Opuscula Mathematica • Vol. 28 • No. 1 • 2008

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# EXPONENTIAL TYPE VECTORS IN WIENER ALGEBRAS ON A BANACH BALL

**Abstract.** We consider Wiener type algebras on an open Banach ball. In particular, we prove that such algebras consist of functions analytic in this ball. We also consider a property of one-parameter groups generated by an isometric group acting on a Banach ball. We establish that the subspace of exponential type vectors of its generators form a dense subalgebra in a Wiener algebra and a generator is a derivation on this subspace.

**Keywords:** unitary one-parametric group, Wiener type algebras.

**Mathematics Subject Classification:** Primary 46E50; Secondary 46G25.

## 1. INTRODUCTION

As it is well known [8], Nelson's classical theorem implies the following assertion: if a closed linear operator $iA$ on a Hilbert space generates the unitary group $e^{itA}$, then the set of its analytic vectors is dense. Let $\big(X, \|\,.\,\|\big)$ be a Banach space over the field of complex numbers $\mathbb{C}$,

$$A:\, \mathcal{D}(A) \subset X \to X$$

a closed unbounded linear operator with a dense domain $\mathcal{D}(A)$ and $\{A^k :\, k = 2, 3, \ldots\}$ powers of $A$ with domains

$$\mathcal{D}(A^k) := \Big\{x \in \mathcal{D}(A^{k-1}) : Ax \in \mathcal{D}(A^{k-1})\Big\}.$$

An element

$$x \in \mathcal{D}(A^\infty) := \bigcap_{k\geq 1} \mathcal{D}(A^k)$$

is called a vector of finite exponential type of the operator $A$ if

$$\varlimsup_{k\to\infty} \big\|A^k x\big\|^{1/k} < \infty.$$

The set of all vectors of finite exponential type will be denoted by $\mathcal{E}(A)$. In his work [9] Ya. Radyno proved the density of the exponential type vectors for generators of strongly continuous one-parameter operator groups in a Banach space. It is easy to see that the set of the exponential type vectors is an essentially smaller subset of the set of analytic vectors; therefore, we can treat the last fact as a generalization of Nelson's theorem on a wide class of groups defined in Banach spaces (not necessarily Hilbert spaces).

In the present paper, we consider the exponential type vectors for some class of one-parameter operator groups defined on Wiener type algebras of bounded analytic functions in an infinite dimensional Banach ball.

It should be noted that the definition of the Wiener type algebra of analytic functions in a Banach ball introduced in this paper follows the pattern set in [4]. In [4], the authors introduced a definition of the Wiener type algebra of analytic functions in a Hilbert ball.

An idea of a construction of such spaces uses the fact that the dual space to the symmetric Banach space $n$-tensor product endowed with a cross-norm is isometric to a subspace of $n$-homogeneous polynomials on a Banach space. Considering a special cross-norm, known as Hilbertian cross-norm, the authors have obtained a natural Hilbert structure on the tensor product of the Hilbert spaces. After symmetrization of such tensor product, they have obtained a predual space to a Hilbert space of continuous polynomials. We can note that the direct $\ell_1$-sum of symmetric Hilbertian tensor products plays an important role in quantum mechanics, where it forms a dense subspace in the well-known symmetric Fock space or a so-called Boson space [8].

In the present paper, we generalize, first, the construction of Wiener type algebras of bounded analytic functions in the case of Banach spaces and, second, the Ya. Radyno's theorem on the density of the exponential type vectors for generators of some strongly continuous operator groups acting on such Wiener type algebras. Clearly, results obtained here may also be treated as a generalization of the above mentioned Nelson theorem on the case of Boson spaces.

In Section 3, we explain that a notion of an algebra in the definition is used correctly and we check up that the functions considered are actually analytic. Let us recall that a complex-valued function on an open subset of a Banach space $X$ is analytic if it is locally bounded and its restriction to every complex one-dimensional affine subspace of $X$ is analytic. In other words, $f$ is analytic on $D$ if $f$ is locally bounded and if for every $x_0 \in D$ and direction $x \in X$, the function $\lambda \to f(x_0 + \lambda x)$ depends analytically on $\lambda$ for $\lambda$ near 0. From the corresponding facts in one complex variable it follows that sums, products, and uniform limits of analytic functions are analytic [5].

In Sections 4 and 5, on a Wiener type algebra, we define a group as a change of variables. We investigate some properties of such group. In particular, we prove that the subspace of exponential type vectors of the generator of this group is dense, which is, in a way, a generalization of Nelson's theorem for such groups. In this respect, we used results of [1].

