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A FINITE DIFFERENCE METHOD FOR NONLINEAR PARABOLIC-ELLIPTIC SYSTEMS OF SECOND-ORDER PARTIAL DIFFERENTIAL EQUATIONS

Abstract. This paper deals with a finite difference method for a wide class of weakly coupled nonlinear second-order partial differential systems with initial condition and weakly coupled nonlinear implicit boundary conditions. One part of each system is of the parabolic type (degenerated parabolic equations) and the other of the elliptic type (equations with a parameter) in a cube in $\mathbf{R}^{1+n}$. A suitable finite difference scheme is constructed. It is proved that the scheme has a unique solution, and the numerical method is consistent, convergent and stable. The error estimate is given. Moreover, by the method, the differential problem has at most one classical solution. The proof is based on the Banach fixed-point theorem, the maximum principle for difference functional systems of the parabolic type and some new difference inequalities. It is a new technique of studying the mixed-type systems. Examples of physical applications and numerical experiments are presented.

Keywords: partial differential equation, parabolic-elliptic system, finite difference method, finite difference scheme, consistence, convergence, stability, error estimate, uniqueness.

Mathematics Subject Classification: Primary 65M06, 65M12, 65M15; Secondary 35M10, 39A10, 39B72.

1. INTRODUCTION

The aim of the paper is to give a finite difference method of approximate solving of systems of weakly coupled strongly nonlinear second-order partial differential equations with initial condition and different boundary conditions, in particular weakly coupled nonlinear implicit conditions. One part of each system is of the parabolic type (degenerated parabolic equations) and the other of the elliptic type (equations with a parameter) in $\Omega := [0,T] \times (0,\delta)^n \subset \mathbf{R}^{1+n}$. The nonlinearity in the systems is admitted with respect to second derivatives. It is a novelty for such mixed-type systems. The existing study of the problem is devoted to parabolic-elliptic systems with different, linear or nonlinear, boundary conditions, but our conditions include most of them.

The general weakly coupled nonlinear systems of the parabolic type or the elliptic type have been treated by numerous authors, and various methods have been proposed for the study of different aspects of the problem, such as the uniqueness of the classical solution, differential inequalities, difference inequalities and a finite difference method for the computation of an approximate solution (cf. [7, 10–12, 15, 16, 25, 26]). The paper is motivated by the question whether these classical results can be transferred from single-type systems to systems of mixed types. Numerical methods for general weakly coupled functional differential systems of the first-order have, for instance, been considered by D. Jaruszewska-Walczak and Z. Kamont [6].

There are a lot of well-known mathematical models describing physical phenomena by means of parabolic-elliptic systems with different initial-boundary conditions. Weakly coupled system (3.1) realizes the process of incompressible fluid flow in a porous medium [1–3]. P. Segall [20] used it for computing poroelastic stress changes due to fluid extraction. System (3.1) supports a description of the process of heat exchange with flow of a substance when temperature changes are small – modifications of the very important Navier-Stokes system. Parabolic-elliptic systems similar to (3.1) are also used in medicine, in the theory of chemotaxis (the Keller-Segal model) [21]. The mentioned systems occur in certain problems of astrophysics (the evolutional version of Chandrasekhar's model), hydromechanics (statistics of whirls in Euler's equations) and statistical mechanics (the Vlasov-Poisson-Boltzmann equation) [4, 8, 14]. R.C. MacCamy and M. Suri [9] use them to describe rotary currents in electrodynamics. The parabolic-elliptic systems arise in a groundwater flow problem [5], a model of evolution of water waves (the Davey-Stewartson systems) [28] and in the theory of magnetism (the Myrzakulov equations) [13]. Another example is the Poisson-Schrödinger nonstationary system in the theory of semiconductors.

Such systems also have numerous various applications. Unfortunately though, they have been less examined than systems of the parabolic, elliptic or hyperbolic types. It is mainly caused by their specific mixed structure.

In the present paper, we construct a finite difference scheme for nonlinear differential system (3.1) with nonlinear implicit initial-boundary value conditions (3.2). It is proved that, under suitable assumptions on functions and steps of a mesh, difference scheme (4.1) has a unique solution – the algorithm of its numerical solving is included, moreover, the method is consistent, convergent and stable (stability follows immediately from the convergence). The error estimate of the approximate solution is given. Proof is based on the Banach fixed-point theorem, the maximum principle for difference functional systems of the parabolic type formulated by M. Malec [10] and some new difference inequalities. At the end of the paper, we present a numerical example.

The assumptions generally concern the Lipschitz continuous of reaction and boundary functions, the quasi-monotone property of the reaction functions and the domination of the main diagonal in some symmetric matrices. They are typical of such investigations of single-type systems (cf. [7, 10–12]).

It follows, from the convergence of the numerical method, that the problem considered has at most one classical solution. Theorems on the existence and uniqueness of such solutions for some special parabolic-elliptic systems are given for example in [4, 8, 14] and [19].

We add that a finite difference method for parabolic-elliptic systems has been studied among others by M.S. Mock [14], Z.Z. Sun [22, 23] and L. Sapa [17], but in a case of quasi-linear systems of a special form only, without mixed derivatives. L. Sapa [18] has considered a finite difference method for a general class of such systems but with the Dirichlet condition only. Unfortunately, these more classical difference methods and techniques of proof cannot be adapted to strongly nonlinear systems with different, in particular implicit, boundary conditions. A quite simple example given by (6.1), (6.2) illustrates this fact well.

2. NOTATION AND DEFINITIONS

2.1. SETS AND SPACES OF FUNCTIONS

Denote by $\mathbf{R}^n$ the *Euclidean space* and define the following sets

$$E := (0, \delta)^n \subset \mathbf{R}^n, \qquad \Omega := [0, T] \times E, \tag{2.1}$$

where $0 < \delta < +\infty$ and $0 < T < +\infty$.

Define also the sets

$$\overline{E} := [0, \delta]^n, \qquad \overline{\Omega} := [0, T] \times \overline{E}. \tag{2.2}$$

Let

$$\Gamma := [0, T] \times \partial E \tag{2.3}$$

be the *side surface* of *the cube* Ω, where ∂E is the *boundary of* E. In Γ we distinguish the subset

$$\widetilde{\Gamma} := (0, T] \times \partial E \tag{2.4}$$

and the families of subsets of the form

$$\Gamma_{mi} := \Big\{(t, x) \in \Gamma : \quad x_i = (m-1)\delta\Big\},$$
$$\widetilde{\Gamma}_{mi} := \Big\{(t, x) \in \widetilde{\Gamma} : \quad x_i = (m-1)\delta\Big\}$$

for $m = 1, 2$ and $i = 1, \dots, n$, where $t \in \mathbf{R}$, $x = (x_1, \dots, x_n) \in \mathbf{R}^n$.

Next, denote by

$$I_1 := \{1, \dots, q\}, \qquad I_2 := \{q + 1, \dots, p\}, \qquad I := I_1 \cup I_2 \tag{2.5}$$

the sets of indices, where p and q, $q < p$, are given natural numbers.

Moreover, define the sets

$$\begin{gathered} \Delta := \Omega \times \mathbf{R}^p \times \mathbf{R}^n \times \mathbf{R}^{n^2}, \\ \Theta := \Gamma \times \mathbf{R}^p \times \mathbf{R}, \qquad \widetilde{\Theta} := \widetilde{\Gamma} \times \mathbf{R}^q \times \mathbf{R}, \\ \Theta_{mi} := \Gamma_{mi} \times \mathbf{R}^p \times \mathbf{R}, \qquad \widetilde{\Theta}_{mi} := \widetilde{\Gamma}_{mi} \times \mathbf{R}^q \times \mathbf{R} \end{gathered} \tag{2.6}$$

for $m = 1, 2$, $i = 1, \ldots, n$, and the sets

$$\Delta_1 := \Omega \times \mathbf{R}^{2p} \times \mathbf{R}^{2n} \times \mathbf{R}^{2n^2}, \qquad \Theta_{1mi} := \Gamma_{mi} \times \mathbf{R}^{2p} \times \mathbf{R}^2, \qquad \widetilde{\Theta}_{1mi} := \widetilde{\Gamma}_{mi} \times \mathbf{R}^{2q} \times \mathbf{R}^2 \tag{2.7}$$

for $m = 1, 2$, $i = 1, \ldots, n$.

A continuous mapping $u = (u_l)_{l \in I} : A \to \mathbf{R}^p$ whose derivatives $\frac{\partial u_l}{\partial t}$, $\frac{\partial u_l}{\partial x_i}$, $\frac{\partial^2 u_l}{\partial x_j x_i}$, $l \in I$, $i, j = 1, \ldots, n$, are continuous on $A \subset \overline{\Omega}$ will be called *regular on* A. We briefly write $u \in C_{reg}(A, \mathbf{R}^p)$. We define the space $C_{reg}(A, \mathbf{R}^q)$ in the same way.

The set

$$B\left(\overline{\Omega}\right) := \left\{ z = (z_l)_{l \in I_1} \,\middle|\, z_l : \overline{\Omega} \to \mathbf{R}, \ \sup_{(t,x) \in \overline{\Omega}} \left| z_l\left(t, x\right) \right| < +\infty, \ l \in I_1 \right\} \tag{2.8}$$

is the *set of functions bounded on* $\overline{\Omega}$.

For a fixed $t \in [0, T]$,

$$\|z\|\left(t\right) := \max_{l \in I_1} \left\{ \sup_{x \in \overline{E}} \left| z_l\left(t, x\right) \right| \right\} \tag{2.9}$$

stands for a semi-norm in the space $B(\overline{\Omega})$, where $z = (z_l)_{l \in I_1} \in B(\overline{\Omega})$.

2.2. DIFFERENTIAL OPERATORS

Let $D_t := \frac{\partial}{\partial t}$ and let $D_i := \frac{\partial}{\partial x_i}$, $D_{ij} := \frac{\partial^2}{\partial x_j \partial x_i}$ for $i, j = 1, \ldots, n$. Put $D_x := (D_1, \ldots, D_n)$ and $D_x^2 := (D_{11}, \ldots, D_{1n}, \ldots, D_{n1}, \ldots, D_{nn})$. The *operator of the first derivative in the internal normal to the boundary* Γ (see (2.3)) is denoted by D_ν.

Let $\varphi_{lmi} : \widetilde{\Theta}_{mi} \to \mathbf{R}$ for $l \in I_1$ and $\psi_{lmi} : \Theta_{mi} \to \mathbf{R}$ for $l \in I_2$ (see (2.6)) be arbitrarily given functions, where $m = 1, 2$, $i = 1, \ldots, n$, and let $\varphi_l := (\varphi_{lmi})$ for $l \in I_1$, $\psi_l := (\psi_{lmi})$ for $l \in I_2$.

