text{on } E'_h,$$

and

$$\|\Gamma_h^{(r,m)}\| \leq \alpha(h) \qquad \text{on } \ E'_h.$$

An easy computation shows that

$$\|\partial_t v^{(r,m)} - \delta_0 v_h^{(r,m)}\| \leq \frac{h_0}{2}\, C\left(1 + \frac{1}{n}\sum_{j=1}^{n} M_j^2\right)$$

and

$$\|\partial_{x_j} v^{(r,m)} - \delta_j v_h^{(r,m)}\| \leq \frac{h_0}{2} C M_j \qquad \text{for } \ 1 \leq j \leq n.$$

According to the above estimates and Lemma (3), there is

$$\|\Gamma_h^{(r,m)}\| \leq Ah_0 + Bh_0^2 \quad \text{ on } E'_h$$

Then

$$\eta(h) = -\frac{Ah_0 + Bh_0^2}{L(1+c_0)} + \left(\alpha_0(h) + \frac{Ah_0 + Bh_0^2}{L(1+c_0)}\right) e^{\,L(1+c_0)\,a}.$$

and assertion (38) follows. □

3.2. δ_0, δ DEFINED BY (27)–(29)

Now we consider functional differential problem (1), (2) and the difference functional problem consisting of (34) and initial boundary condition (7). This time, δ_0 and $\delta = (\delta_1, \ldots, \delta_n)$ are defined by (27)–(29).

Theorem 7. *Suppose that Assumption $H[\varrho, f]$ is satisfied and*

1) $h \in \Delta$ *and*

$$1 - h_0 \sum_{j=1}^{n} \frac{1}{h_j} |\, \varrho_{ij}(t, x, w)| \geq 0 \qquad \text{on } \ E \times C(D, R^k)$$

and there is $M = (M_1, \ldots, M_n) \in R_+^n$ such that $h' \leq M h_0$;

2) *the function $u_h \colon E_h^* \to R$ is a solution of the problem* (34), (7) *with δ_0 and δ given by* (27)–(29)*;*
3) $v \colon E^* \to R^k$ *is a solution of* (1), (2)*;*
4) *the function $v|_{\overline{E}}$ is of class C^1 and $c_0 \in R_+$ is such a constant that*

$$\|\, \partial_t v(t, x) \,\| \ , \quad \|\, \partial_{x_j} v(t, x) \,\| \leq c_0 \qquad \text{on } \overline{E}, \quad 1 \leq j \leq n;$$

5) *there is $\alpha_0 \colon \Delta \to R_+$ such that condition* (35) *is satisfied.*

Then there is $\eta \colon \Delta \to R_+$ such that

$$\|\, u_h^{(r,m)} - v_h^{(r,m)} \,\| \leq \eta(h) \qquad \text{on } E_h \quad \text{and} \quad \lim_{h \to 0} \eta(h) = 0. \tag{41}$$

Proof. We use Theorem 2 to prove the above assertion. Let $\Gamma\colon E'_h \to R^k$ be a function given by (37) with δ defined by (28), (29). From Assumption $H[\varrho, f]$ and Lemma (4), it follows that there is a function $\alpha\colon \Delta \to R_+$ such that

$$\|\,\Gamma_h^{(i,m)}\| \le \alpha(h) \qquad \text{on } E'_h \qquad \text{and} \qquad \lim_{h\to 0} \alpha(h) = 0.$$

Then the assumptions of Theorem 2 are satisfied and assertion (41) follows from (31). □

Now we give an error estimate for method (34), (7) and δ_0, $\delta = (\delta_1, \ldots, \delta_n)$ defined by (27)–(29).

