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DEFORMATION MINIMAL BENDING OF COMPACT MANIFOLDS: CASE OF SIMPLE CLOSED CURVES

Abstract. The problem of minimal distortion bending of smooth compact embedded connected Riemannian n-manifolds M and N without boundary is made precise by defining a deformation energy functional Φ on the set of diffeomorphisms $\mathrm{Diff}(M, N)$. We derive the Euler-Lagrange equation for Φ and determine smooth minimizers of Φ in case M and N are simple closed curves.

Keywords: minimal deformation, distortion minimal, geometric optimization.

Mathematics Subject Classification: 58E99.

1. INTRODUCTION

Two diffeomorphic compact embedded hypersurfaces admit infinitely many diffeomorphisms, which we view as prescriptions for bending one hypersurface into the other. We ask which diffeomorphic bendings have minimal distortion with respect to a natural bending energy functional that will be precisely defined. We determine the Euler-Lagrange equation for the general case of hypersurfaces in Euclidean spaces and solve the problem for one-dimensional manifolds embedded in the plane. The existence of minima for the general case is a difficult open problem. An equivalent problem for a functional that measures the total energy of deformation due to stretching was solved in [2]. Some related discussions on the minimization problem are presented in [4, 7–9].

2. MINIMAL DISTORTION DIFFEOMORPHISMS

Let M and N denote compact, connected and oriented n-manifolds without boundary that are embedded in $\mathbb{R}^{n+1}$ and equip them with the natural Riemannian metrics g_M and g_N inherited from the usual metric of $\mathbb{R}^{n+1}$. These Riemannian manifolds (M, g_M) and (N, g_N) have the volume forms ω_M and ω_N induced by their Riemannian metrics. We assume that M and N are diffeomorphic, denote the class of (C^∞) diffeomorphisms from M to N by $\mathrm{Diff}(M,N)$, the (total space of the) tangent bundle of M by TM, the cotangent bundle by TM^*, and the sections of an arbitrary vector bundle V by $\Gamma(V)$. For $h \in \mathrm{Diff}(M,N)$, we use the standard notation h^* for the pull-back map associated with h and h_* for its push-forward map.

Definition 2.1. *The strain tensor $S \in \Gamma(TM^* \otimes TM^*)$ corresponding to $h \in \mathrm{Diff}(M,N)$ is defined to be*

$$S = h^* g_N - g_M \tag{1}$$

(*cf.* [5,7]).

Recall the natural bijection between covectors in T^*M and vectors in TM (see [3]): To each covector $\alpha_p \in T_pM^*$ assign the vector $\alpha_p^\# \in T_pM$ that is implicitly defined by the relation

$$\alpha_p = (g_M)_p(\alpha_p^\#, \cdot).$$

Using this correspondence, we introduce the Riemannian metric g_M^* on TM^* by

$$g_M^*(\alpha,\beta) = g_M(\alpha^\#, \beta^\#),$$

where the base points are suppressed.

There is a natural Riemannian metric G on $TM^* \otimes TM^*$ given by $G = g_M^* \otimes g_M^*$. To compute this metric in local coordinates, let (U,ϕ) be a local coordinate system on M. Using the coordinates of $\mathbb{R}^n$, the map $\phi : U \to \mathbb{R}^n$ can be expressed in the form

$$\phi(p) = \big(x^1(p), \dots, x^n(p)\big).$$

As usual, $\big(x^1(p), \dots, x^n(p)\big)$ are the local coordinates of $p \in M$ and the n-tuple of functions $(x^1, x^2, \dots, x^n)$ is the local coordinate system with respect to (U,ϕ). Because ϕ is a homeomorphism from U onto $\phi(U)$, we identify $p \in U$ and $\phi(p) \in \mathbb{R}^n$ via ϕ. Let us define $\left(\frac{\partial}{\partial x^i}\right)_p = \frac{\partial \phi^{-1}}{\partial x^i}\big(\phi(p)\big)$. The set of vectors $\Big((\frac{\partial}{\partial x^1})_p, \dots, (\frac{\partial}{\partial x^n})_p\Big)$ forms a basis of the tangent space T_pM. Its dual basis $\big((dx^1)_p, \dots, (dx^n)_p\big)$ is a basis of T_pM^*, i.e.,

$$(dx^i)_p\Big(\big(\frac{\partial}{\partial x^j}\big)_p\Big) = \delta^i_j, \quad 1 \le i,j \le n.$$