Suppose that functions $f = (f_l)_{l \in I} : \Delta \to \mathbf{R}^p$, $\varphi = (\varphi_l)_{l \in I_1} : \widetilde{\Theta} \to \mathbf{R}^q$ and $\psi = (\psi_l)_{l \in I_2} : \Theta \to \mathbf{R}^{p-q}$ are given. For such the functions, we define the differential operators

$$\begin{aligned} &F : C_{reg}(\Omega, \mathbf{R}^p) \to \mathbf{R}^q, && F = (F_l)_{l \in I_1}, \\ &G : C_{reg}(\Omega, \mathbf{R}^p) \to \mathbf{R}^{p-q}, && G = (G_l)_{l \in I_2}, \\ &\Phi : C_{reg}(\widetilde{\Gamma}, \mathbf{R}^q) \to \mathbf{R}^q, && \Phi = (\Phi_l)_{l \in I_1}, \\ &\Psi : C_{reg}(\Gamma, \mathbf{R}^p) \to \mathbf{R}^{p-q}, && \Psi = (\Psi_l)_{l \in I_2}, \end{aligned} \tag{2.10}$$

with the following components

$$\begin{aligned} &F_l[u](t,x) := D_t u_l(t,x) - f_l(t, x, u(t,x), D_x u_l(t,x), D_x^2 u_l(t,x)) && \text{for } \ l \in I_1, \\ &G_l[u](t,x) := f_l(t, x, u(t,x), D_x u_l(t,x), D_x^2 u_l(t,x)) && \text{for } \ l \in I_2, \\ &\Phi_l[u](t,x) := \varphi_l(t, x, u(t,x), D_\nu u_l(t,x)) && \text{for } \ l \in I_1, \\ &\Psi_l[u](t,x) := \psi_l(t, x, u(t,x), D_\nu u_l(t,x)) && \text{for } \ l \in I_2. \end{aligned} \tag{2.11}$$

2.3. DISCRETIZATION

Define a mesh on the set $\overline{\Omega}$ (see (2.2)) in the following way. Let N_1 and N, $N \geq 2$, be some natural numbers and put

$$k := \frac{T}{N_1}, \qquad h := \frac{\delta}{N}. \tag{2.12}$$

We will call the set of discrete points

$$S_{kh} := \left\{(t^\mu, x_1^{m_1}, \ldots, x_n^{m_n}) \in \overline{\Omega} : t^\mu = \mu k,\ x_i^{m_i} = m_i h,\ i = 1, \ldots, n\right\}, \tag{2.13}$$

where $\mu = 0, 1, \ldots, N_1$ and $m_i = 0, 1, \ldots, N$ for $i = 1, \ldots, n$, the *uniform rectangular mesh on* $\overline{\Omega}$ with the *time step* k and *spatial step* h. Elements of S_{kh} are called *knot points* or briefly *knots*. For simplicity of notation, we write x^M instead of $(t^\mu, x_1^{m_1}, \ldots, x_n^{m_n}) \in S_{kh}$, where $M = (\mu, m) \in \mathbf{Z}^{1+n}$ and $m = (m_1, \ldots, m_n) \in \mathbf{Z}^n$; $\mathbf{Z}$ is the set of integer numbers.

There exists a one-to-one correspondence between the mesh S_{kh} and the set of multi-indices

$$Z := \left\{M \in \mathbf{Z}^{1+n} : \quad 0 \leq \mu \leq N_1,\ 0 \leq m_i \leq N,\ i = 1, \ldots, n\right\} \tag{2.14}$$

if steps k, h are fixed. Accordingly in the further part of the paper a knot $x^M \in S_{kh}$ is identified with a suitable multi-index $M \in Z$.

We assume that the set Z is well ordered (the order is arbitrary) for any steps k and h.

In Z we distinguish the following subsets

$$\begin{aligned}
Z_\mu &:= \{M = (\mu, m) \in Z : \quad 0 \leq m_i \leq N,\ i = 1, \ldots, n\}, \\
Z_{\mu 1} &:= \{M \in Z_\mu : \quad 0 \leq m_i \leq N - 1,\ i = 1, \ldots, n\}, \\
Z_{\mu 2} &:= \{M \in Z_\mu : \quad 1 \leq m_i \leq N,\ i = 1, \ldots, n\}, \\
Z_\mu^0 &:= Z_{\mu 1} \cap Z_{\mu 2}, \quad \partial Z_\mu := Z_\mu \setminus Z_\mu^0
\end{aligned} \tag{2.15}$$

for $\mu = 0, 1, \ldots, N_1$. Note that Z_μ is the set of multi-indices of all knots of the mesh S_{kh}, Z_μ^0 is the set of multi-indices of knots of the mesh belonging to Ω and ∂Z_μ is the set of multi-indices of knots of the mesh belonging to Γ, for any $\mu \in \{0, 1, \ldots, N_1\}$.

We define recurrently the sets $Z_{\mu 1i}$ and $Z_{\mu 2i}$ for $\mu = 0, 1, \ldots, N_1$, $i = 1, \ldots, n$, as follows

$$\begin{aligned}
&Z_{\mu 11} := \{M \in \partial Z_\mu : \quad m_1 = 0\}, \quad Z_{\mu 21} := \{M \in \partial Z_\mu : \quad m_1 = N\}, \\
&Z_{\mu 1i} := \{M \in \partial Z_\mu : \quad m_i = 0\} \setminus \left(\bigcup_{k=1}^{i-1} (Z_{\mu 1k} \cup Z_{\mu 2k})\right), \\
&Z_{\mu 2i} := \{M \in \partial Z_\mu : \quad m_i = N\} \setminus \left(\bigcup_{k=1}^{i-1} (Z_{\mu 1k} \cup Z_{\mu 2k})\right),
\end{aligned} \tag{2.16}$$

where $\mu = 0, 1, \ldots, N_1$, $i = 2, \ldots, n$. It is evident that the above sets form a decomposition of the boundary ∂Z_μ, $\mu = 0, 1, \ldots, N_1$, into separable sets.

We also define the contiguity $S(l,M)$ as a set of multi-indices $T \in Z$, $T \neq M$, such that knots $x^T \in S_{kh}$ are used to approximate derivatives $D_x u_l(x^M)$, $D_x^2 u_l(x^M)$ and $D_\nu u_l(x^M)$ in $l-$th equations of systems (3.1) and (3.2), respectively.

2.4. SPACES OF MESH FUNCTIONS, DIFFERENCE AND STEP OPERATORS

A *mesh function* it is any function $a : B \ni M \to a^M \in \mathbf{R}$, where B is any subset of Z. We denote the space of all such functions by $F(B, \mathbf{R})$ and call it the *space of mesh functions*. The spaces of a system of such functions are denoted similarly by: $F(B, \mathbf{R}^p)$, $F(B, \mathbf{R}^q)$, $F(B, \mathbf{R}^{p-q})$.

In the space of mesh functions $F(B, \mathbf{R}^p)$, $B \subset Z$, we introduce the *maximum norm*

$$\|a\| := \max_{l \in I} \left\{ \max_{M \in B} \left| a_l^M \right| \right\}, \tag{2.17}$$

where $a = (a_l)_{l \in I} \in F(B, \mathbf{R}^p)$, $a_l : B \ni M \to a_l^M \in \mathbf{R}$ for $l \in I$. We define norms in $F(B, \mathbf{R}^q)$ and $F(B, \mathbf{R}^{p-q})$ in the same manner.

We will call the functions

$$\bigcup_{\mu=0}^{N_1} Z_{\mu 1} \ni M \to i(M) \in Z, \qquad \bigcup_{\mu=0}^{N_1} Z_{\mu 2} \ni M \to -i(M) \in Z,$$

$$\bigcup_{\mu=0}^{N_1-1} Z_{\mu} \ni M \to +M \in Z,$$

where

$$i(M) := (\mu, m_1, \ldots, m_{i-1}, m_i + 1, m_{i+1}, \ldots, m_n),$$
$$-i(M) := (\mu, m_1, \ldots, m_{i-1}, m_i - 1, m_{i+1}, \ldots, m_n),$$

$$+M := (\mu + 1, m)$$

for $i = 1, \ldots, n$, the *shift functions* (cf. [10–12]).

Denote by a^{M-}, a^{Mi}, a^{Mi-}, a^{-Mi}, a^{-Mij} and a^{+Mij} the *difference quotients* defined by

$$\begin{aligned}
a^{M-} &:= \frac{1}{k}(a^{+M} - a^M) && \text{for } M \in \bigcup_{\mu=0}^{N_1-1} Z_\mu, \\
a^{Mi} &:= \frac{1}{2h}(a^{i(M)} - a^{-i(M)}) && \text{for } M \in \bigcup_{\mu=0}^{N_1} Z_\mu^0, \\
a^{Mi-} &:= \frac{1}{h}(a^{i(M)} - a^M) && \text{for } M \in \bigcup_{\mu=0}^{N_1} Z_{\mu 1i}, \\
a^{-Mi} &:= \frac{1}{h}(a^M - a^{-i(M)}) && \text{for } M \in \bigcup_{\mu=0}^{N_1} Z_{\mu 2i},
\end{aligned} \tag{2.18}$$

$$a^{-Mij} := \frac{1}{2h^2}(a^{i(M)} + a^{j(M)} + a^{-i(M)} + a^{-j(M)} - 2a^M - a^{i(-j(M))} - a^{-i(j(M))}),$$
$$a^{+Mij} := \frac{1}{2h^2}(-a^{i(M)} - a^{j(M)} - a^{-i(M)} - a^{-j(M)} + 2a^M + a^{i(j(M))} + a^{-i(-j(M))})$$

for $M \in \bigcup_{\mu=0}^{N_1} Z_\mu^0$, $i,j = 1,\ldots,n$, on the space $F(Z,\mathbf{R})$. These operators will be used to approximate derivatives in equations (3.1) and boundary conditions (3.2).

For any system of mesh functions $a = (a_l)_{l\in I} \in (Z,\mathbf{R}^p)$, we introduce a notation

$$a^M := (a_l^M)_{l\in I} \in \mathbf{R}^p, \qquad \widetilde{a}^M := (a_l^M)_{l\in I_1} \in \mathbf{R}^q, \tag{2.19}$$

where $M \in Z$.

Let

$$\begin{aligned} a_l^{MI} &:= (a_l^{M1},\ldots,a_l^{Mn}),\\ a_l^{MII} &:= (a_l^{M11},\ldots,a_l^{M1n},\ldots,a_l^{Mn1},\ldots,a_l^{Mnn}) \end{aligned} \tag{2.20}$$

for $l \in I$ and $M \in \bigcup_{\mu=0}^{N_1} Z_\mu^0$ be vectors whose coefficients are the difference quotients given by (2.18), where a_l^{Mij} has to be chosen equal either to a_l^{-Mij} or to a_l^{+Mij}, depending on what is specified further, in assumption F_8 of Section 3.