## 2. PRELIMINARIES

Let $(X, \|\cdot\|)$ be a complex Banach space, $B = \{x \in X \colon \|x\| < 1\}$ the open unit ball and $X'$ the normed dual space of $X$. The algebraic tensor product

$$X^{\otimes n} := X \otimes \ldots \otimes X$$

consists of finite sums $u = \sum_j x_{1j} \otimes \cdots \otimes x_{nj}$, where $x_{ij} \in X$, with the usual rules for algebraic manipulation. The dual of $X^{\otimes n}$ is the space of all $n$-linear functionals on $X \times \ldots \times X$, where the linear functional $\tilde{F}$ corresponding to the $n$-linear functional $F$ is given by

$$\tilde{F}\left(\sum_j x_{1j} \otimes \cdots \otimes x_{nj}\right) = \sum_j F(x_{1j}, \cdots, x_{nj}).$$

There are a number of useful norms that can be introduced into the algebraic tensor power $X^{\otimes n}$. The projective tensor product norm is given by

$$\|u\|_\pi = \inf \sum_j \|x_{1j}\| \cdots \|x_{nj}\|,$$

where the infimum is taken over all representations of $u$ as a finite sum $u = \sum_j x_{1j} \otimes \cdots \otimes x_{nj}$. This is evidently the largest norm with the property

$$\|x_1 \otimes \cdots \otimes x_n\|_\pi \leq \|x_1\| \cdots \|x_n\|$$

for all $x_1, \ldots, x_n \in X$. We denote the completion of $X^{\otimes n}$ with respect to this norm by $X_\pi^{\otimes n}$ and refer to it as the projective tensor power.

According to [5], the projective tensor product norm satisfies the cross-property, i.e.,

$$\|x_1 \otimes \cdots \otimes x_n\|_\pi = \|x_1\| \cdots \|x_n\|$$

for all $x_1, \ldots, x_n \in X$ and the dual of $X_\pi^{\otimes n}$ is isometrically isomorphic to the space of continuous $n$-linear functionals on $X \times \ldots \times X$.

In the sequel, Grothendieck's following assertion [6] will frequently be used: for each element $u \in X_\pi^{\otimes n}$ there exists $x_{ij} \in X$ such that

$$u = \sum_j x_{1j} \otimes \cdots \otimes x_{nj},$$

where the series $\sum_j x_{1j} \otimes \cdots \otimes x_{nj}$ absolutely converges, i.e., $\sum_j \|x_{1j}\| \cdots \|x_{nj}\| < \infty$. Analogically, we complete the projective tensor power $X_\pi^{\prime\otimes n}$ of dual spaces. We use an apostrophe to denote a linear functional and the values of the linear functional $f_n' \in X_\pi^{\prime\otimes n}$ are denoted by $\langle u \mid f_n' \rangle$ for all $u \in X_\pi^{\otimes n}$.

Let $\mathcal{G}_n$ be the permutation group of the set $\{1, \ldots, \mathrm{n}\}$ and

$$S_n \colon x_1 \otimes \ldots \otimes x_n \longrightarrow x_1 \odot \ldots \odot x_n := \frac{1}{n!} \sum_{s \in \mathcal{G}_n} x_{s(1)} \otimes \ldots \otimes x_{s(n)}$$

be a symmetric projection defined on $X_\pi^{\otimes n}$ with the codomain $X_\pi^{\odot n}$.

Then $X_\pi^{\otimes n} = X_\pi^{\odot n} \oplus \mathcal{N}(S_n)$. We use the notation $x^{\odot n} = x \odot \ldots \odot x \in X^{\odot n}$ for all $x \in X$. Note that $\|S_n\| \le 1$, since

$$\begin{aligned}\|S_n u\|_\pi &= \Big\| S_n\Big(\sum_j x_{1j}\otimes\cdots\otimes x_{nj}\Big)\Big\|_\pi = \Big\|\sum_j S_n(x_{1j}\otimes\cdots\otimes x_{nj})\Big\|_\pi = \\ &= \Big\|\sum_j \frac{1}{n!}\sum_s x_{s(1)j}\otimes\ldots\otimes x_{s(n)j}\Big\|_\pi \le \sum_j \frac{1}{n!}\sum_s \|x_{s(1)j}\otimes\ldots\otimes x_{s(n)j}\|_\pi \le \\ &\le \inf\sum_j \frac{1}{n!}\sum_s \|x_{s(1)j}\|\cdots\|x_{s(n)j}\| = \inf\sum_j \frac{n!}{n!}\|x_{1j}\|\cdots\|x_{nj}\| = \|u\|_\pi\end{aligned}$$

for all $u \in X^{\otimes n}$. It is clear that the equality

$$X_\pi^{'\otimes n} = X_\pi^{'\odot n} \oplus \mathcal{N}(S_n') \tag{1}$$

holds for the projection $S_n' \colon X_\pi^{'\otimes n} \longrightarrow X_\pi^{'\odot n}$ and

$$\|S_n'\| \le 1.$$

## 3. WIENER TYPE ALGEBRAS

Each element $f_n' \in X_\pi^{'\odot n}$ corresponds to the unique $n$-homogeneous continuous polynomial $f_n$ such that