Using the Einstein summation convention, a tensor $B \in \Gamma(TM^* \otimes TM^*)$ has local coordinate representation $B = b_{ij} dx^i \otimes dx^j$, where $b_{ij} = B(\partial/\partial x^i, \partial/\partial x^j)$. The local coordinate representation of the Riemannian metric G is

$$G(B,B) = b_{ij} b_{kl} g_M^*(dx^i, dx^k) g_M^*(dx^j, dx^l) = b_{ij} b_{kl} [g_M]^{ik} [g_M]^{jl}, \tag{2}$$

where $[g_M]^{ij}$ is the (i,j) entry of the inverse matrix of $\big([g_M]_{ij}\big)$.

Definition 2.2. *The deformation energy functional $\Phi : \mathrm{Diff}(M,N) \to \mathbb{R}_+$ is defined to be*

$$\Phi(h) = \int_M G(h^* g_N - g_M, h^* g_N - g_M)\, \omega_M. \tag{3}$$

The following invariance property of the functional Φ is obvious because the isometries of $\mathbb{R}^{n+1}$ are compositions of translations and rotations, which produce no deformations.

Lemma 2.3. *If $k \in \operatorname{Diff}(N)$ is an isometry of N (i.e., $k^* g_N = g_N$), then $\Phi(k \circ h) = \Phi(h)$.*

3. THE FIRST VARIATION

We will compute the Euler-Lagrange equation for the deformation energy functional Φ. To do this, we will consider smooth variations.

Definition 3.1. *A C^∞ function $F(t,p) = h_t(p)$ defined on $(-\varepsilon, \varepsilon) \times M$ is called a smooth variation of a diffeomorphism $h \in \operatorname{Diff}(M,N)$ if*

1. *$h_t \in \operatorname{Diff}(M,N)$ for all $t \in (-\varepsilon, \varepsilon)$ and*
2. *$h_0 = h$.*

The tangent space $T_h \operatorname{Diff}(M,N)$ is identified with the set $\Gamma(h^{-1}TN)$ of all the smooth sections of the induced bundle $h^{-1}TN$ with fiber $T_{h(p)}N$ over the point p of the manifold M (cf. [6]). Indeed, each smooth variation $F : (-\varepsilon, \varepsilon) \times M \to N$ corresponds to a curve $t \mapsto F(t,p) = h_t(p)$ in $\operatorname{Diff}(M,N)$.

Definition 3.2. *Let $F : (-\varepsilon, \varepsilon) \times M \to N$ be a smooth variation of a diffeomorphism $h \in \operatorname{Diff}(M,N)$. The variational vector field $V \in \Gamma(h^{-1}TN)$ is defined by*

$$V(p) = \frac{d}{dt} h_t \Big|_{t=0} (p) = \frac{\partial}{\partial t} F(0,p)$$

for $p \in M$.

Since the tangent space $T_h \operatorname{Diff}(M,N)$ consists of all the variational vector fields of the diffeomorphism h, it follows that $T_h \operatorname{Diff}(M,N)$ is a subset of $\Gamma(h^{-1}TN)$. On the other hand, suppose that a vector field $V \in \Gamma(h^{-1}TN)$ is given. We can easily construct a variation of h with the variational vector field V. Indeed, let ψ_t be the flow of the vector field $X = V \circ h^{-1} \in \Gamma(TN)$. The smooth variation $F(t,p) = \psi_t \circ h(p)$ of the diffeomorphism $h \in \operatorname{Diff}(M,N)$ has the variational vector field $V(p) = \frac{d}{dt}(\psi_t \circ h)(p) = X \circ h(p) = V(p)$ as required. Hence,

$$T_h \operatorname{Diff}(M,N) = \Gamma(h^{-1}TN).$$

We will consider all variations of $h \in \operatorname{Diff}(M,N)$ of the form $F(t,p) = h \circ \phi_t(p)$, where ϕ_t is the flow of a vector field $X \in \Gamma(TM)$. The variational vector field corresponding to the variation F is $V = h_* X$. Since h is a diffeomorphism, it is easy to see that the variational vector fields of the variations of the form $h \circ \psi_t$ exhaust all possible variational vector fields.