Define the discrete operators

$$\begin{aligned} &S^0 : F(Z,\mathbf{R}^p) \to F(Z_0,\mathbf{R}^q),\\ &S^1 : F(Z,\mathbf{R}^p) \to F(Z\backslash \left(Z_{N_1}^0 \cup \partial Z_0\right), \mathbf{R}^q), \quad S^1 := (S_l)_{l\in I_1},\\ &S^2 : F(Z,\mathbf{R}^p) \to F(Z,\mathbf{R}^{p-q}), \qquad\qquad\quad S^2 := (S_l)_{l\in I_2} \end{aligned} \tag{2.21}$$

by putting

$$S^{0aM} := \widetilde{a}^M - u_0(x^M) \quad \text{for} \quad M \in Z_0, \tag{2.22}$$

$$S_l^{aM} := \begin{cases} a_l^{M-} - f_l(x^M, a^M, a_l^{MI}, a_l^{MII}) & \text{for} \quad M \in \bigcup_{\mu=0}^{N_1-1} Z_\mu^0,\\ \varphi_{l1i}(x^M, \widetilde{a}^{i(M)}, a_l^{Mi-}) & \text{for} \quad M \in \bigcup_{\mu=1}^{N_1} Z_{\mu 1i},\\ \varphi_{l2i}(x^M, \widetilde{a}^{-i(M)}, a_l^{-Mi}) & \text{for} \quad M \in \bigcup_{\mu=1}^{N_1} Z_{\mu 2i},\\ \quad (i=1,\ldots,n) \end{cases} \tag{2.23}$$

for $l \in I_1$ and

$$S_l^{aM} := \begin{cases} f_l(x^M, a^M, a_l^{MI}, a_l^{MII}) & \text{for} \quad M \in \bigcup_{\mu=0}^{N_1} Z_\mu^0,\\ \psi_{l1i}(x^M, a^M, a_l^{Mi-}) & \text{for} \quad M \in \bigcup_{\mu=0}^{N_1} Z_{\mu 1i},\\ \psi_{l2i}(x^M, a^M, a_l^{-Mi}) & \text{for} \quad M \in \bigcup_{\mu=0}^{N_1} Z_{\mu 2i},\\ \quad (i=1,\ldots,n) \end{cases} \tag{2.24}$$

for $l \in I_2$; $a = (a_l)_{l\in I} \in F(Z, \mathbf{R}^p)$. The function $u_0 = (u_{0l})_{l\in I_1} : \overline{E} \to \mathbf{R}^q$ appears in (3.2).

Finally, define the *step operator* $\mathbf{S} : F(Z, \mathbf{R}^q) \to B(\overline{\Omega})$, $\mathbf{S} = (\mathbf{S}_l)_{l\in I_1}$ by the formula

$$\mathbf{S}_l[a](t,x) := \sum_{M\in Z} \chi_M(t,x) a_l^M \quad \text{for} \quad (t,x) \in \overline{\Omega},\ l \in I_1, \tag{2.25}$$

where $a = (a_l)_{l\in I_1} \in F(Z, \mathbf{R}^q)$ and

$$\chi_M(t,x) := \begin{cases} 1 & \text{for} \quad (t,x) \in J_M, \\ 0 & \text{for} \quad (t,x) \in \overline{\Omega} \setminus J_M, \end{cases} \tag{2.26}$$

$$J_M := \big\{(t,x) \in \overline{\Omega} :\ \mu k \le t < (\mu+1)k,\ m_i h \le x_i < (m_i+1)h\ ,\ i = 1,\ldots,n\big\}. \tag{2.27}$$

We briefly write $\mathbf{S}[a] = \mathbf{a}$.

Remark 2.1. *The step operator given by* (2.25)–(2.27) *has been used extensively in* [10, 11] *and* [12] *to study systems of difference functional inequalities and to approximate the functional term in systems of differential functional equations. In this paper it is used in the construction of some difference functional inequalities in a proof of the convergence of the finite difference method for systems of differential equations, without a functional term, which are a key-step in our proof. It is a new application of the above step operator.*

3. DIFFERENTIAL PROBLEM

Let functions $f = (f_l)_{l\in I} : \Delta \to \mathbf{R}^p$, $\varphi = (\varphi_l)_{l\in I_1} : \widetilde{\Theta} \to \mathbf{R}^q$, $\psi = (\psi_l)_{l\in I_2} : \Theta \to \mathbf{R}^{p-q}$ be the functions given in Section 2.2 and let $u_0 = (u_{0l})_{l\in I_1} : \overline{E} \to \mathbf{R}^q$ (the *initial function*) be given. We consider a system of weakly coupled nonlinear differential equations of the form

$$\begin{cases} F[u](t,x) = 0 & \text{for} \quad (t,x) \in \Omega, \\ G[u](t,x) = 0 & \text{for} \quad (t,x) \in \Omega \end{cases} \tag{3.1}$$

with the *initial condition* and *nonlinear implicit boundary conditions*

$$\begin{cases} \widetilde{u}(0,x) = u_0(x) & \text{for} \quad x \in \overline{E}, \\ \Phi[u](t,x) = 0 & \text{for} \quad (t,x) \in \widetilde{\Gamma}, \\ \Psi[u](t,x) = 0 & \text{for} \quad (t,x) \in \Gamma, \end{cases} \tag{3.2}$$

where $\widetilde{u} := (u_l)_{l\in I_1}$.

We need the following assumptions on the functions f, φ, ψ and regularity of a solution u of (3.1), (3.2).

Assumption *F*:

F_1. There exist bounded functions $\alpha_{ls},\ \beta_{li},\ \gamma_{lij} : \Delta_1 \to \mathbf{R}$, $l, s \in I$, $i, j = 1, \ldots, n$, such that for any two points $(t, x, y, z, w), (t, x, \overline{y}, \overline{z}, \overline{w}) \in \Delta$,

$$f_l(t, x, y, z, w) - f_l(t, x, \overline{y}, \overline{z}, \overline{w}) = \sum_{s\in I} \alpha_{ls}(P)(y_s - \overline{y}_s) + \sum_{i=1}^{n} \beta_{li}(P)(z_i - \overline{z}_i) + \\ + \sum_{i,j=1}^{n} \gamma_{lij}(P)(w_{ij} - \overline{w}_{ij}) \tag{3.3}$$

for $l \in I$, where $P = (t, x, y, \overline{y}, z, \overline{z}, w, \overline{w}) \in \Delta_1$ (see (2.6), (2.7)).

F_2. The matrices $(\gamma_{lij}(P))_{i,j=1,\ldots,n}$ are symmetric for all indices $l \in I$ and points $P \in \Delta_1$; and for (l, i, j) fixed, $\gamma_{lij}(P) \geq 0$ for all $P \in \Delta_1$ or $\gamma_{lij}(P) \leq 0$ for all $P \in \Delta_1$.

F_3. There exist constants $L_1,\ L_2,\ N_l,\ G_{li},\ g_{li} > 0$, L, $H_{li} \geq 0$ and $K_l < 0$, $l \in I_2$, $i = 1, \ldots, n$, such that the functions $\alpha_{ls},\ \beta_{li},\ \gamma_{lij}$, $l, s \in I$, $i, j = 1, \ldots, n$, fulfil in Δ_1 the following conditions

$$\alpha_{ls} \geq 0 \qquad \text{for } l, s \in I_1,\ l \neq s, \tag{3.4}$$

$$\sum_{s\in I_1} \alpha_{ls} \leq L \qquad \text{for } l \in I_1, \tag{3.5}$$

$$\alpha_{ll} + \sum_{\substack{s\in I_2\\ s\neq l}} |\alpha_{ls}| \leq -N_l \qquad \text{for } l \in I_2, \tag{3.6}$$

$$K_l \leq \alpha_{ll} \qquad \text{for } l \in I_2, \tag{3.7}$$

$$|\alpha_{ls}| \leq L_1 \qquad \text{for } l \in I_1 \text{ and } s \in I_2, \tag{3.8}$$

$$|\alpha_{ls}| \leq L_2 \qquad \text{for } l \in I_2 \text{ and } s \in I_1, \tag{3.9}$$

$$|\beta_{li}| \leq H_{li} \qquad \text{for } l \in I_2, \tag{3.10}$$

$$\gamma_{lii} - \sum_{\substack{j=1\\ j\neq i}}^{n} |\gamma_{lij}| > 0 \qquad \text{for } l \in I_1, \tag{3.11}$$

$$g_{li} \leq \gamma_{lii} - \sum_{\substack{j=1\\ j\neq i}}^{n} |\gamma_{lij}| \qquad \text{for } l \in I_2, \tag{3.12}$$

$$\gamma_{lii} \leq G_{li} \qquad \text{for } l \in I_2,\ i = 1, \ldots, n. \tag{3.13}$$

F_4. There exist bounded functions $\delta_{lmis},\ \rho_{lmi} : \widetilde{\Theta}_{1mi} \to \mathbf{R}$, $l, s \in I_1$, $m = 1, 2$, $i = 1, \ldots, n$, such that for any two points $(t, x, y, z),\ (t, x, \overline{y}, \overline{z}) \in \widetilde{\Theta}_{mi}$,

$$\varphi_{lmi}(t, x, y, z) - \varphi_{lmi}(t, x, \overline{y}, \overline{z}) = \sum_{s\in I} \delta_{lmis}(P)(y_s - \overline{y}_s) + \rho_{lmi}(P)(z - \overline{z}) \tag{3.14}$$

for $l \in I_1$, $m = 1, 2$ and $i = 1, \ldots, n$, where $P = (t, x, y, \overline{y}, z, \overline{z}) \in \widetilde{\Theta}_{1mi}$ (see (2.6), (2.7)).

F_5. There exist bounded functions $\delta_{lmis},\ \rho_{lmi} : \Theta_{1mi} \to \mathbf{R}$, $l \in I_2$, $s \in I$, $m = 1, 2$, $i = 1, \ldots, n$, such that for any two points $(t, x, y, z),\ (t, x, \overline{y}, \overline{z}) \in \Theta_{mi}$,

$$\psi_{lmi}(t, x, y, z) - \psi_{lmi}(t, x, \overline{y}, \overline{z}) = \sum_{s\in I} \delta_{lmis}(P)(y_s - \overline{y}_s) + \rho_{lmi}(P)(z - \overline{z}) \tag{3.15}$$

for $l \in I_2$, $m = 1, 2$ and $i = 1, \ldots, n$, where $P = (t, x, y, \overline{y}, z, \overline{z}) \in \Theta_{1mi}$.

F_6. There exist constants G, $I_{lmi} > 0$, $R_{lmi} \geq 0$ and $S_{lmi} < 0$, $l \in I_2$, $m = 1, 2$, $i = 1, \dots, n$, such that the functions δ_{lmis}, ρ_{lmi}, $l, s \in I$, $m = 1, 2$, $i = 1, \dots, n$, fulfil in $\widetilde{\Theta}_{1mi}$ and Θ_{1mi}, respectively, the following conditions

$$(-1)^m \delta_{lmis} \leq 0 \qquad \text{for } l, s \in I_1,\ l \neq s, \tag{3.16}$$

$$\delta_{lmil} + \sum_{\substack{s \in I_2 \\ s \neq l}} |\delta_{lmis}| \leq -I_{lmi} \qquad \text{for } l \in I_2, \tag{3.17}$$

$$S_{lmi} \leq \delta_{lmil} \qquad \text{for } l \in I_2, \tag{3.18}$$

$$|\delta_{lmis}| \leq L_2 \qquad \text{for } l \in I_2 \text{ and } s \in I_1, \tag{3.19}$$

$$G \leq (-1)^m \rho_{lmi}^{-1} \sum_{s \in I_1} \delta_{lmis} \qquad \text{for } l \in I_1, \tag{3.20}$$

$$\rho_{lmi} \geq 1 \qquad \text{for } l \in I_1, \tag{3.21}$$

$$0 \leq (-1)^{m-1} \rho_{lmi} \leq R_{lmi} \qquad \text{for } l \in I_2, \tag{3.22}$$

where L_2 is given in F_3, $m = 1, 2$, $i = 1, \dots, n$.

F_7. A function $u \in C_{reg}(\overline{\Omega}, \mathbf{R}^p)$ is a *regular solution* of differential problem (3.1), (3.2).