$$f_n(x) := \langle x^{\odot n} \mid f_n'\rangle$$

for all $x \in X$ and we denote the vector space of such polynomials by

$$\mathcal{P}_\pi^n(X) = \{f_n \colon f_n' \in X_\pi^{'\odot n}\}.$$

In particular, we put $\mathcal{P}_\pi^0(X) = \mathbb{C}$ and $x^{\odot 0} = 1 \in \mathbb{C}$. So $\mathcal{P}_\pi^n(X) = X_\pi^{'\odot n}$ algebraically. This is the algebraic isomorphism $f_n \longmapsto f_n'$ which is realized by the restriction of functionals belonging to $X_\pi^{'\odot n}$ to the total set $\{x^{\odot n} \colon x \in X\}$ in the space $X_\pi^{\odot n}$. The total property of the set $\{x^{\odot n} \colon x \in X\}$ in the space $X_\pi^{\odot n}$ follows from the polarization formula [3], which says that for any Banach space $Y$ and any symmetric $n$-linear mapping $L \in \mathcal{L}_s({}^nX, Y)$ there holds

$$L(x_1,\ldots,x_n) = \frac{1}{2^n n!}\sum_{1\le i\le n}\sum_{\epsilon_i=\pm1}\epsilon_1\ldots\epsilon_n L\circ\Delta_n\Big(\sum_{j=1}^n \epsilon_j x_j\Big), \tag{2}$$

where $\Delta_n : X \ni x \mapsto (x,\ldots,x) \in X^n$ is the diagonal mapping. Thus

$$L(x_1,\ldots,x_n) = \frac{1}{2^n n!}\sum_{1\le i\le n}\sum_{\epsilon_i=\pm1}\epsilon_1\ldots\epsilon_n L\Big(\sum_{j=1}^n \epsilon_j x_j,\ldots,\sum_{j=1}^n \epsilon_j x_j\Big).$$

Let us replace the mapping $L$ with a symmetric tensor product; then we receive

$$x_1 \odot \cdots \odot x_n = \frac{1}{2^n n!} \sum_{1\le i\le n} \sum_{\epsilon_i=\pm 1} \epsilon_1 \ldots \epsilon_n \Big( \sum_{j=1}^{n} \epsilon_j x_j \odot \cdots \odot \sum_{j=1}^{n} \epsilon_j x_j \Big) =$$
$$= \frac{1}{2^n n!} \sum_{1\le i\le n} \sum_{\epsilon_i=\pm 1} \epsilon_1 \ldots \epsilon_n \Big( \sum_{j=1}^{n} \epsilon_j x_j \Big)^{\odot n}.$$

Let us denote $y_{\epsilon_1,\ldots,\epsilon_n} := \sum_{j=1}^{n} \epsilon_j x_j$. Then we obtain

$$x_1 \odot \cdots \odot x_n = \frac{1}{2^n n!} \sum_{1\le i\le n} \sum_{\epsilon_i=\pm 1} \epsilon_1 \ldots \epsilon_n y^{\odot n}_{\epsilon_1,\ldots,\epsilon_n}.$$

In the sequel, we define the norm

$$\|f_n\| := \|f'_n\|_\pi$$

on $\mathcal{P}^n_\pi(X)$, where the element $f'_n$ belongs to the subspace $X'^{\odot n}_\pi$. In consequence, we obtain the isometry of $\mathcal{P}^n_\pi(X)$ and $X'^{\odot n}_\pi$. The space of all such polynomials is defined as the complex linear span

$$\operatorname{span}\big\{\mathcal{P}^n_\pi(X) \colon n \in \mathbb{Z}_+\big\} := \mathcal{P}_\pi(X).$$

**Definition 1.** *Following* [4], *we call a Wiener type algebra the* $\ell_1$*-sum*

$$W_\pi(B) := \Big\{ f = \sum_{n\in\mathbb{Z}_+} f_n \colon f_n \in \mathcal{P}^n_\pi(X) \Big\}$$

*with the finite norm* $\|f\| = \sum_{n\in\mathbb{Z}_+} \|f_n\|$.

The role of $B$ in the above notation will be explained in the next proposition. We investigate some properties of $W_\pi(B)$ and verify that the notion of an algebra in the definition is used correctly.

**Proposition 1.** $W_\pi(B)$ *is a Banach subalgebra with the unit of the algebra of all bounded analytic functions in* $B$.