Lemma 8. *Suppose that*

1) *the function* $\varrho\colon E\times C(D,R^k) \to M_{k\times n}$ *and* $f\colon E\times C(D,R^k)\to R^k$ *are continuous and there is* $L>0$ *such that*

$$\|\,\varrho(t,x,w) - \varrho(t,x,\bar{w})\| \le L\|w-\bar{w}\|_D,$$

$$\|\,f(t,x,w) - f(t,x,\bar{w})\| \le L\|w-\bar{w}\|_D$$

on $E\times C(D,R^k)$*;*
2) *assumptions* 1), 2), 3), 5) *of Theorem 7 are satisfied;*
3) $v\colon E^*\to R^k$ *is a solution of* (1), (2) *and* v *is of class* C^2 *on* E^**;*
4) $c_0, C, d \in R_+$ *are such constants that*

$$\|\,\partial_{tt}v(t,x)\|,\ \|\,\partial_{tx_j}v(t,x)\|,\ \|\,\partial_{x_jx_l}v(t,x)\| \le C \quad \text{on } D, \quad 1\le j,l\le n,$$

$$\|\varrho_j(t,x,v_{(t,x)})\| \le d \quad \text{on } E \qquad \text{for } 1\le j\le n$$

and

$$\|\,\partial_{x_j}v(t,x)\| \le c_0 \quad \text{on } E,\ 1\le j\le n.$$

Then

$$\|\,u_h^{(r,m)} - v_h^{(r,m)}\,\| \le \eta(h) \qquad \text{on } E_h, \tag{42}$$

where $\eta(h)$ *is given by* (39) *with*

$$A = \frac{1}{2}C\Big[1 + d\sum_{j=1}^{n} M_j\Big]$$

and B *is defined in* (40).

The proof of Lemma 8 is similar to that of Lemma 6. Details are omitted.

4. NUMERICAL EXAMPLE

For $n = 2$, we put

$$\begin{aligned} E &= [0,1] \times [-1,1] \times [-1,1], \\ D &= \{0\} \times \Big[-\frac{1}{2}, \frac{1}{2}\Big] \times \Big[-\frac{1}{2}, \frac{1}{2}\Big], \\ D_0 &= \Big[-\frac{1}{2}, \frac{1}{2}\Big] \times \Big[-\frac{1}{2}, \frac{1}{2}\Big]. \end{aligned}$$

Denote by $x = (x_1, x_2)$ and by z the unknown function in the variables (t, x), and consider the differential integral equation

$$\begin{aligned} \partial_t z(t,x) = &- \left[\int_{D_0} z(t, x+s)\, ds - z(t,x) + \frac{1}{6}(1+t)\right] \partial_{x_1} z(t,x) + \\ &+ \left[\int_{D_0} z(t, x+s)\, ds - 4z(t, \frac{x}{2}) - \frac{1}{6}(1+t)\right] \partial_{x_2} z(t,x) + \\ &+ 3\int_{D_0} z(t, x+s)\, ds - \frac{3}{2} z(t,x) - 6z(t, \frac{x}{2}) - \frac{1}{2}(1+t) + x_1^2 + x_2^2, \end{aligned} \tag{43}$$

with the initial boundary condition

$$z(t,x) = (1+t)(x_1^2 + x_2^2) \qquad \text{for } (t,x) \in E_0 \cup \partial_0 E \tag{44}$$

where

$$\begin{aligned} E_0 &= \{0\} \times [-2,2] \times [-2,2], \\ \partial_0 E &= [0,1] \times \Big[\big([-2,2] \times [-2,2]\big) \setminus \big([-2,2] \times [-2,2]\big)\Big]. \end{aligned}$$