Let us restrict the domain of the functional Φ to $\mathrm{Diff}(M,N)$. The diffeomorphism h is a critical point of Φ if

$$\frac{d}{dt}\Phi(h\circ\phi_t)|_{t=0} = D\Phi(h)h_*Y = \int_M G(h^*g_N - g_M, L_Y h^*g_N) = 0 \tag{4}$$

for all $Y \in \Gamma(TM)$, where L_Y denotes the Lie derivative in the direction Y.

Let $\beta \in \Gamma(TM^* \otimes TM^*)$ have the local representation $\beta_{ij}dx^i \otimes dx^j$. We will use the following formula for the components of the Lie derivative $L_X\beta$ of β in the direction of the vector field X:

$$[L_X\beta]_{ij} = X^k\frac{\partial \beta_{ij}}{\partial x^k} + \beta_{kj}\frac{\partial X^k}{\partial x^i} + \beta_{ik}\frac{\partial X^k}{\partial x^j}. \tag{5}$$

4. SOLUTION FOR ONE DIMENSIONAL MANIFOLDS

In this section M and N are smooth simple closed curves in $\mathbb{R}^2$. Their arclengths are denoted $L(M)$ and $L(N)$ respectively, and they are supposed to have base points $p \in M$ and $q \in N$. We will determine the minimum of the functional

$$\Phi(h) = \int_M G\big(h^*g_N - g_M, h^*g_N - g_M\big)\omega_M \tag{6}$$

over the admissible set

$$\mathcal{A} = \{h \in \mathrm{Diff}(M,N) : h(p) = q\}. \tag{7}$$

There exist unique arc length parametrizations $\gamma : [0, L(M)] \to M$ and $\xi : [0, L(N)] \to N$ of M and N respectively, which correspond to the positive orientations of the curves M and N in the plane, and are such that $\gamma(0) = p$, $\xi(0) = q$. Notice that $[g_M]_{11}(t) = |\dot\gamma(t)|^2 = 1 = [g_M]^{11}(t)$ for $t \in [0, L(M)]$ and $[h^*g_N]_{11}(t) = |Dh\big(\gamma(t)\big)\dot\gamma(t)|^2$. Using formula (2) for the metric G, we can rewrite functional (6) in local coordinates:

$$\Phi(h) = \int_0^{L(M)} \Big(\big|Dh\big(\gamma(t)\big)\dot\gamma(t)\big|^2 - 1\Big)^2 dt. \tag{8}$$

Let us denote the local representation of a diffeomorphism $h \in \mathrm{Diff}(M,N)$ by $u = \xi^{-1}\circ h \circ \gamma$. The function u is a diffeomorphism on the open interval $\big(0, L(M)\big)$ and can be continuously extended onto the closed interval $[0, L(M)]$ as follows. If h is orientation preserving, we can extend u to a continuous function on $[0, L(M)]$ by defining $u(0) = 0$ and $u(L(M)) = L(N)$. In this case $\dot u > 0$. If h is orientation reversing, we define $u(0) = L(N)$ and $u(L(M)) = 0$.