*Proof.* From equality (1) there follows that the space $X'^{\odot n}_\pi$ is complete. Since the space

$$\mathcal{P}^n_\pi(X) = X'^{\odot n}_\pi$$

is complete, their $\ell_1$-sum is complete, too. Put $x = \zeta a$, where $\|a\| = 1$ and $|\zeta| < 1$. The series

$$f(x) = \sum_{n\in\mathbb{Z}_+} \zeta^n f_n(a)$$

converges absolutely and uniformly on any closed ball $B_r = \{\zeta a \colon |\zeta| \le r < 1, \|a\| = 1\}$, since

$$\sum_{n\in\mathbb{Z}_+} |\zeta|^n \sup_{\|a\|=1} |f_n(a)| \le \sum_{n\in\mathbb{Z}_+} |\zeta|^n \|f_n\| \le \|f\|.$$

Thus the function $f$ is bounded, continuous and analytic in $B$ (cf. [7]). The equality

$$S_{n+m}x^{\odot(n+m)} = x^{\odot(n+m)}$$

and the total property of the set

$$\{x^{\odot(n+m)} \colon x \in X\}$$

in the space $X_\pi^{\odot(n+m)}$ imply that the equalities

$$\begin{aligned}(f_n g_m)(x) &= f_n(x) g_m(x) = \langle x^{\odot n} \mid f'_n\rangle\langle x^{\odot m} \mid g'_m\rangle = \langle x^{\odot(n+m)} \mid f'_n \otimes g'_m\rangle = \\ &= \langle S_{n+m}x^{\odot(n+m)} \mid f'_n \otimes g'_m\rangle = \langle x^{\odot(n+m)} \mid S'_{n+m}(f'_n \otimes g'_m)\rangle = \\ &= \langle x^{\odot(n+m)} \mid f'_n \odot g'_m\rangle\end{aligned}$$

uniquely define a functional

$$f'_n \odot g'_m := S'_{n+m}(f'_n \otimes g'_m) \in X_\pi^{'\odot(n+m)}$$

for any $f'_n \in X_\pi^{'\odot n}$ and $g'_m \in X_\pi^{'\odot m}$. Therefore, the $(n+m)$-homogeneous polynomial $f_n \odot g_m$ is well defined and belongs to $\mathcal{P}_\pi^{n+m}(X)$. Moreover,

$$\begin{aligned}\|f_n g_m\| &= \|f'_n \odot g'_m\|_\pi = \|S'_{n+m}(f'_n \otimes g'_m)\|_\pi \leq \\ &\leq \|S'_{n+m}\|\|(f'_n \otimes g'_m)\|_\pi \leq \|f'_n \otimes g'_m\|_\pi = \|f'_n\|_\pi \|g'_m\|_\pi = \|f_n\|\,\|g_m\|.\end{aligned}$$

It follows that

$$\begin{aligned}\|fg\| &= \Big\| \sum_{n\in\mathbb{Z}_+} \sum_{k=0}^{n} f_k g_{n-k} \Big\| \leq \sum_{n\in\mathbb{Z}_+} \sum_{k=0}^{n} \|f_k\|\,\|g_{n-k}\| = \\ &= \Big( \sum_{n\in\mathbb{Z}_+} \|f_n\| \Big) \Big( \sum_{m\in\mathbb{Z}_+} \|g_m\| \Big) = \|f\|\,\|g\|,\end{aligned}$$

and, in particular, the product

$$fg = \sum_{n\in\mathbb{Z}_+} \left( \sum_{k=0}^{n} f_k g_{n-k} \right)$$

of analytic functions $f = \sum_{n\in\mathbb{Z}_+} f_n \in W_\pi(B)$ and $g = \sum_{n\in\mathbb{Z}_+} g_n \in W_\pi(B)$ also belongs to $W_\pi(B)$. It is easy to see that the scalar unit $1 \in \mathcal{P}_\pi^0(X)$ is also the unit of the algebra $W_\pi(B)$. □

## 4. ONE-PARAMETER OPERATOR GROUPS

Let us consider a one-parameter group of linear bounded operators $\mathbb{R} \ni t \longmapsto U_t$ acting on the reflexive Banach space $X$. On the Wiener algebra $W_\pi(B)$, we define the operator

$$\widehat{U}_t f(x) := f(U_t x),$$

where $f \in W_\pi(B)$ and $x \in B$.

**Proposition 2.** *Let $U_t$ be an isometric group of linear operators on $X$. Then the operator-valued function $\mathbb{R} \ni t \longmapsto \widehat{U}_t$ on the Wiener algebra $W_\pi(B)$ is a contraction group of algebraic automorphisms.*

*Proof.* We denote by $U'_t \colon X' \longrightarrow X'$ the corresponding group of adjoint operators. Put $\widehat{U}_t^{\odot 0} = 1$ and

$$\widehat{U}_t^{\odot n} f_n(x) := \langle x^{\odot n} \mid U_t^{'\otimes n} f'_n \rangle$$

for all $f'_n \in X_\pi^{'\odot n}$ and $x \in B$, where

$$U_t^{'\otimes n} := \underbrace{U'_t \otimes \ldots \otimes U'_t}_{n}.$$

For all $x \in X$ and $f_n \in \mathcal{P}_\pi^n(X)$, the following equalities hold

$$\widehat{U}_t^{\odot n} f_n(x) := \langle x^{\odot n} \mid U_t^{'\otimes n} f'_n \rangle = \langle U_t^{\otimes n} x^{\odot n} \mid f'_n \rangle = \langle (U_t x)^{\odot n} \mid f'_n \rangle = f_n(U_t x). \tag{3}$$