The difference method for the problem is of the form

$$\begin{aligned} \delta_t z^{(r,i,j)} = &- \Big[I_h^{(r,i,j)} - z^{(r,i,j)} + \frac{1}{6}(1 + t^{(r)})\Big] \delta_1 z^{(r,i,j)} + \\ &+ \Big[I_h^{(r,i,j)} - 4z^{(r,i,j)} - \frac{1}{6}(1 + t^{(r)})\Big] \delta_2 z^{(r,i,j)} + \\ &+ 3I_h^{(r,i,j)} - \frac{3}{2} z^{(r,i,j)} - 6T_h z_{[r,i,\frac{j}{2}]} - \frac{1}{2}(1 + t^{(r)}) + (x_1^{(i)})^2 + (x_2^{(j)})^2 \end{aligned} \tag{45}$$

and

$$z^{(r,i,j)} = (1 + t^{(r)})\big((x_1^{(i)})^2 + (x_2^{(j)})^2 \big) \qquad \text{for } (t^{(r)}, x_1^{(i)}, x_2^{(j)}) \in E_0 \cup \partial_0 E, \tag{46}$$

where $\delta_0 z^{(r,i,j)}$, $\delta_1 z^{(r,i,j)}$ and $\delta_2 z^{(r,i,j)}$ are defined by (27)–(29) and

$$I_h^{(r,i,j)} = \int_{D_0} T_h z_{[r,i,j]}(p,s)\, dp\, ds$$

with T_h defined in Section 3 and $h = (h_0, h_1, h_2)$.

The function $v(t,x) = (1+t)(x_1^2+x_2^2)$ is the solution of (43), (44). Let $u_h\colon E_h^* \to R$ be the solution of (45), (46) and $\varepsilon = |u_h - v|$. The values $\varepsilon(0.3,x)$, $\varepsilon(0.5,x)$, $\varepsilon(0.7,x)$, $\varepsilon(0.9,x)$ and $u_h(0.3,x)$, $u_h(0.5,x)$, $u_h(0.7,x)$, $u_h(0.9,x)$ are listed in Tables 1 and 2 for $h_0 = 0.0005$, $h_1 = 0.005$ and $h_2 = 0.005$.

Table 1

$x^{(j)}$	$y^{(k)}$	$t^{(r)} = 0.3$		$t^{(r)} = 0.5$	
		u_h	ε_h	u_h	ε_h
-0.5	-0.5	0.6507	$7.106\ 10^{-4}$	0.7505	$5.483\ 10^{-4}$
-0.5	0.0	0.3267	$1.732\ 10^{-3}$	0.3774	$2.394\ 10^{-3}$
-0.5	0.5	0.6499	$7.457\ 10^{-5}$	0.7500	$4.606\ 10^{-5}$
0.0	-0.5	0.3254	$3.673\ 10^{-4}$	0.3753	$2.756\ 10^{-4}$
0.0	0.0	-0.0001	$6.930\ 10^{-5}$	-0.0001	$1.421\ 10^{-4}$
0.0	0.5	0.3218	$3.161\ 10^{-3}$	0.3704	$4.589\ 10^{-3}$
0.5	-0.5	0.6500	$2.407\ 10^{-5}$	0.7500	$2.869\ 10^{-5}$
0.5	0.0	0.3227	$2.278\ 10^{-3}$	0.3713	$3.699\ 10^{-3}$
0.5	0.5	0.6433	$6.728\ 10^{-3}$	0.7399	$1.006\ 10^{-2}$

Table 2

$x^{(j)}$	$y^{(k)}$	$t^{(r)} = 0.7$		$t^{(r)} = 0.9$	
		u_h	ε_h	u_h	ε_h
-0.5	-0.5	0.8500	$3.074\ 10^{-5}$	0.9492	$7.995\ 10^{-4}$
-0.5	0.0	0.4278	$2.799\ 10^{-3}$	0.4781	$3.064\ 10^{-3}$
-0.5	0.5	0.8501	$5.237\ 10^{-5}$	0.9503	$2.964\ 10^{-4}$
0.0	-0.5	0.4250	$2.427\ 10^{-5}$	0.4745	$5.080\ 10^{-4}$
0.0	0.0	-0.0002	$2.094\ 10^{-4}$	-0.0003	$2.707\ 10^{-4}$
0.0	0.5	0.4194	$5.636\ 10^{-3}$	0.4686	$6.433\ 10^{-3}$
0.5	-0.5	0.8500	$1.112\ 10^{-5}$	0.9500	$2.752\ 10^{-5}$
0.5	0.0	0.4200	$4.957\ 10^{-3}$	0.4690	$6.004\ 10^{-3}$
0.5	0.5	0.8375	$1.248\ 10^{-2}$	0.9359	$1.412\ 10^{-2}$

Let ε_{max} be the largest and ε_{mean} the mean value of all ε for a given $t^{(r)}$ (Tab. 3).