Since

$$\Big|\frac{d}{dt}(h\circ\gamma)(t)\Big|^2 = \Big|\frac{d}{dt}(\xi\circ u)(t)\Big|^2 = \dot u^2(t)\Big|\dot\xi\big(u(t)\big)\Big|^2 = \dot u^2(t)$$

for $t \in \big(0, L(M)\big)$, the original problem of the minimization of functional (6) can be reduced to the minimization of the functional

$$\Psi(u) = \int_0^{L(M)} (\dot{u}^2 - 1)^2 dt \tag{9}$$

over the admissible sets

$$\mathcal{B} = \Big\{ u \in C^2\big([0, L(M)], [0, L(N)]\big) : u(0) = 0, u(L(M)) = L(N) \Big\}$$

and

$$\mathcal{C} = \Big\{ u \in C^2\big([0, L(M)], [0, L(N)]\big) : u(0) = L(N), u(L(M)) = 0 \Big\}.$$

The minima will be shown to correspond to diffeomorphisms in $\mathrm{Diff}(M, N)$.

Lemma 4.1. *Suppose that $L(N) \geq L(M)$.*

(i) *The function $v(t) = L(N)/L(M)t$, where $t \in [0, L(M)]$, is the unique minimum of the functional Ψ over the admissible set $\mathcal{B}$.*
(ii) *The function $w(t) = -L(N)/L(M)t + L(N)$, where $t \in [0, L(M)]$, is the unique minimum of the functional Ψ over the admissible set $\mathcal{C}$.*

Proof. Since the proofs of (i) and (ii) are almost identical, we will only present the proof of statement (i).

The Euler-Lagrange equation for functional (9) is

$$4\ddot{u}(3\dot{u}^2 - 1) = 0. \tag{10}$$

The only solution of the above equation that belongs to the admissible set $\mathcal{B}$ is $v(t) = \frac{L(N)}{L(M)}t$, where $t \in [0, L(M)]$. Note that v corresponds to a diffeomorphism in $\mathrm{Diff}(M, N)$.

We will show that the critical point v minimizes the functional Ψ; that is,

$$\Psi(u) \geq \Psi(v) = \frac{(L(N)^2 - L(M)^2)^2}{L(M)^3} \tag{11}$$

for all $u \in \mathcal{B}$. Using Hölder's inequality

$$L(N) = u(L(M)) = \int_0^{L(M)} \dot{u}(s)\, ds \leq \big[L(M) \int_0^{L(M)} \dot{u}^2(s)\, ds\big]^{1/2},$$

we have that

$$\frac{L(N)^2}{L(M)} \leq \int_0^{L(M)} \dot{u}^2(s)\, ds.$$

Thus, in view of the hypothesis that $L(N) \geq L(M)$,

$$\int_0^{L(M)} (\dot{u}^2(s) - 1)\, ds = \int_0^{L(M)} \dot{u}^2(s)\, ds - L(M) \geq \frac{L(N)^2 - L(M)^2}{L(M)} \geq 0. \tag{12}$$

After squaring both sides of inequality (12), we obtain the inequality

$$\Big(\int_0^{L(M)} (\dot{u}^2(s) - 1)\, ds\Big)^2 \geq \frac{(L(N)^2 - L(M)^2)^2}{L(M)^2}. \tag{13}$$

Applying Hölder's inequality to $\Phi(u)$ and taking into account inequality (13), we obtain inequality (11). Hence, the function $v(t) = L(N)/L(M)t$, where $t \in [0, L(M)]$, minimizes the functional Ψ over the admissible set $\mathcal{B}$. $\square$

Remark 4.2. *Let us write the Euler-Lagrange equation* (4) *for the one-dimensional case and compare it with equation* (10).

Recall that

$$[g_M]_{11}(t) = 1, \qquad [h^* g_N]_{11}(t) = \dot{u}(t)^2$$

and use formula (5) *to compute*

$$[L_Y h^* g_N]_{11}(t) = 2\dot{u}(t)\big(\ddot{u}(t)y(t) + \dot{y}(t)\dot{u}(t)\big) = 2\dot{u}(t)\frac{d}{dt}\big(\dot{u}(t)y(t)\big),$$

where $y(t)$ is the local coordinate of the vector field $Y = y\frac{\partial}{\partial t}$, i.e., y is a smooth periodic function on $[0, L(M)]$, which can be taken to be in $C_c^\infty([0, L(M)])$. Using the previous computation and formulas (2) *and* (4)*, we obtain the following Euler-Lagrange equation:*