They unique by define the group of linear bounded operators on the space of $n$-homogeneous polynomials $\mathcal{P}_\pi^n(X)$:

$$\widehat{U}_0^{\odot n} f_n(x) = f_n(U_0 x) = f_n(Ix) = f_n(x) \tag{4}$$

and

$$\widehat{U}_{t+s}^{\odot n} f_n(x) = f_n(U_{t+s} x) = f_n(U_t U_s x) = \widehat{U}_t^{\odot n} f_n(U_s x) = \widehat{U}_t^{\odot n} \widehat{U}_s^{\odot n} f_n(x).$$

Moreover, the property $S'_n U_t^{'\otimes n} = U_t^{'\otimes n} S'_n$ implies the relations,

$$\begin{aligned} \|\widehat{U}_t^{\odot n} f_n\| &= \|U_t^{'\otimes n} f'_n\|_\pi = \|U_t^{'\otimes n} S'_n f'_n\|_\pi = \|S'_n U_t^{'\otimes n} f'_n\|_\pi \le \\ &\le \|S'_n\| \|U_t^{'\otimes n}\| \|f'_n\|_\pi = \|S'_n\| \|U'_t\|^n \|f'_n\|_\pi \le \|f'_n\|_\pi = \|f_n\|. \end{aligned} \tag{5}$$

Applying equality (3), we obtain the relation

$$\widehat{U}_t f = \sum_{n \in \mathbb{Z}_+} \widehat{U}_t^{\odot n} f_n \tag{6}$$

for any $f = \sum_{n \in \mathbb{Z}_+} f_n \in W_\pi(B)$, since

$$\widehat{U}_t f(x) = f(U_t x) = \sum_{n \in \mathbb{Z}_+} f_n(U_t x) = \sum_{n \in \mathbb{Z}_+} \widehat{U}_t^{\odot n} f_n(x). \tag{7}$$

Therefore, from (5) there follows

$$\|\widehat{U}_t f\| = \sum_{n \in \mathbb{Z}_+} \|\widehat{U}_t^{\odot n} f_n\| \le \sum_{n \in \mathbb{Z}_+} \|f_n\| = \|f\|. \tag{8}$$

Hence, $\|\widehat{U}_t\| \leq 1$ for all $t \in \mathbb{R}$. It is clear that

$$\widehat{U}_t(fg) = \sum_{n\in\mathbb{Z}_+} \Big( \sum_{k=0}^{n} \widehat{U}_t^{\odot k} f_k \widehat{U}_t^{\odot(n-k)} g_{n-k} \Big) = (\widehat{U}_t f)(\widehat{U}_t g),$$

since the following equalities hold:

$$\begin{aligned}
\widehat{U}_t^{\odot k} f_k(x) \widehat{U}_t^{\odot(n-k)} g_{n-k}(x) &= \langle x^{\odot k} \mid U_t^{'\otimes k} f_k' \rangle \langle x^{\odot(n-k)} \mid U_t^{'\otimes(n-k)} g_{n-k}' \rangle = \\
&= \langle x^{\odot n} \mid U_t^{'\otimes k} f_k' \otimes U_t^{'\otimes(n-k)} g_{n-k}' \rangle = \\
&= \langle S_n x^{\odot n} \mid U_t^{'\otimes k} f_k' \otimes U_t^{'\otimes(n-k)} g_{n-k}' \rangle = \\
&= \langle x^{\odot n} \mid S_n'(U_t^{'\otimes k} f_k' \otimes U_t^{'\otimes(n-k)} g_{n-k}') \rangle = \\
&= \langle x^{\odot n} \mid U_t^{'\otimes n}(f_k' \odot g_{n-k}') \rangle = \\
&= \widehat{U}_t^{\odot n} \langle x^{\odot n} \mid f_k' \odot g_{n-k}' \rangle = \\
&= \widehat{U}_t^{\odot n} f_k(x) g_{n-k}(x).
\end{aligned}$$

□

Let the unbounded linear operator $A$ with a dense domain $\mathcal{D}(A) \subset X$ be the generator of the group $U_t$. Denote by $A'$ the generator of the adjoint group $U_t'$ with a domain $\mathcal{D}(A') \subset X'$. On the algebraic tensor power $\mathcal{D}(A')^{\otimes n}$, which is a dense subspace of $X^{'\otimes n}$, the following operators are well defined

$$A_j' := \underbrace{I' \otimes \ldots \otimes I' \otimes A'}_{j} \otimes \underbrace{I' \otimes \ldots \otimes I'}_{n-j},$$

for $j \geq 1$ and $A_0' := I'$, where $I'$ is the identical operator on $X'$. Let $\mathcal{D}(A')^{\otimes n}$ be the domain and $\mathcal{D}(A')^{\odot n}$ the codomain of the projection $S_n'$. By $\widehat{A}$ we denote the generator of $\widehat{U}_t$ on the Wiener algebra $W_\pi(B)$; let

$$\mathcal{D}(\widehat{A}) := \Big\{ f \in W_\pi(B) \colon \lim_{t\to 0} \frac{\widehat{U}_t f - f}{t} \text{ exists} \Big\}$$

be its domain.