F_8. The difference quotients a_l^{Mij} have the form

$$a_l^{Mij} = \begin{cases} a_l^{-Mij}, & \text{if} \quad i = j \quad \text{or} \quad \gamma_{lij} \leq 0, \\ a_l^{+Mij}, & \text{if} \quad i \neq j \quad \text{and} \quad \gamma_{lij} \geq 0 \end{cases} \tag{3.23}$$

for $l \in I$, $i, j = 1, \dots, n$ and $M \in \bigcup_{\mu=0}^{N_1} Z_\mu^0$ (see (2.20), (2.18)).

Remark 3.1. *Assumptions F_1, F_4 and F_5 are equivalent to the Lipschitz condition, but they are more useful in the other assumptions. Moreover, if the reaction functions f_l and the boundary functions φ_l, ψ_l are differentiable, then the bounded functions in these assumptions may be equal, by the mean value theorem, to their suitable derivatives.*

Remark 3.2. *If assumptions F_2 and F_3* (see (3.11), (3.12)) *on the strong domination of the main diagonal in the symmetric matrices $(\gamma_{lij}(P))_{i,j=1,\dots,n}$ for $l \in I$ and $P \in \Delta_1$ are satisfied, then differential system* (3.1) *is of the parabolic-elliptic type* (*the degenerated parabolic-elliptic system with a parameter t*) *in the class of functions $u \in C_{reg}(\overline{\Omega}, \mathbf{R}^p)$. This follows from the fact that the matrices $(\gamma_{lij}(P))_{i,j=1,\dots,n}$ are positive defined and from the definition of ellipticity of the functions f_l in* [24], *p.* 132 (see also [7] and [27], p. 182).

4. DIFFERENCE PROBLEM

We give a definition of the difference scheme which will be applied to approximate a solution of differential problem (3.1), (3.2).

Definition 4.1. *A difference scheme for differential problem* (3.1), (3.2) *is the system of algebraic equations*

$$\begin{cases} S^{0a} = 0, \\ S^{1a} = 0, \\ S^{2a} = 0, \end{cases} \tag{4.1}$$

where $a \in F(Z, \mathbf{R}^p)$ (see (2.21)–(2.24)).

In the further part of the paper, we use the following assumptions on steps k and h of the mesh S_{kh}.

Assumption *K*:

K_1. The time step k and spatial step h are such that

$$h^{-1}\Big(\gamma_{lii} - \sum_{\substack{j=1\\ j\neq i}}^{n} |\gamma_{lij}|\Big) - \frac{1}{2}|\beta_{li}| \geq 0, \tag{4.2}$$

$$1 + k\alpha_{ll} - 2kh^{-2} \sum_{i=1}^{n} \gamma_{lii} \geq 0 \tag{4.3}$$

for $l \in I_1$, $i = 1, \ldots, n$ and for all points belonging to Δ_1 (see (2.7), (3.3)).

K_2. The step h fulfils the inequalities

$$\rho_{lmi} + (-1)^{m-1} h\delta_{lmil} \geq 0 \tag{4.4}$$

for $l \in I_1$, $m = 1, 2$, $i = 1, \ldots, n$ and for all points in the sets $\widetilde{\Theta}_{1mi}$, $m = 1, 2$, $i = 1, \ldots, n$, respectively (see (2.7), (3.14)).

K_3. The inequalities

$$h^{-1} g_{li} - \frac{1}{2} H_{li} \geq 0, \tag{4.5}$$

$l \in I_2$, $i = 1, \ldots, n$, hold, where g_{li} and H_{li} are the constants defined in F_3 (see (3.10), (3.12)).

Remark 4.1. *If Assumption F holds, then there exists a sequence of steps k, h which fulfil Assumption K and* $(k, h) \to (0, 0)$.

5. THEORETICAL STUDY OF THE SCHEME

5.1. EXISTENCE AND UNIQUENESS OF THE SOLUTION OF THE DIFFERENCE SCHEME

Suppose that Assumption F holds and let $\mathbf{A}_\mu = (\mathbf{A}_{\mu l})_{l\in I} \in F(Z_\mu, \mathbf{R}^p)$ be arbitrary for $\mu = 0, 1, \ldots, N_1$. Define $N_1 + 1$ of the difference operators $F^\mu : F(Z_\mu, \mathbf{R}^p) \to F(Z_\mu, \mathbf{R}^{p-q})$, $F^\mu = (F_l^\mu)_{l\in I_2}$, $\mu = 0, 1, \ldots, N_1$, by setting

$$F_l^{\mu \mathbf{A}_\mu M} := \begin{cases} f_l(x^M, a^M, a_l^{MI}, a_l^{MII}) & \text{for} \quad M \in Z_\mu^0, \\ \psi_{l1i}\left(x^M, a^M, a_l^{Mi-}\right) & \text{for} \quad M \in Z_{\mu 1i}, \\ \psi_{l2i}\left(x^M, a^M, a_l^{-Mi}\right) & \text{for} \quad M \in Z_{\mu 2i}, \\ \quad (i = 1, \ldots, n), \end{cases} \tag{5.1}$$

where $a_l^M := \mathbf{A}_{\mu l}^M$ and $a^M := (a_l^M)_{l\in I} \in \mathbf{R}^p$ (see (2.19)).

Let $k_l = k_l(h) > 0$, $l \in I_2$, be arbitrary real numbers such that

$$k_l < \left(\min \left\{ N_l, \min_{\substack{m=1,2 \\ i=1,\dots,n}} \{I_{lmi}\} \right\} \right)^{-1}, \tag{5.2}$$

$$k_l \le \left(2h^{-2} \sum_{i=1}^{n} G_{li} - K_l \right)^{-1}, \tag{5.3}$$

$$k_l \le \left(h^{-1} R_{lmi} - S_{lmi} \right)^{-1} \tag{5.4}$$

for $l \in I_2$, $m = 1, 2$, $i = 1, \dots, n$, where N_l, I_{lmi}, G_{li}, K_l, R_{lmi}, S_{lmi} are the constants given in assumptions F_3 and F_6. Denote by $M^{(p-q)(N+1)^n \times (p-q)(N+1)^n}$ the set of $(p-q)(N+1)^n \times (p-q)\ (N+1)^n$ nonsingular real matrices and define the matrices $C = C(k_{q+1}, \dots, k_p) = (c_{(l,M)(s,T)})_{\substack{(l,M) \in I_2 \times Z_\mu \\ (s,T) \in I_2 \times Z_\mu}} \in M^{(p-q)(N+1)^n \times (p-q)(N+1)^n}$, $\mu = 0, 1, \dots, N_1$, in the following way

$$c_{(l,M)(s,T)} := k_l \delta_l^s \delta_M^T \tag{5.5}$$

for $l, s \in I_2$, $M, T \in Z_\mu$, $\mu = 0, 1, \dots, N_1$, where δ_l^s and δ_M^T are the Dirac delta functions.

Next, we define the discrete operators $\Phi^{\mu C \widetilde{V}_\mu} : F(Z_\mu, \mathbf{R}^{p-q}) \to F(Z_\mu, \mathbf{R}^{p-q})$, $\Phi^{\mu C \widetilde{V}_\mu} = \left(\Phi_l^{\mu C \widetilde{V}_\mu} \right)_{l \in I_2}$, $\mu = 0, 1, \dots, N_1$, associated with the discrete operators F^μ and matrices C, by the formula

$$\Phi^{\mu C \widetilde{V}_\mu A_\mu} := A_\mu + C F^{\mu \mathbf{A}_\mu}, \tag{5.6}$$

where $A_\mu = (A_{\mu l})_{l \in I_2} \in F(Z_\mu, \mathbf{R}^{p-q})$, $\mathbf{A}_\mu := (\widetilde{V}_\mu, A_\mu)$; $\widetilde{V}_\mu = (\widetilde{V}_{\mu l})_{l \in I_1} \in F(Z_\mu, \mathbf{R}^q)$ is an arbitrary parameter.

Lemma 5.1. *If Assumption F holds, then for any spatial step h of the mesh S_{kh} and $p - q$ numbers $k_l = k_l(h) > 0$, $l \in I_2$, given by inequalities* (5.2)–(5.4),

$$H \in (0, 1), \tag{5.7}$$

where

$$H = H(k_{q+1}, \dots, k_p) := \max_{l \in I_2} \left\{ 1 - k_l \min \left\{ N_l, \min_{\substack{m=1,2 \\ i=1,\dots,n}} \{I_{lmi}\} \right\} \right\} \tag{5.8}$$

and N_l, I_{lmi} are the numbers defined by (3.6), (3.17).

Proof. Dependence (5.7) is a consequence of definition (5.8), inequality (5.2) and the fact that the numbers N_l, I_{lmi} and k_l are positive. □

Lemma 5.2. *If Assumptions F and K are satisfied, then for a fixed* $\mu \in \{0, 1, \ldots, N_1\}$ *and a parameter* $\widetilde{V}_\mu = (\widetilde{V}_{\mu l})_{l \in I_1} \in F(Z_\mu, \mathbf{R}^q)$ *the inequality*

$$\left\| \Phi^{\mu C \widetilde{V}_\mu A_\mu} - \Phi^{\mu C \widetilde{V}_\mu B_\mu} \right\| \le H \, \|A_\mu - B_\mu\| \tag{5.9}$$

is true for all $A_\mu = (A_{\mu l})_{l \in I_2}$, $B_\mu = (B_{\mu l})_{l \in I_2} \in F(Z_\mu, \mathbf{R}^{p-q})$, *where* $\Phi^{\mu C \widetilde{V}_\mu}$ *are the operators given by* (5.6) *and* H *is the constant in* (5.7).

Proof. Fix $\mu \in \{0, 1, \ldots, N_1\}$ and a parameter $\widetilde{V}_\mu \in F(Z_\mu, \mathbf{R}^q)$.

Put $\mathbf{A}_\mu := (\widetilde{V}_\mu, A_\mu)$, $\mathbf{B}_\mu := (\widetilde{V}_\mu, B_\mu)$ for arbitrary mesh functions $A_\mu, B_\mu \in F(Z_\mu, \mathbf{R}^{p-q})$.

For simplicity of notation, let

$$R_\mu := A_\mu - B_\mu, \tag{5.10}$$

$$D_\mu := \Phi^{\mu C \widetilde{V}_\mu A_\mu} - \Phi^{\mu C \widetilde{V}_\mu B_\mu}. \tag{5.11}$$

Then, from definitions (5.5), (5.6), (5.10) and (5.11), it follows that

$$d_l^M = r_l^M + k_l (F_l^{\mu \mathbf{A}_\mu M} - F_l^{\mu \mathbf{B}_\mu M}) \quad \text{for} \quad l \in I_2,\ M \in Z_\mu, \tag{5.12}$$

where $r_l^M := R_{\mu l}^M$, $d_l^M := D_{\mu l}^M$.