$$\int_0^{L(M)} (\dot{u}^2 - 1)\dot{u}\frac{d}{dt}(\dot{u}y)\, dt = -\int_0^{L(M)} \frac{d}{dt}\big((\dot{u}^2 - 1)\dot{u}\big)\dot{u}y\, dt = 0$$

for all $y \in C_c^\infty([0, L(M)])$. The latter equation yields

$$\frac{d}{dt}\big((\dot{u}^2 - 1)\dot{u}\big)\dot{u} = \dot{u}\ddot{u}(3\dot{u}^2 - 1) = 0, \tag{14}$$

which has the same solutions in the admissible sets $\mathcal{B}$ and $\mathcal{C}$ as equation (10) *does.*

Proposition 4.3. *Suppose that M and N are smooth simple closed curves in $\mathbb{R}^2$ with arc lengths $L(M)$ and $L(N)$ and base points $p \in M$ and $q \in N$; γ and ξ are arc length parametrizations of M and N with $\gamma(0) = p$ and $\xi(0) = q$ that induce positive orientations; and the functions v and w are as in Lemma 4.1. If $L(N) \geq L(M)$, then the functional $\Phi(h)$ defined in display* (6) *has exactly two minimizers in the admissible set*

$$\mathcal{A} = \{h \in \mathrm{Diff}(M, N) : h(p) = q\} :$$

the orientation preserving minimizer

$$h_1 = \xi \circ v \circ \gamma^{-1}$$

and the orientation reversing minimizer

$$h_2 = \xi \circ w \circ \gamma^{-1}$$

(*where we consider γ as a function defined on $\big[0, L(M)\big)$ so that $\gamma^{-1}(p) = 0$*). *Moreover, the minimal value of the functional Φ is*

$$\Phi_{min} = \frac{(L(N)^2 - L(M)^2)^2}{L(M)^3}. \tag{15}$$

Example 4.4. *For $R > 0$, the radial map $h : \mathbb{R}^2 \to \mathbb{R}^2$ is defined to be $h(z) = Rz$. If M is a simple closed curve, $N := h(M)$ and $R > 1$, then h is a minimum of Φ on* $\mathrm{Diff}(M, N)$. *To see this fact, let $\gamma(t) = \big(x(t), y(t)\big)$, $t \in [0, L(M)]$, be an arc length parametrization of M. It is easy to see that $\xi(t) = R\big(x(t/R), y(t/R)\big)$, $t \in [0, RL(M)]$ parametrizes $N = h(M)$ by its arc length. By Proposition 4.3, the minimizer h_1 is*

$$h_1(z) = \xi\big(v \circ \gamma^{-1}(z)\big) = \xi\big(R\gamma^{-1}(z)\big) = \xi(Rt) = R\gamma(t) = Rz$$

for all $z \in M$. Hence, h_1 is the radial map.

Lemma 4.5. *If $L(N) < L(M)$, then the functional Ψ has no minimum in the admissible set $\mathcal{B}$.*

Proof. Let $\phi : [0, L(M)] \to \mathbb{R}$ be a continuous piecewise linear function such that $\phi(0) = 0$, $\phi(L(M)) = L(N)$, and $\dot\phi(t) = \pm 1$ whenever $t \in (0, L(M))$ and the derivative is defined. The graph of ϕ looks like a zig-zag. It is easy to see that ϕ is an element of the Sobolev space $W^{1,4}(0, L(M))$ (one weak derivative in the Lebesgue space L^4). By the standard properties of $W^{1,4}(0, L(M))$ with its usual norm $\|\cdot\|_{1,4}$, there exists a sequence of smooth functions $\phi_k \in C^\infty[0, L(M)]$ (each of which satisfies the boundary conditions $\phi_k(0) = 0$ and $\phi_k(L(M)) = L(N)$) such that $\|\phi_k - \phi\|_{1,4} \to 0$ as $k \to \infty$. Moreover, there is some constant $C > 0$ such that $\int_0^{L(M)} (\dot\phi_k^2 - \dot\phi^2)^2\, dx \le C\|\phi_k - \phi\|_{1,4}^2$. It is easy to see that