**Proposition 3.** *Let $U_t$ be an isometric group of linear operators in $X$. Then, on the subspace of elements $f = \sum_{n\in\mathbb{Z}_+} f_n \in W_\pi(B)$ such that $f_n' \in \mathcal{D}(A')^{\odot n}$ for all $n \in \mathbb{N}$, the generator $\widehat{A}$ of the group $\widehat{U}_t$ takes the form*

$$\widehat{A} f(x) = \sum_{n\in\mathbb{Z}_+} \Big\langle x^{\odot n} \,\Big|\, \sum_{j=0}^{n} A_j' f_n' \Big\rangle$$

*for any $x \in B$.*

*Proof.* As it is known [10], the equalities

$$\frac{d}{dt} U_t' y = A' U_t' y = U_t' A' y$$

hold for all $y \in \mathcal{D}(A')$. This, in particular, implies that $A' \colon \mathcal{D}(A') \longrightarrow \mathcal{D}(A')$, i.e., that the domain of the generator is $A'$-invariant. Due to this property, for any $y_j \in \mathcal{D}(A')$, $(j = 1, \dots, n)$, the generator satisfies the equalities

$$\begin{aligned}
\frac{d}{dt}(U'_t y_1 \otimes \ldots \otimes U'_t y_n) &= \sum_{j=1}^{n} U'_t y_1 \otimes \ldots \otimes \frac{d}{dt} U'_t y_j \otimes \ldots \otimes U'_t y_n = \\
&= \sum_{j=1}^{n} U'_t y_1 \otimes \ldots \otimes A' U'_t y_j \otimes \ldots \otimes U'_t y_n = \\
&= \Big(\sum_{j=1}^{n} A'_j\Big)(U'_t y_1 \otimes \ldots \otimes U'_t y_n) = \\
&= U_t^{'\otimes n}\Big(\sum_{j=1}^{n} y_1 \otimes \ldots \otimes A' y_j \otimes \ldots \otimes y_n\Big) = \\
&= \Big(\sum_{j=1}^{n} A'_j\Big) U_t^{'\otimes n}(y_1 \otimes \ldots \otimes y_n).
\end{aligned}$$

The linear span of elements $y_1 \otimes \ldots \otimes y_n$ coincides with $\mathcal{D}(A')^{\otimes n}$. So,

$$\frac{d}{dt} U_t^{'\otimes n} f'_n = \Big(\sum_{j=1}^{n} A'_j\Big) U_t^{'\otimes n} f'_n,$$

thus

$$\frac{d}{dt} U_t^{'\otimes n} f'_n \mid_{t=0} = \Big(\sum_{j=1}^{n} A'_j\Big) f'_n$$

for all $f'_n \in \mathcal{D}(A')^{\otimes n}$. As $S'_n U_t^{'\otimes n} = U_t^{'\otimes n} S'_n$, we obtain

$$\frac{d}{dt} U_t^{'\otimes n} f'_n \mid_{t=0} = \Big(\sum_{j=1}^{n} A'_j\Big) f'_n$$

for all $f'_n \in \mathcal{D}(A')^{\odot n}$ and $n \in \mathbb{Z}_+$. Finally, applying (6), we obtain

$$\begin{aligned}
\frac{d}{dt}\widehat{U}_t f(x) \mid_{t=0} &= \frac{d}{dt} \sum_{n \in \mathbb{Z}_+} \widehat{U}_t^{\odot n} f_n(x) \mid_{t=0} = \sum_{n \in \mathbb{Z}_+} \frac{d}{dt} \widehat{U}_t^{\odot n} f_n(x) \mid_{t=0} = \\
&= \sum_{n \in \mathbb{Z}_+} \Big\langle x^{\odot n} \mid \frac{d}{dt} U_t^{'\otimes n} f'_n \mid_{t=0} \Big\rangle = \sum_{n \in \mathbb{Z}_+} \Big\langle x^{\odot n} \mid \sum_{j=0}^{n} A'_j f'_n \Big\rangle = \widehat{A} f(x)
\end{aligned}$$

for any $f = \sum_{n \in \mathbb{Z}_+} f_n \in W_\pi(B)$ and $x \in B$. $\square$