We now define real numbers $c_{l,s}^{M,T} = c_{l,s}^{M,T}(P_l^M)$ depending on points P_l^M, $l \in I_2$, $s \in I$, $M \in Z_\mu$, $T \in S(l, M) \cup \{M\}$ (see Section 2.3), as follows: if $M \in Z_\mu^0$, then

$$c_{l,s}^{M,T} := \begin{cases} \alpha_{ll} - 2h^{-2} \sum\limits_{i=1}^{n} \gamma_{lii} + h^{-2} \sum\limits_{\substack{i,j=1 \\ i \ne j}}^{n} |\gamma_{lij}| & \text{for} \quad T = M,\ s = l, \\ \alpha_{ls} \quad \text{for} \quad T = M,\ s \ne l, \\ h^{-1} \left[h^{-1} \left(\gamma_{lii} - \sum\limits_{\substack{j=1 \\ j \ne i}}^{n} |\gamma_{lij}| \right) + \frac{1}{2}(-1)^\nu \beta_{li} \right] & \text{for } T = (-1)^\nu i(M),\ s = l, \\ \frac{1}{2} h^{-2} |\gamma_{lij}| \quad \text{for} \quad T = (-1)^\nu i\left((-1)^\nu e(l,i,j) j(M)\right),\ s = l,\ i \ne j, \\ \qquad (\nu = 1, 2,\ i, j = 1, \ldots, n), \end{cases} \tag{5.13}$$

where

$$e(l, i, j) := \begin{cases} -1, & \text{if} \quad i = j \quad \text{or} \quad \gamma_{lij} \le 0, \\ 1, & \text{if} \quad i \ne j \quad \text{and} \quad \gamma_{lij} \ge 0, \end{cases} \tag{5.14}$$

$P_l^M = (x^M, a^M, b^M, a_l^{MI}, b_l^{MI}, a_l^{MII}, b_l^{MII}) \in \Delta_1$; if $M \in Z_{\mu mi}$, $m = 1, 2$, $i = 1, \ldots, n$, then

$$c_{l,s}^{M,T} := \begin{cases} \delta_{lmil} + (-1)^m h^{-1} \rho_{lmi} & \text{for} \quad T = M,\ s = l, \\ \delta_{lmis} \quad \text{for} \quad T = M,\ s \ne l, \\ (-1)^{m+1} h^{-1} \rho_{lmi} \quad \text{for} \quad T = (-1)^{m+1} i(M),\ s = l, \end{cases} \tag{5.15}$$

where $P_l^M = (x^M, a^M, b^M, a_l^{Mi-}, b_l^{Mi-}) \in \Theta_{11i}$ for $M \in Z_{\mu 1i}$ and $P_l^M = (x^M, a^M, b^M, a_l^{-Mi}, b_l^{-Mi}) \in \Theta_{12i}$ for $M \in Z_{\mu 2i}$. In this definition: $\alpha_{ls} = \alpha_{ls}(P_l^M)$, $\beta_{li} = \beta_{li}(P_l^M)$, $\gamma_{lij} = \gamma_{lij}(P_l^M)$, $\delta_{lmis} = \delta_{lmis}(P_l^M)$, $\rho_{lmi} = \rho_{lmi}(P_l^M)$.

Further, by virtue of assumptions F_1, F_2, F_4, F_5, definitions (5.1), (2.18), (2.19)–(3.23) and (5.13), (5.15), we get

$$F_l^{\mu \mathbf{A}_\mu M} - F_l^{\mu \mathbf{B}_\mu M} = \sum_{s \in I_1} c_{l,s}^{M,M} (v_s^M - v_s^M) + \sum_{s \in I_2} c_{l,s}^{M,M} r_s^M + \sum_{T \in S(l,M)} c_{l,l}^{M,T} r_l^T \tag{5.16}$$

for $l \in I_2$ and $M \in Z_\mu$, where $v_s^M := \widetilde{V}_{\mu s}^M$.

(5.16) and (5.12) imply

$$d_l^M = (1 + k_l c_{l,l}^{M,M}) r_l^M + k_l \left(\sum_{\substack{s \in I_2 \\ s \neq l}} c_{l,s}^{M,M} r_s^M + \sum_{T \in S(l,M)} c_{l,l}^{M,T} r_l^T \right) \tag{5.17}$$

for $l \in I_2$ and $M \in Z_\mu$.

To prove the statement of the lemma, we take $\tau \in I_2$ and $A \in Z_\mu$ such that

$$\left| d_\tau^A \right| = \| D_\mu \| . \tag{5.18}$$

We consider two cases:

a) $A \in Z_\mu^0$,

b) $A \in Z_{\mu m i}$ for some $m \in \{1, 2\}$ and $i \in \{1, \dots, n\}$.

In case a), from definition (5.13) of the coefficients $c_{l,s}^{M,T}$, according to assumptions (3.7), (3.13) and inequality (5.3), we conclude that

$$\begin{aligned} 1 + k_\tau c_{\tau,\tau}^{A,A} = 1 + k_\tau \left(\alpha_{\tau\tau} - 2h^{-2} \sum_{i=1}^n \gamma_{\tau ii} + h^{-2} \sum_{\substack{i,j=1 \\ i \neq j}}^n |\gamma_{\tau ij}| \right) \geq \\ \geq 1 + k_\tau \left(K_\tau - 2h^{-2} \sum_{i=1}^n G_{\tau i} \right) \geq 0. \end{aligned} \tag{5.19}$$

Moreover, assumptions (3.10), (3.12) and (4.5) give

$$h^{-1} \left(\gamma_{\tau ii} - \sum_{\substack{j=1 \\ j \neq i}}^n |\gamma_{\tau ij}| \right) + \frac{1}{2} (-1)^\nu \beta_{\tau i} \geq h^{-1} g_{\tau i} - \frac{1}{2} H_{\tau i} \geq 0 \tag{5.20}$$

for $\nu = 1, 2$. Applying formulas (5.13), (5.14), (5.17), (5.18), the above inequalities, assumption (3.6) and definition (5.8), we can write

$$\begin{aligned}
\|D_\mu\| \le & \left[\left|1 + k_\tau c^{A,A}_{\tau,\tau}\right| + k_\tau \left(\sum_{\substack{s\in I_2 \\ s\ne\tau}} \left|c^{A,A}_{\tau,s}\right| + \sum_{T\in S(\tau,A)} \left|c^{A,T}_{\tau,\tau}\right| \right)\right] \|R_\mu\| = \\
&= \left[1 + k_\tau \left(c^{A,A}_{\tau,\tau} + \sum_{\substack{s\in I_2 \\ s\ne\tau}} \left|c^{A,A}_{\tau,s}\right| + \sum_{T\in S(\tau,A)} \left|c^{A,T}_{\tau,\tau}\right| \right)\right] \|R_\mu\| = \\
&= \Bigg[1 + k_\tau \Bigg(\alpha_{\tau\tau} - 2h^{-2} \sum_{i=1}^{n} \gamma_{\tau ii} + h^{-2} \sum_{\substack{i,j=1 \\ i\ne j}}^{n} |\gamma_{\tau ij}| + \sum_{\substack{s\in I_2 \\ s\ne\tau}} |\alpha_{\tau s}| + \\
&\quad + \frac{1}{2} h^{-1} \sum_{i=1}^{n}\sum_{\nu=1}^{2} (-1)^\nu \beta_{\tau i} + h^{-2} \sum_{i=1}^{n}\sum_{\nu=1}^{2} \left(\gamma_{\tau ii} - \sum_{\substack{j=1 \\ j\ne i}}^{n} |\gamma_{\tau ij}| \right) + \\
&\quad + \frac{1}{2} h^{-2} \sum_{\substack{i,j=1 \\ i\ne j}}^{n}\sum_{\nu=1}^{2} |\gamma_{\tau ij}| \Bigg)\Bigg] \|R_\mu\| = \left[1 + k_\tau \left(\alpha_{\tau\tau} + \sum_{\substack{s\in I_2 \\ s\ne\tau}} |\alpha_{\tau s}| \right)\right] \|R_\mu\| \le \\
&\le (1 - k_\tau N_\tau)\|R_\mu\| \le H\|R_\mu\|.
\end{aligned}$$

In case b), using definition (5.15) of the coefficients $c^{M,T}_{l,s}$, assumptions (3.18), (3.22) and inequality (5.4) gives

$$\begin{aligned}
1 + k_\tau c^{A,A}_{\tau,\tau} &= 1 + k_\tau(\delta_{\tau mi\tau} + (-1)^m h^{-1} \rho_{\tau mi}) \ge \\
&\ge 1 + k_\tau (S_{\tau mi} - h^{-1} R_{\tau mi}) \ge 0.
\end{aligned} \tag{5.21}$$

As a consequence of (5.15), (5.17), (5.18), (5.21), assumptions (3.17), (3.22) and definition (5.8), the following estimate is true

$$\begin{aligned}
\|D_\mu\| \le & \left[\left|1 + k_\tau c^{A,A}_{\tau,\tau}\right| + k_\tau \left(\sum_{\substack{s\in I_2 \\ s\ne\tau}} \left|c^{A,A}_{\tau,s}\right| + \sum_{T\in S(\tau,A)} \left|c^{A,T}_{\tau,\tau}\right| \right)\right] \|R_\mu\| = \\
&= \left[1 + k_\tau \left(c^{A,A}_{\tau,\tau} + \sum_{\substack{s\in I_2 \\ s\ne\tau}} \left|c^{A,A}_{\tau,s}\right| + \sum_{T\in S(\tau,A)} \left|c^{A,T}_{\tau,\tau}\right| \right)\right] \|R_\mu\| = \\
&= \left[1 + k_\tau \left(\delta_{\tau mi\tau} + (-1)^m h^{-1} \rho_{\tau mi} + \sum_{\substack{s\in I_2 \\ s\ne\tau}} |\delta_{\tau mis}| + (-1)^{m+1} h^{-1} \rho_{\tau mi} \right)\right] \|R_\mu\| = \\
&= \left[1 + k_\tau \left(\delta_{\tau mi\tau} + \sum_{\substack{s\in I_2 \\ s\ne\tau}} |\delta_{\tau mis}| \right)\right] \|R_\mu\| \le (1 - k_\tau I_{\tau mi}) \|R_\mu\| \le H \|R_\mu\| .
\end{aligned}$$

Thus the proof is finished. □

Theorem 5.1. *If the assumptions of Lemma* 5.2 *hold and*

$$\varphi_{lmi} \in C^1\left(\widetilde{\Theta}_{mi}\right) \quad \text{for} \quad l \in I_1,\ m = 1,2,\ i = 1,\dots,n, \tag{5.22}$$

then difference scheme (4.1) *has the unique solution in the space* $F(Z, \mathbf{R}^p)$.

Proof. To prove Theorem 5.1, first, using Algorithm 1 formulated below, we construct a solution of difference scheme (4.1) and then show that it is unique.

Algorithm 1.

Step 1. Put $\mu := 0$ and $a_l^M := u_{0l}(x^M)$ for $l \in I_1$, $M \in Z_0$.
Step 2. If $\mu > 0$, then solve the system of $q[(N+1)^n - (N-1)^n]$ algebraic equations

$$\begin{cases} \varphi_{l1i}\left(x^M, \widetilde{a}^{i(M)}, a_l^{Mi-}\right) = 0 & \text{for} \quad M \in Z_{\mu 1i}, \\ \varphi_{l2i}\left(x^M, \widetilde{a}^{-i(M)}, a_l^{-Mi}\right) = 0 & \text{for} \quad M \in Z_{\mu 2i}, \\ \quad (l \in I_1,\ i = 1,\dots,n) & \end{cases} \tag{5.23}$$

in $q[(N+1)^n - (N-1)^n]$ unknowns a_l^M, $l \in I_1$, $M \in \partial Z_\mu$.
Step 3. Solve the system of $(p-q)(N+1)^n$ algebraic equations

$$\begin{cases} f_l\left(x^M, a^M, a_l^{MI}, a_l^{MII}\right) = 0 & \text{for} \quad M \in Z_\mu^0, \\ \psi_{l1i}\left(x^M, a^M, a_l^{Mi-}\right) = 0 & \text{for} \quad M \in Z_{\mu 1i}, \\ \psi_{l2i}\left(x^M, a^M, a_l^{-Mi}\right) = 0 & \text{for} \quad M \in Z_{\mu 2i}, \\ \quad (l \in I_2,\ i = 1,\dots,n) & \end{cases} \tag{5.24}$$

in $(p-q)(N+1)^n$ unknowns a_l^M, $l \in I_2$, $M \in Z_\mu$.
Step 4. If $\mu = N_1$, then FINISH.
Step 5. 5. Solve the system of $q(N-1)^n$ algebraic equations

$$a_l^{M-} = f_l\left(x^M, a^M, a_l^{MI}, a_l^{MII}\right) \quad \text{for} \quad M \in Z_\mu^0,\ l \in I_1 \tag{5.25}$$

in $q(N-1)^n$ unknowns a_l^{+M}, $l \in I_1$, $M \in Z_\mu^0$. Then put $\mu := \mu + 1$ and go to *Step* 2.