Table 3

$t^{(r)}$	ε_{max}	ε_{mean}
0.1	$6.284\ 10^{-3}$	$6.246\ 10^{-4}$
0.2	$1.101\ 10^{-2}$	$1.153\ 10^{-3}$
0.3	$1.437\ 10^{-2}$	$1.602\ 10^{-3}$
0.4	$1.671\ 10^{-2}$	$1.986\ 10^{-3}$
0.5	$1.843\ 10^{-2}$	$2.324\ 10^{-3}$
0.6	$1.983\ 10^{-2}$	$2.632\ 10^{-3}$
0.7	$2.115\ 10^{-2}$	$2.926\ 10^{-3}$
0.8	$2.329\ 10^{-2}$	$3.369\ 10^{-3}$
0.9	$2.416\ 10^{-2}$	$3.523\ 10^{-3}$
1.0	$2.667\ 10^{-2}$	$3.847\ 10^{-3}$

The computation was performed on a PC computer.

REFERENCES

[1] P. Brandi, Z. Kamont, A. Salvadori *Aproximate solutions of the mixed problems for first order partial differential-functional equations,* Atti Sem. Mat. Fis. Univ. Modena **39** (1991), 277–302.

[2] H. Bruner, *The numerical treatment of ordinary and partial Volterra integro-differential equations,* Proceed. First Internat. Colloq. on Numerical Anal., Plovdiv, 17–23 August 1992 (D. Bainov, V. Covacher., eds.), 13–26, Tokyo: VSP Ultrecht, 1993.

[3] E. Godlewski, P. Raviart, *Numerical Approximation of Hyperbolic Systems of Conservation Laws,* Springer, Berlin, 1996.

[4] W. Hakbusch, *Extrapolation applied to certain discretization method solving the initial value problem for hyperbolic differential equations,* Numer. Math. **28** (1977), 121–142.

[5] Z. Kamont, K. Prządka, *Difference methods for nonlinear partial differential equations of the first order,* Ann. Polon. Math. **48** (1988), 227–246.

[6] Z. Kamont, *Hyperbolic functional differential inequalities and applications,* Kluver Acad. Publ., Dordrecht, Boston, London, 1989.

[7] Z. Kamont, *Finite difference approximations for first order partial differential functional equations,* Ukr. Math. Journ. **46** (1994), 985–996.

[8] Z. Kamont, K. Prządka, *Difference methods for first order partial differential functional equations with initial boundary conditions,* Journ. Vycisl. Mat. i Mat. Fis. **31** (1991), 1476–1488.

[9] Z. Kowalski, *A difference method for nonlinear partial differential equations of the first order,* Ann. Polon. Math. **18** (1960), 235–242.

[10] Z. Kowalski, *On the difference method for certain hyperbolic systems of non-linear partial differential equations of the first order,* Bull. Acad. Polon. Sci., Ser. sci. math. astr. phys. **16** (1968) 4, 297–310.

[11] K.M. Maghomedov, A.S. Holodov, *Set-characteristics Numerical Methods,* Moscow, 1988 (in Russian).

[12] T. Meis, U. Marcowitz, *Numerical Solutions of Partial Differential Equations,* Acad. Press., New York, 1981.

[13] K. Prządka, *Difference methods for nonlinear partial differential functional equations of the first order,* Math. Nachr. **138** (1988), 105–123.

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