## 5. EXPONENTIAL TYPE VECTORS

It is known [2,9] that if $T$ is an unbounded linear operator on a Banach space $Y$ with a domain $\mathcal{D}(T)$, $\{T^n : n \in \mathbb{N}\}$ are integer powers of $T$ with domain

$$\begin{aligned} \mathcal{D}(T^n) &:= \big\{ y \in \mathcal{D}(T^{n-1}) : Ty \in \mathcal{D}(T^{n-1}) \big\}, \\ \mathcal{D}(T^\infty) &:= \bigcap_{n \geq 1} \mathcal{D}(T^n), \end{aligned}$$

and $T^0 = I$ is the identity operator on $Y$, then an element $y \in \mathcal{D}(T^\infty)$ is called a vector of exponential type $\nu$ of the operator $T$ if there exists a constant $c = c(y)$ such that

$$\|T^n y\| \leq \nu^n c$$

for all $n \in \mathbb{Z}_+$. We denote the set of all vectors of exponential type $\nu$ of $T$ by $\mathcal{E}^\nu(T)$. Then, as it is known [1],

$$\mathcal{E}(T) := \bigcup_{\nu > 0} \mathcal{E}^\nu(T).$$

It is clear that the subspaces $\mathcal{E}^\nu(T)$ and $\mathcal{E}(T)$ are $T$-invariant [1].

Let $\mathcal{E}(A')$ be the subspace of exponential type vectors for $A'$ and $\mathcal{E}(\widehat{A})$ an analogical subspace for $\widehat{A}$. Denote by $\mathcal{E}(A')^{\otimes n}$ the domain and by $\mathcal{E}(A')^{\odot n}$ the codomain of the projection $S'_n$.

**Theorem 1.** *Let $X$ be a reflexive space and $U_t$ an isometric $(C_0)$-group in $X$. Then*

$1^\circ$ *$\widehat{U}_t$ is a contraction $(C_0)$-group;*

$2^\circ$ *the generator $\widehat{A}$ of $\widehat{U}_t$ is a closed operator with a domain $\mathcal{D}(\widehat{A})$ dense in $W_\pi(B)$;*

$3^\circ$ *the subspace*

$$\operatorname{span}\{\mathcal{E}(A')^{\odot n} : n \in \mathbb{Z}_+\},$$

*which contains in $\mathcal{E}(\widehat{A})$, is a dense subalgebra of $W_\pi(B)$, $\widehat{A}$-invariant and the corresponding restriction $\widehat{A}\,|_{\mathcal{E}(\widehat{A})}$ is a derivation on $\mathcal{E}(\widehat{A})$.*

*Proof.* Since $X$ is reflexive, the adjoint unbounded linear operator $A'$ is the generator of the adjoint group $U'_t$ and $A'$ has a dense domain $\mathcal{D}(A') \subset X'$. As it is known, in this case, $U'_t$ is a $(C_0)$-group (see [10, Chapter IX]). This and relation (6) imply

$$\lim_{t \to 0} \widehat{U}_t f = \sum_{n \in \mathbb{Z}_+} \lim_{t \to 0} \widehat{U}_t^{\odot n} f_n = \sum_{n \in \mathbb{Z}_+} f_n = f$$

for any $f = \sum\limits_{n \in \mathbb{Z}_+} f_n \in W_\pi(B)$. In fact,

$$\begin{aligned} \lim_{t \to 0} \widehat{U}_t f(x) &= \lim_{t \to 0} \sum_{n \in \mathbb{Z}_+} \widehat{U}_t^{\odot n} f_n(x) = \sum_{n \in \mathbb{Z}_+} \lim_{t \to 0} \widehat{U}_t^{\odot n} f_n(x) = \\ &= \sum_{n \in \mathbb{Z}_+} \langle x^{\odot n} \mid \lim_{t \to 0} U_t^{'\otimes n} f'_n \rangle = \sum_{n \in \mathbb{Z}_+} \langle x^{\odot n} \mid f'_n \rangle = \sum_{n \in \mathbb{Z}_+} f_n(x) = f(x). \end{aligned}$$

Thus $1^\circ$ is proved.

The Hille-Yosida theorem implies 2° immediately.

For a proof of 3°, let us denote

$$ {}_nA' := \sum_{j=1}^{n} A'_j. $$

We will verify that the subspace $\mathcal{E}(A')^{\otimes n}$ contains to $\mathcal{E}({}_nA')$. From the definition of exponential type vectors of $A'$ it follows that for any $y_j \in \mathcal{E}(A')$ there exists $\nu_j > 0$ such that $y_j \in \mathcal{E}^{\nu_j}(A')$. Then for each $y_j \in \mathcal{E}^{\nu_j}(A')$ there exists a constant $c_j$ such that $\|A'^k y_j\| \le \nu_j^k c_j$ for all $k \in \mathbb{Z}_+$. From the identity

$$ {}_nA'^k(y_1 \otimes \ldots \otimes y_n) = \sum_{k_1+\ldots+k_n=k} \frac{k!}{k_1!\ldots k_n!} A'^{k_1}y_1 \otimes \ldots \otimes A'^{k_n}y_n $$