We start to construct the desired solution $v \in F(Z, \mathbf{R}^p)$ of (4.1).

Put $\mu := 0$ and $a_l^M := u_{0l}(x^M)$, $l \in I_1$, $M \in Z_0$, in *Step* 1 of Algorithm 1.

Then we omit *Step* 2, because $\mu = 0$, and go to *Step* 3. We wish to find a solution $V_0 = (V_{0l})_{l \in I_2} \in F(Z_0, \mathbf{R}^{p-q})$ of (5.24) for $\mu = 0$. Put

$$\widetilde{V}_0 = (\widetilde{V}_{0l})_{l \in I_1} \in F(Z_0, \mathbf{R}^q), \quad \widetilde{V}_{0l}^M := a_l^M \tag{5.26}$$

for $l \in I_1$ and $M \in Z_0$. It is easily seen that system of equations (5.24) is equivalent to the equation

$$F^{0\mathbf{A}_0} = 0 \tag{5.27}$$

for $\mu = 0$, where the operator F^0 is defined by (5.1), $\mathbf{A}_0 := (\widetilde{V}_0, A_0) \in F(Z_0, \mathbf{R}^p)$, $\widetilde{V}_0$ is the parameter given by (5.26), and $A_0 \in F(Z_0, \mathbf{R}^{p-q})$ is any mesh function.

Then non-singularity of the matrix C in (5.5) implies that (5.27) is equivalent to an equation of the form

$$\Phi^{0C\widetilde{V}_0 A_0} = A_0 \tag{5.28}$$

(see (5.6)). By Lemmas 5.2 and 5.1 and the Banach fixed-point theorem, it follows that the last equation has the unique solution V_0. This is also the unique solution of (5.24) for $\mu = 0$.

Next we go to *Step* 5, taking $a_l^M := V_{0l}^M$ for $l \in I_2$, $M \in Z_0$, because $\mu = 0 < N_1$ and *step* 4 is omitted. As system of equations (5.25) for $\mu = 0$ is the explicit difference scheme, we compute numbers a_l^{+M} for $l \in I_1$, $M \in Z_0^0$ uniquely. Put $\mu := 1$ and go to *Step* 2.

By consideration of assumptions (3.21), (5.22) and the implicit function theorem, in *Step* 2, in $F(\partial Z_1, \mathbf{R}^q)$ there is exactly one solution of system (5.23) for $\mu = 1$.

Then we set the parameter

$$\widetilde{V}_1 = (\widetilde{V}_{1l})_{l\in I_1} \in F(Z_1, \mathbf{R}^q), \quad \widetilde{V}_{1l}^M := a_l^M \tag{5.29}$$

for $l \in I_1$ and $M \in Z_1$, where the numbers a_l^M are computed above, and go to *Step* 3. The procedure is repeated until $\mu = N_1$ in *Step* 4.

Thus, the system of mesh functions

$$v := (\widetilde{V}_\mu, V_\mu)_{\mu=0,\dots,N_1} \tag{5.30}$$

is the solution of difference scheme (4.1), where V_μ and $\widetilde{V}_\mu$, $\mu = 0, 1, \dots, N_1$, are uniquely determined as above with use of Algorithm 1.

Suppose now that difference scheme (4.1) has another solution w. From the form of (4.1), there exists $\mu_0 \in \{0, 1, \dots, N_1\}$ such that system of equations (5.23) for $\mu = \mu_0$, $\mu_0 > 0$, or system of equations (5.24) for $\mu = \mu_0$ has at least two different solutions in $F(\partial Z_{\mu_0}, \mathbf{R}^q)$ or $F(Z_{\mu_0}, \mathbf{R}^{p-q})$, respectively. But we have proved, constructing v in (5.30), that each of these systems has exactly one solution for each μ. This gives a contradiction.

This completes the proof of Theorem 5.1. □

5.2. CONVERGENCE OF THE DIFFERENCE METHOD

In this part we deal with the convergence of the method considered. The error estimate of the approximate solution of differential problem (3.1), (3.2) will be given. To this end, we first formulate and prove some lemmas and prove that the difference method is consistent.

Lemma 5.3. *Fix $\mu \in \{0, 1, \dots, N_1\}$. Let real functions $c_{l,s}^{M,M}$ and $c_{l,l}^{M,T}$, $l \in I_2$, $s \in I$, $M \in Z_\mu$, $T \in S(l, M)$, defined in arbitrary domains and a mesh function $D_\mu \in F(Z_\mu, \mathbf{R}^{p-q})$ be given. Suppose that a mesh function $\mathbf{R}_\mu = (\widetilde{R}_\mu, R_\mu) \in F(Z_\mu, \mathbf{R}^q) \times F(Z_\mu, \mathbf{R}^{p-q})$ is a solution of the system of algebraic equations*

$$\sum_{s\in I_1} c_{l,s}^{M,M} r_s^M + \sum_{s\in I_2} c_{l,s}^{M,M} r_s^M + \sum_{T\in S(l,M)} c_{l,l}^{M,T} r_l^T = d_l^M \tag{5.31}$$

for $l \in I_2$, $M \in Z_\mu$, *where* $d_l^M := D_{\mu l}^M$, $r_s^M := \mathbf{R}_{\mu s}^M$, $r_l^T := R_{\mu l}^T$, *and let indices* $\tau = \tau(\mu) \in I_2$, $A = A(\mu) \in Z_\mu$ *fulfil the condition*

$$\left|r_\tau^A\right| = \|R_\mu\|. \tag{5.32}$$

If, moreover,

(1) *there exists a constant* $\overline{\lambda} = \overline{\lambda}(\mu) > 0$ *such that in the domains of the coefficients* $c_{\tau,s}^{A,A}$ *and* $c_{\tau,\tau}^{A,T}$, $s \in I_2$, $T \in S(\tau, A)$, *the inequality*

$$\left|c_{\tau,\tau}^{A,A}\right| - \left(\sum_{\substack{s\in I_2\\ s\neq\tau}} \left|c_{\tau,s}^{A,A}\right| + \sum_{T\in S(\tau,A)} \left|c_{\tau,\tau}^{A,T}\right|\right) \geq \overline{\lambda} \tag{5.33}$$

holds,

(2) *there exists a constant* $\overline{L} = \overline{L}(\mu) \geq 0$ *such that in the domains of the coefficients* $c_{\tau,s}^{A,A}$, $s \in I_1$, *the estimate*

$$\left|c_{\tau,s}^{A,A}\right| \leq \overline{L} \tag{5.34}$$

is true,

then

$$\left|r_\tau^A\right| \leq \overline{\lambda}^{-1}\left(\left|d_\tau^A\right| + \overline{L}\sum_{s\in I_1}\left|r_s^A\right|\right), \tag{5.35}$$

$$\|R_\mu\| \leq \overline{\lambda}^{-1}\left(\|D_\mu\| + q\overline{L}\left\|\widetilde{R}_\mu\right\|\right). \tag{5.36}$$

Proof. Fix $\mu \in \{0, 1, \ldots, N_1\}$ and note that inequality (5.36) is a consequence of relation (5.35).

Applying assumption (5.33) to system (5.31), it is easy to verify that

$$\begin{aligned}
\left|d_\tau^A - \sum_{s\in I_1} c_{\tau,s}^{A,A} r_s^A\right| &= \left|c_{\tau,\tau}^{A,A} r_\tau^A + \sum_{\substack{s\in I_2\\ s\neq\tau}} c_{\tau,s}^{A,A} r_s^A + \sum_{T\in S(\tau,A)} c_{\tau,\tau}^{A,T} r_\tau^T\right| \geq \\
&\geq \left|c_{\tau,\tau}^{A,A}\right|\left|r_\tau^A\right| - \left|\sum_{\substack{s\in I_2\\ s\neq\tau}} c_{\tau,s}^{A,A} r_s^A\right| - \left|\sum_{T\in S(\tau,A)} c_{\tau,\tau}^{A,T} r_\tau^T\right| \geq \\
&\geq \left[\left|c_{\tau,\tau}^{A,A}\right| - \left(\sum_{\substack{s\in I_2\\ s\neq\tau}} \left|c_{\tau,s}^{A,A}\right| + \sum_{T\in S(\tau,A)} \left|c_{\tau,\tau}^{A,T}\right|\right)\right]\left|r_\tau^A\right| \geq \\
&\geq \overline{\lambda}\left|r_\tau^A\right|.
\end{aligned} \tag{5.37}$$

The formula (5.37) and assumption (5.34) lead to

$$\left|r_\tau^A\right| \le \overline{\lambda}^{-1} \left| d_\tau^A - \sum_{s\in I_1} c_{\tau,s}^{A,A} r_s^A \right| \le \overline{\lambda}^{-1} \left(\left|d_\tau^A\right| + \sum_{s\in I_1} \left|c_{\tau,s}^{A,A}\right| \left|r_s^A\right| \right) \le$$
$$\le \overline{\lambda}^{-1} \left(\left|d_\tau^A\right| + \overline{L} \sum_{s\in I_1} \left|r_s^A\right| \right),$$

giving (5.35) and concluding the proof. □

Lemma 5.4. *If the assumptions of Lemma* 5.2 *are satisfied, then for a fixed* $\mu \in \{0, 1, \dots, N_1\}$ *and all mesh functions* $\mathbf{A}_\mu = (\widetilde{A}_\mu, A_\mu)$, $\mathbf{B}_\mu = (\widetilde{B}_\mu, B_\mu) \in F(Z_\mu, \mathbf{R}^q) \times F(Z_\mu, \mathbf{R}^{p-q})$,

$$\|A_\mu - B_\mu\| \le \lambda\Big(\left\|F^{\mu \mathbf{A}_\mu} - F^{\mu \mathbf{B}_\mu}\right\| + qL_2 \left\|\widetilde{A}_\mu - \widetilde{B}_\mu\right\| \Big), \tag{5.38}$$

where

$$\lambda := \left(\min_{l\in I_2} \left\{ N_l, \min_{\substack{m=1,2\\ i=1,\dots,n}} \{I_{lmi}\} \right\} \right)^{-1}, \tag{5.39}$$

F^μ *are the operators given by* (5.1), L_2 *is the constant in assumptions* F_3, F_6 *and* N_l, I_{lmi} *are the numbers defined by* (3.6), (3.17).