$$|\Phi(\phi_k) - \Phi(\phi)| \le C_1\|\phi_k - \phi\|_{1,4}$$

for some constant $C_1 > 0$. Taking into account the equality $\Psi(\phi) = 0$, we conclude that $\Psi(\phi_k) \to 0$ as $k \to \infty$. Thus, $\{\phi_k\}_{k=1}^\infty$ is a minimizing sequence for the functional Ψ in the admissible set $\mathcal{B}$. On the other hand, there is no function $f \in \mathcal{B}$ such that $\Psi(f) = 0 = \inf_{g\in\mathcal{B}} \Psi(g)$. Therefore, if $L(N) < L(M)$, the functional Φ has no minimum in the admissible set $\mathcal{B}$. □

Corollary 4.6. *If $L(N) < L(M)$, then the functional Φ has no minimum in the admissible set*

$$Q = \{h \in C^2(M, N) : h \textit{ is orientation preserving and } h(p) = q\}.$$

Let us interpret the result of Lemma 4.5. Let $h = \xi\circ\phi\circ\gamma^{-1}$, where $\phi : [0, L(M)] \to \mathbb{R}$ is defined in the proof of Lemma 4.5 and γ, ξ are arc length (positive orientation) parametrizations of the curves M and N viewed as periodic functions on $\mathbb{R}$. In case $L(N) < L(M)$, the action of the function h on the curve M can be described as follows. The curve M is cut into segments $\{M_i\}_{i=1}^k$, $k \in \mathbb{N}$, such that $\dot\phi$ has a constant value (1 or (-1)) on $\gamma^{-1}(M_i)$. Each segment M_i is wrapped around the curve N counterclockwise or clockwise depending on whether $\dot\phi$ equals 1 or (-1) on $\gamma^{-1}(M_i)$ respectively. Since $L(N)$ is less than $L(M)$, some points of N will be covered by segments of M several times. During this process, the segments of the curve M need not be stretched. Hence, as measured by the functional Φ, no strain is produced, i.e. $\Phi(h) = 0$.

The statement of Corollary 4.6 leaves open an interesting question: Does the functional Φ have a minimum in the admissible set $\mathcal{A}$? Some results in this direction are presented in the next section.

5. SECOND VARIATION

We will derive a necessary condition for a diffeomorphism $h \in \mathrm{Diff}(M,N)$ to be a minimum of the functional Φ. Let $h_t = h \circ \phi_t$ be a family of diffeomorphisms in $\mathrm{Diff}(M,N)$, where ϕ_t is the flow of a vector field $Y \in \Gamma(TM)$. Using the Lie derivative formula (see [1]), we derive the equations $\frac{d}{dt}(h_t^* g_N) = \phi_t^* L_Y h^* g_N$ and $\frac{d}{dt}(\phi_t^* L_Y h^* g_N) = \phi_t^* L_Y L_Y h^* g_N$. If there exists $\delta > 0$ such that $\Phi(h_t) > \Phi(h)$ for all $|t| < \delta$ and for all variations h_t of h, then h is called a relative minimum of h. If $h \in \mathrm{Diff}(M,N)$ is a relative minimum of Φ, then $\frac{d^2}{dt^2}\Phi(h_t)|_{t=0} > 0$.

Using the previous computations of Lie derivatives, the second variation of Φ is

$$\begin{aligned}\frac{1}{2}\frac{d^2}{dt^2}\Phi(h_t)|_{t=0} = \int_M & G(L_Y h^* g_N, L_Y h^* g_N)\omega_M + \\ + \int_M & G(L_Y L_Y h^* g_N, h^* g_N - g_M)\omega_M.\end{aligned} \tag{16}$$

Lemma 5.1. *Let M and N be simple closed curves parametrized by functions γ and ξ satisfying all the properties stated in Lemma 4.3. If $h \in \mathrm{Diff}(M,N)$ minimizes the functional Φ in the admissible set $\mathcal{A}$, then the local representation $u = \xi^{-1} \circ h \circ \gamma$ of h satisfies the inequality*

$$\dot{u}^2(t) \geq \frac{1}{3} \tag{17}$$

for all $t \in \big(0, L(M)\big)$.