we obtain

$$ \begin{aligned} \|{}_nA'^k(y_1 \otimes \ldots \otimes y_n)\|_\pi &\le \sum_{k_1+\ldots+k_n=k} \frac{k!}{k_1!\ldots k_n!} \|A'^{k_1}y_1\| \ldots \|A'^{k_n}y_n\| \le \\ &\le \prod c_j \sum_{k_1+\ldots+k_n=k} \frac{k!}{k_1!\ldots k_n!} \nu_1^{k_1} \ldots \nu_n^{k_n} = c\nu^k, \end{aligned} $$

where $c := \prod_{j=1}^{n} c_j$ and $\nu := \sum_{j=1}^{n} \nu_j$. Therefore, $y_1 \otimes \ldots \otimes y_n \in \mathcal{E}^{\nu}({}_nA')$, i.e.,

$$ \mathcal{E}^{\nu_1}(A') \otimes \ldots \otimes \mathcal{E}^{\nu_n}(A') \subset \mathcal{E}^{\nu}({}_nA'), $$

hence

$$ \mathcal{E}(A')^{\otimes n} = \bigcup_{\nu_1,\ldots,\nu_n} \mathcal{E}^{\nu_1}(A') \otimes \ldots \otimes \mathcal{E}^{\nu_n}(A') \subset \bigcup_{\nu} \mathcal{E}^{\nu}({}_nA') = \mathcal{E}({}_nA'). $$

Since $U'_t$ is a contraction $(C_0)$-group [9],

$$ \overline{\mathcal{E}(A')} = X'. $$

As it is known [6], there exists a topological isomorphism

$$ (X_\pi'^{\otimes n})' \simeq L(X' \times \ldots \times X'; \mathbb{C}), $$

where $L(X' \times \ldots X'; \mathbb{C})$ is the space of continuous $n$-linear functionals on the Cartesian product.

Let the functional $f \in \left(X_\pi'^{\otimes n}\right)'$ be such that

$$ f(y_1, \ldots, y_n) = 0 $$

for all $y_j \in \mathcal{E}(A')$, $(j = 1, \ldots, n)$. By the previous isomorphism, for any $y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_n \in \mathcal{E}(A')$, the linear functional

$$ F_{y_j} : X' \ni y_j \longmapsto f(y_1, \ldots, y_j, \ldots, y_n) $$

belongs to the bidual space $X''$ and satisfies the equality

$$F_{y_j}(y_j) = 0$$

for all $y_j \in \mathcal{E}(A')$. By a density of $\mathcal{E}(A')$ in $X'$,

$$F_{y_j} = 0$$

for all $y_j \in X'$ and $j = 1, \dots, n$. Hence, $f = 0$ and the Hahn-Banach theorem imply that $\mathcal{E}(A')^{\otimes n}$ is dense in $X_\pi'^{\otimes n}$. Thus, for all $n \in \mathbb{Z}_+$ the subspace $\mathcal{E}(A')^{\odot n}$ is dense in $X_\pi'^{\odot n}$, as a codomain of a dense subspace $\mathcal{E}(A')^{\otimes n}$ relative to the projection $S_n'$. In consequence,

$$\operatorname{span}\{\mathcal{E}(A')^{\odot n} : n \in \mathbb{Z}_+\}$$

is dense in $W_\pi(B)$. It is clear that if $f_n' \in \mathcal{E}(A')^{\odot n}$ and $g_k' \in \mathcal{E}(A')^{\odot k}$ then $S_{n+k}'(f_n' \otimes g_k') \in \mathcal{E}(A')^{\odot(n+k)}$, hence the linear $\operatorname{span}\{\mathcal{E}(A')^{\odot n} : n \in \mathbb{Z}_+\}$ is the subalgebra of $W_\pi(B)$.

By analyticity of functions $f \in \mathcal{E}(\widehat{A})$ in the ball $B$, the derivation $d_x f(y)$ for all $x, y \in B$ is well defined. It is easy to verify that the generator $\widehat{A}$ satisfies the equality

$$(\widehat{A}f)(x) = d_x f(Ax),$$

where $d_x f(Ax)$ is the derivative of $f$ at a point $x \in \mathcal{E}(A) \cap B$ for all $Ax \in \mathcal{E}(A)$. In fact,

$$\begin{aligned} \widehat{A}f(U_t x) &= \widehat{A}\widehat{U}_t f(x) = \widehat{U}_t' f(x) = \frac{d}{dt}(\widehat{U}_t f(x)) = \\ &= \frac{d}{dt}(f(U_t x)) = d_{U_t x} f\Big(\frac{d}{dt} U_t x\Big) = d_{U_t x} f(AU_t x). \end{aligned}$$

Therefore, $\widehat{A}$ is a derivation on the subalgebra $\mathcal{E}(\widehat{A})$. □

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*Received: June 30, 2006.*
*Revised: June 29, 2007.*
*Accepted: July 2, 2007.*