Proof. Fix $\mu \in \{0, 1, \dots, N_1\}$ and let $\mathbf{A}_\mu = (\widetilde{A}_\mu, A_\mu)$, $\mathbf{B}_\mu = (\widetilde{B}_\mu, B_\mu) \in F(Z_\mu, \mathbf{R}^q) \times F(Z_\mu, \mathbf{R}^{p-q})$ be arbitrary mesh functions.

Put

$$\widetilde{R}_\mu := \widetilde{A}_\mu - \widetilde{B}_\mu,\ R_\mu := A_\mu - B_\mu,\ \mathbf{R}_\mu := (\widetilde{R}_\mu, R_\mu), \tag{5.40}$$

$$D_\mu := F^{\mu \mathbf{A}_\mu} - F^{\mu \mathbf{B}_\mu}. \tag{5.41}$$

Now, reasoning similarly as in the proof of Lemma 5.2, we can write

$$F_l^{\mu \mathbf{A}_\mu M} - F_l^{\mu \mathbf{B}_\mu M} = \sum_{s\in I_1} c_{l,s}^{M,M} r_s^M + \sum_{s\in I_2} c_{l,s}^{M,M} r_s^M + \sum_{T\in S(l,M)} c_{l,l}^{M,T} r_l^T \tag{5.42}$$

for $l \in I_2$ and $M \in Z_\mu$ (see (5.16)), where $c_{l,s}^{M,M}$, $c_{l,l}^{M,T}$ are the numbers defined for $\mathbf{A}_\mu$, $\mathbf{B}_\mu$, analogously as in (5.13), (5.15), and $r_s^M := \mathbf{R}_{\mu s}^M$, $r_l^T := R_{\mu l}^T$.

Therefore, it is obvious that the mesh function $\mathbf{R}_\mu$ in (5.40) is a solution of a system of algebraic equations of the form

$$\sum_{s\in I_1} c_{l,s}^{M,M} r_s^M + \sum_{s\in I_2} c_{l,s}^{M,M} r_s^M + \sum_{T\in S(l,M)} c_{l,l}^{M,T} r_l^T = d_l^M \tag{5.43}$$

for $l \in I_2$ and $M \in Z_\mu$, where $d_l^M := D_{\mu l}^M$. This system is of the type (5.31) in Lemma 5.3.

We show that if the indices $\tau \in I_2$ and $A \in Z_\mu$ are defined as in (5.32), then the assumptions of Lemma 5.3 are satisfied for

$$\overline{\lambda} := \lambda^{-1}, \quad \overline{L} := L_2. \tag{5.44}$$

Indeed, estimate (5.34) follows from definition (5.44), and formulas (3.9) in assumption F_3 and (3.19) in assumption F_6.

Next, to prove inequality (5.33) we consider two cases:

a) $A \in Z^0_\mu$,

b) $A \in Z_{\mu m i}$ for some $m \in \{1,2\}$ and $i \in \{1,\ldots,n\}$.

In case a), addition of assumptions (3.6), (3.10), (3.12), (4.5), formula (5.20), definitions (5.13), (5.44) and relation (5.39) yields

$$\begin{aligned}
&\left|c^{A,A}_{\tau,\tau}\right| - \left(\sum_{\substack{s\in I_2\\ s\neq\tau}} \left|c^{A,A}_{\tau,s}\right| + \sum_{T\in S(\tau,A)} \left|c^{A,T}_{\tau,\tau}\right|\right) = \\
&= \left|\alpha_{\tau\tau} - 2h^{-2}\sum_{i=1}^{n}\gamma_{\tau ii} + h^{-2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}|\gamma_{\tau ij}|\right| - \sum_{\substack{s\in I_2\\ s\neq\tau}}|\alpha_{\tau s}| - \\
&\quad - h^{-1}\sum_{i=1}^{n}\sum_{\nu=1}^{2}\left|h^{-1}\left(\gamma_{\tau ii} - \sum_{\substack{j=1\\ j\neq i}}^{n}|\gamma_{\tau ij}|\right) + \frac{1}{2}(-1)^{\nu}\beta_{\tau i}\right| - \frac{1}{2}h^{-2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\sum_{\nu=1}^{2}|\gamma_{\tau ij}| = \\
&= \left|\alpha_{\tau\tau} - h^{-2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}|\gamma_{\tau ij}| - 2h^{-2}\sum_{i=1}^{n}\left(\gamma_{\tau ii} - \sum_{\substack{j=1\\ j\neq i}}^{n}|\gamma_{\tau ij}|\right)\right| - \\
&\quad - \sum_{\substack{s\in I_2\\ s\neq\tau}}|\alpha_{\tau s}| - 2h^{-2}\sum_{i=1}^{n}\left(\gamma_{\tau ii} - \sum_{\substack{j=1\\ j\neq i}}^{n}|\gamma_{\tau ij}|\right) - h^{-2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}|\gamma_{\tau ij}| = \\
&= -\alpha_{\tau\tau} + h^{-2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}|\gamma_{\tau ij}| + 2h^{-2}\sum_{i=1}^{n}\left(\gamma_{\tau ii} - \sum_{\substack{j=1\\ j\neq i}}^{n}|\gamma_{\tau ij}|\right) - \\
&\quad - \sum_{\substack{s\in I_2\\ s\neq\tau}}|\alpha_{\tau s}| - 2h^{-2}\sum_{i=1}^{n}\left(\gamma_{\tau ii} - \sum_{\substack{j=1\\ j\neq i}}^{n}|\gamma_{\tau ij}|\right) - h^{-2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}|\gamma_{\tau ij}| = \\
&= -\alpha_{\tau\tau} - \sum_{\substack{s\in I_2\\ s\neq\tau}}|\alpha_{\tau s}| \geq N_\tau \geq \overline{\lambda}.
\end{aligned}$$

In case b), from assumptions (3.17) and (3.22), definitions (5.15) and (5.44) and relation (5.39), we get

$$\begin{aligned}
&\left|c^{A,A}_{\tau,\tau}\right| - \left(\sum_{\substack{s\in I_2\\ s\neq\tau}} \left|c^{A,A}_{\tau,s}\right| + \sum_{T\in S(\tau,A)} \left|c^{A,T}_{\tau,\tau}\right| \right) = \\
&\qquad = \left|\delta_{\tau mi\tau} + (-1)^m h^{-1}\rho_{\tau mi}\right| - \sum_{\substack{s\in I_2\\ s\neq\tau}} |\delta_{\tau mis}| - \left|(-1)^{m+1}h^{-1}\rho_{\tau mi}\right| = \\
&\qquad = -\delta_{\tau mi\tau} - (-1)^m h^{-1}\rho_{\tau mi} - \sum_{\substack{s\in I_2\\ s\neq\tau}} |\delta_{\tau mis}| - (-1)^{m+1}h^{-1}\rho_{\tau mi} = \\
&\qquad = -\delta_{\tau mi\tau} - \sum_{\substack{s\in I_2\\ s\neq\tau}} |\delta_{\tau mis}| \geq I_{\tau mi} \geq \overline{\lambda}.
\end{aligned}$$

Owing to a) and b), the statement of this lemma is a result of Lemma 5.3. □

Definition 5.1. *We say that a sequence of difference schemes of form* (4.1) *approximates differential problem* (3.1), (3.2) *on its regular solution* $u \in C_{reg}(\overline{\Omega}, \mathbf{R}^p)$ *or briefly that difference method* (4.1) *is consistent if*

$$\lim_{(k,h)\to(0,0)} \max\left\{\left\|S^{0U}\right\|, \left\|S^{1U}\right\|, \left\|S^{2U}\right\|\right\} = 0,$$

where $U \in F(Z, \mathbf{R}^p)$ *are the restrictions of* u *to the meshes* S_{kh}.

Theorem 5.2. *If Assumption F holds, then difference method* (4.1) *is consistent in the sense of Definition* 5.1.

Proof. The consistence of difference method (4.1) follows immediately from the regularity of u and continuity of the mappings f_l, φ_{lmi} and ψ_{lmi} with respect to y, z, w in suitable sets (see F_1, F_4, F_5). □

We now go to the main problem of the paper, the problem of the convergence of difference method (4.1).

Let $U \in F(Z, \mathbf{R}^p)$ be the restriction of the regular solution $u \in C_{reg}(\overline{\Omega}, \mathbf{R}^p)$ of differential problem (3.1), (3.2) (see assumption F_7) to the mesh S_{kh}, i.e. $U^M_l := u_l(x^M)$ for $l \in I$, and let $v \in F(Z, \mathbf{R}^p)$ be the solution of difference scheme (4.1) (see Theorem 5.1).

Definition 5.2. *Difference method* (4.1) *is uniformly convergent if*

$$\lim_{(k,h)\to(0,0)} \|r\| = 0,$$

where $r := U - v \in F(Z, \mathbf{R}^p)$ *is the error of this method.*

Let $\mathbf{U}_\mu = (\widetilde{U}_\mu, U_\mu) \in F(Z_\mu, \mathbf{R}^q) \times F(Z_\mu, \mathbf{R}^{p-q})$ be the restriction of U to the intersection Z_μ of Z, $\mu = 0, 1, \ldots, N_1$.

Next, we define the mesh functions $\varepsilon^\mu \in F(Z_\mu, \mathbf{R}^{p-q})$ by the formula

$$\varepsilon_l^{\mu M} := F_l^{\mu \mathbf{U}_\mu M} \quad \text{for} \quad l \in I_2,\ M \in Z_\mu,\ \mu = 0, 1, \ldots, N_1, \tag{5.45}$$

where F^μ are the operators in (5.1).

Lemma 5.5. *If the assumptions of Theorem* 5.1 *hold, then*

$$\left| r_l^M \right| \le \lambda \left(\|\varepsilon^\mu\| + qL_2 \left\| \widetilde{R}_\mu \right\| \right) \tag{5.46}$$

for $l \in I_2$, $M \in Z_\mu$, $\mu = 0, 1, \ldots, N_1$, *where* r *is the error of the method,* $\widetilde{R}_\mu \in F(Z_\mu, \mathbf{R}^q)$, $\widetilde{R}_{\mu l}^M := r_l^M$ *for* $l \in I_1$, ε^μ *are the functions defined by* (5.45), λ *is the constant in* (5.39) *and* L_2 *the constant in assumption* F_3.

Proof. Fix $\mu \in \{0, 1, \ldots, N_1\}$ and define $\mathbf{V}_\mu = (\widetilde{V}_\mu, V_\mu) \in F(Z_\mu, \mathbf{R}^q) \times F(Z_\mu, \mathbf{R}^{p-q})$ as the restriction of v to Z_μ.

Since

$$F^{\mu \mathbf{U}_\mu} - F^{\mu \mathbf{V}_\mu} = \varepsilon^\mu,$$

we have by Lemma 5.4

$$\|U_\mu - V_\mu\| \le \lambda (\|\varepsilon^\mu\| + qL_2 \left\| \widetilde{R}_\mu \right\|),$$

and therefore inequality (5.46). □

Further, we put

$$K := q(p-q) L_1 L_2 \lambda, \tag{5.47}$$

where the constants L_1, L_2 arise in Assumption F and λ is given by (5.39), and introduce the mesh functions $\eta \in F\Big(\bigcup_{\mu=0}^{N_1 - 1} Z_\mu^0, \mathbf{R}^q \Big)$ and $\eta_{mi} \in F\Big(\bigcup_{\mu=1}^{N_1} Z_{\mu m i}, \mathbf{R}^q \Big)$, $m = 1, 2$, $i = 1, \ldots, n$, defined by

$$\begin{aligned} \eta_l^M &:= U_l^{M-} - f_l \left(x^M, U^M, U_l^{MI}, U_l^{MII} \right), \\ \eta_{1il}^M &:= \varphi_{l1i} \left(x^M, \widetilde{U}^{i(M)}, U_l^{Mi-} \right), \\ \eta_{2il}^M &:= \varphi_{l2i} \left(x^M, \widetilde{U}^{-i(M)}, U_l^{-Mi} \right) \end{aligned} \tag{5.48}$$

for $l \in I_1$ and M belonging to suitable sets.