Proof. Using formula (5), we compute

$$[L_Y h^* g_N]_{11} = 2(\dot{u}\ddot{u}y + \dot{u}^2\dot{y})$$

and

$$[L_Y L_Y h^* g_N]_{11} = 2(\ddot{u}^2 y^2 + \dot{u}\,\dddot{u}\,y^2 + 5\dot{u}\,\ddot{u}\,\dot{y}\,y + \dot{u}^2\ddot{y}\,y + 2\dot{u}^2\dot{y}^2).$$

Substituting the latter expressions into formula (16), we obtain the necessary condition

$$W := 4\int_0^{L(M)} \dot{u}^4\,\dot{y}^2\,dt + 4\int_0^{L(M)} \dot{u}^2(\dot{u}^2-1)\,\dot{y}^2\,dt + 2\int_0^{L(M)} \dot{u}^2(\dot{u}^2-1)\,y\,\ddot{y}\,dt + \ldots \geq 0,$$

where the integrands of the omitted terms all contain the factor y. After integration by parts, we obtain the inequality

$$W = \int_0^{L(M)} \Big(4\dot{u}^4 + 4\dot{u}^2(\dot{u}^2-1) - 2\dot{u}^2(\dot{u}^2-1)\Big)\dot{y}^2\,dt + \ldots \geq 0. \tag{18}$$

Define $y(t) = \varepsilon\rho\big(\frac{t}{\varepsilon}\big)\zeta(t)$, where $\rho(t)$ is a periodic "zig-zag" function defined by the expressions

$$\rho(t) = \begin{cases} t, & \text{if } 0 \le t < 1/2, \\ 1 - t, & \text{if } 1/2 \le t < 1, \end{cases} \tag{19}$$

and $\rho(t+1) = \rho(t)$, $\zeta \in C_c^\infty\big(0, L(M)\big)$. Notice that $\dot\rho^2 = 1$ almost everywhere on $\mathbb{R}$ and $\dot y^2 = \zeta^2 + O(\varepsilon)$ when $\varepsilon \to 0$. Substitute y into inequality (18) and pass to the limit as $\varepsilon \to 0$. All the omitted terms in the expression for W tend to zero, because they contain y as a factor. Hence, we obtain the inequality

$$W = \int_0^{L(M)} \big(4\dot u^4 + 2\dot u^2(\dot u^2 - 1)\big)\zeta^2\, dt \ge 0,$$

which (after a standard bump function argument) reduces to the inequality

$$\dot u^2 \ge 1/3 \tag{20}$$

as required. □

Proposition 5.2. *If M and N are simple closed curves such that their corresponding arc lengths $L(M)$ and $L(N)$ satisfy the inequality $\frac{L(N)}{L(M)} < \frac{1}{\sqrt{3}}$, then the functional Φ has no minimum in the admissible set $\mathcal{A}$.*

Proof. If $h \in \mathrm{Diff}(M, N)$ is a minimum of the functional Φ, then h satisfies the Euler-Lagrange equation (4). Let γ and ξ be parametrizations of the curves M and N with all the properties stated in Corollary 4.3. By Remark 4.2, the local representation $u = \xi^{-1} \circ h \circ \gamma$ of h satisfies the ordinary differential equation (14) on $(0, L(M))$. In addition, u must satisfy the boundary conditions $u(0) = 0, u(L(M)) = L(N)$ or $u(0) = L(N), u(L(M)) = 0$. Hence, either $u(t) = L(N)/L(M)t$ or $u(t) = -L(N)/L(M)t + L(N)$. Since h minimizes Φ, by Lemma 5.1 $\dot u^2 \ge 1/3$, or, equivalently, $L(N)/L(M) \ge \frac{1}{\sqrt{3}}$. This contradicts the assumption of the theorem. □

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