Moreover, we define the real valued functions ε, ε_m, $\overline{\varepsilon}$, $\widehat{\varepsilon}_\mu$, $\widehat{\varepsilon}$, ε^* for $m = 1, 2$, $\mu = 0, 1, \ldots, N_1$, depending on steps k and h, by setting

$$\begin{aligned}
\varepsilon(k,h) &:= \|\eta\|, \\
\varepsilon_m(k,h) &:= \max_{i=1,\ldots,n} \|\eta_{mi}\|, \\
\overline{\varepsilon}(k,h) &:= \max_{m=1,2} \{\varepsilon_m(k,h)\}, \\
\widehat{\varepsilon}_\mu(k,h) &:= \varepsilon(k,h) + (p-q)L_1\lambda \|\varepsilon^\mu\|, \\
\widehat{\varepsilon}(k,h) &:= \max_{\mu=0,1,\ldots,N_1} \{\widehat{\varepsilon}_\mu(k,h)\}, \\
\varepsilon^*(k,h) &:= \widehat{\varepsilon}(k,h) + (L+K)\frac{\overline{\varepsilon}(k,h)}{G}
\end{aligned} \tag{5.49}$$

(see (5.48), (3.8), (5.39), (5.45), (3.4), (3.5), (5.47) and (3.20)).

Making use of the above functions, we define the mesh function $y \in F(Z, \mathbf{R})$ as follows

$$y^M := \begin{cases} \frac{\varepsilon^*(k,h)}{L+K}\left\{[1+k(L+K)]^\mu - 1\right\} + \frac{\overline{\varepsilon}(k,h)}{G} & \text{for} \quad M \in Z^0_\mu, \\ (1-hG)\left(\frac{\varepsilon^*(k,h)}{L+K}\left\{[1+k(L+K)]^\mu - 1\right\} + \frac{\overline{\varepsilon}(k,h)}{G}\right) + h\overline{\varepsilon}(k,h) & \\ & \text{for} \quad M \in \partial Z_\mu. \end{cases} \tag{5.50}$$

We will apply it to estimate the error of the difference method.

Remark 5.1. *Suppose that the assumptions of Theorem* 5.1 *are fulfilled. Then there holds*

$$1 - hG \geq 0, \tag{5.51}$$

$$y \geq 0. \tag{5.52}$$

Proof. Observe that assumptions (3.20), (3.16), (3.21) and (4.4) yield

$$\begin{aligned}
1 - hG &\geq 1 + (-1)^{m-1} h \rho^{-1}_{lmi} \sum_{s \in I_1} \delta_{lmis} = \\
&= 1 + (-1)^{m-1} h \rho^{-1}_{lmi} \delta_{lmil} + (-1)^{m-1} h \rho^{-1}_{lmi} \sum_{\substack{s \in I_1 \\ s \neq l}} \delta_{lmis} \geq 0
\end{aligned}$$

for $l \in I_1$, $m = 1, 2$ and $i = 1, \ldots, n$, which gives (5.52). □

As a consequence of the above lemmas, definitions and remark we obtain the following conclusion.

Theorem 5.3. *Let the assumptions of Theorem* 5.1 *hold. Then*

(i) $$\left|r^M_l\right| \leq y^M \quad \text{for} \quad l \in I_1, \; M \in Z, \tag{5.53}$$

(ii) $\left|r_l^M\right| \le \lambda(\|\varepsilon^\mu\| + qL_2 y^M)$ for $l \in I_2,\ M \in Z_\mu,\ \mu = 0,1,\dots,N_1,$ (5.54)

(iii) *difference method* (4.1) *is convergent in the sense od Definition* 5.2, *where* r *is the error of the method* (see (5.50), (5.45), (5.39), (3.9), (3.19)).

Proof. Note that

$$y^M \le \frac{e^{T(L+K)}-1}{L+K}\varepsilon^*(k,h) + \frac{\overline{\varepsilon}(k,h)}{G} + \delta\overline{\varepsilon}(k,h) \quad \text{for} \quad M \in Z. \tag{5.55}$$

Then, it is obvious that the convergence of (4.1) follows immediately from (5.55), the estimates in (i), (ii) and the consistence of the method (see Theorem 5.2).

From Lemma 5.5 and statement (i), statement (ii) follows. Therefore, it remains to show (i).

We use the maximum principle (the monotonicity theorem) of [10] to the following system of difference functional inequalities of the parabolic type

$$|r_l|^{M-} \le \widehat{\varepsilon}(k,h) + \sum_{s\in I_1} \alpha_{ls}\left|r_s^M\right| + \sum_{i=1}^n \beta_{li}\,|r_l|^{Mi} + \sum_{i,j=1}^n \gamma_{lij}\,|r_l|^{Mij} + K\|\,|\mathbf{r}|\,\|(\mu k), \tag{5.56}$$

$$y^{M-} \ge \widehat{\varepsilon}(k,h) + \sum_{s\in I_1} \alpha_{ls} y^M + \sum_{i=1}^n \beta_{li} y^{Mi} + \sum_{i,j=1}^n \gamma_{lij} y^{Mij} + K\|\mathbf{y}\|(\mu k) \tag{5.57}$$

for $l \in I_1$, $M \in \bigcup_{\mu=0}^{N_1-1} Z_\mu^0$ (see (2.9), (2.15), (2.18), (5.47), (5.49) and assumption F_1);

$$\left|r_l^M\right| \le y^M \quad \text{for} \quad l \in I_1,\ M \in Z_0; \tag{5.58}$$

$$|r_l|^{Mi-} \ge -\overline{\varepsilon}(k,h)\rho_{l1i}^{-1} - \rho_{l1i}^{-1} \sum_{s\in I_1} \delta_{l1is}\left|r_s^{i(M)}\right|, \tag{5.59}$$

$$y^{Mi-} \le -\overline{\varepsilon}(k,h)\rho_{l1i}^{-1} - \rho_{l1i}^{-1} \sum_{s\in I_1} \delta_{l1is} y^{i(M)} \tag{5.60}$$

for $l \in I_1$, $M \in \bigcup_{\mu=1}^{N_1} Z_{\mu 1i}$, $i = 1,\dots,n$; and

$$|r_l|^{-Mi} \le \overline{\varepsilon}(k,h)\rho_{l2i}^{-1} - \rho_{l2i}^{-1} \sum_{s\in I_1} \delta_{l2is}\left|r_s^{-i(M)}\right|, \tag{5.61}$$

$$y^{-Mi} \ge \overline{\varepsilon}(k,h)\rho_{l2i}^{-1} - \rho_{l2i}^{-1} \sum_{s\in I_1} \delta_{l2is} y^{-i(M)} \tag{5.62}$$

for $l \in I_1$, $M \in \bigcup_{\mu=1}^{N_1} Z_{\mu 2i}$, $i = 1,\dots,n$ (see (2.16), (2.18), (5.49) and assumptions F_4, F_5). Note that $|r_l|^{M-}$, $|r_l|^{Mi}$, $|r_l|^{Mij}$, $|r_l|^{Mi-}$ and $|r_l|^{-Mi}$ are the suitable difference quotients for the mesh functions $|r_l| \in F(Z, \mathbf{R}^+)$, $|r_l|^M := \left|r_l^M\right|$, and $|\mathbf{r}| = S[|r|]$ where $|r| := (|r_l|)_{l\in I_1}$ (see (2.25)–(2.27)). Moreover, $\alpha_{ls} = \alpha_{ls}(P_l^M)$, $\beta_{li} = \beta_{li}(P_l^M)$, $\gamma_{lij} = \gamma_{lij}(P_l^M)$, $\delta_{lmis} = \delta_{lmis}(P_l^M)$, $\rho_{lmi} = \rho_{lmi}(P_l^M)$, where $P_l^M = (x^M, U^M, v^M, U_l^{MI}, v_l^{MI}, U_l^{MII}, v_l^{MII}) \in \Delta_1$ in (5.56), (5.57); $P_l^M = (x^M, \widetilde{U}^{i(M)}, \widetilde{v}^{i(M)}, U_l^{Mi-}, v_l^{Mi-}) \in \widetilde{\Theta}_{11i}$ in (5.59), (5.60); $P_l^M = (x^M, \widetilde{U}^{-i(M)}, \widetilde{v}^{-i(M)}, U_l^{-Mi}, v_l^{-Mi}) \in \widetilde{\Theta}_{12i}$ in (5.61), (5.62).

Inequality (5.58) is clear.

Next, we fix $l \in I_1$, $\mu \in \{0, 1, \ldots, N_1 - 1\}$, $M \in Z_\mu^0$ and prove inequalities (5.56), (5.57).

Observe that definitions (5.48), (4.1), (2.19), (2.20), (3.23) and assumptions F_1, F_2 lead to

$$\begin{aligned} r_l^{M-} &= \eta_l^M + f_l(x^M, U^M, U_l^{MI}, U_l^{MII}) - f_l(x^M, v^M, v_l^{MI}, v_l^{MII}) = \\ &= \eta_l^M + \sum_{s\in I} \alpha_{ls}(P_l^M) r_s^M + \sum_{i=1}^n \beta_{li}(P_l^M) r_l^{Mi} + \sum_{i,j=1}^n \gamma_{lij}(P_l^M) r_l^{Mij}. \end{aligned} \tag{5.63}$$

After having grouped the suitable expressions in (5.63), in view of assumptions K_1, F_2, F_3 and definition (5.49), we get the estimate

$$\begin{aligned} \left|r_l^{+M}\right| \le{}& k\varepsilon(k,h) + \left(1 + k\alpha_{ll} - 2kh^{-2}\sum_{i=1}^n \gamma_{lii}\right)\left|r_l^M\right| + k\sum_{\substack{s\in I_1\\ s\neq l}} \alpha_{ls}\left|r_s^M\right| + \\ &+ kh^{-1}\sum_{i=1}^n \left[h^{-1}\left(\gamma_{lii} - \sum_{\substack{j=1\\ j\neq i}}^n |\gamma_{lij}|\right) + \frac{1}{2}\beta_{li}\right]\left|r_l^{i(M)}\right| + \\ &+ kh^{-1}\sum_{i=1}^n \left[h^{-1}\left(\gamma_{lii} - \sum_{\substack{j=1\\ j\neq i}}^n |\gamma_{lij}|\right) - \frac{1}{2}\beta_{li}\right]\left|r_l^{-i(M)}\right| + \\ &+ \frac{1}{2}kh^{-2}\sum_{\substack{i,j=1\\ i\neq j}}^n |\gamma_{lij}|\left(2\left|r_l^M\right| + \left|r_l^{i(e(l,i,j)j(M))}\right| + \left|r_l^{-i(-e(l,i,j)j(M))}\right|\right) + \\ &+ k\sum_{s\in I_2} |\alpha_{ls}|\left|r_s^M\right|, \end{aligned} \tag{5.64}$$

where

$$e(l,i,j) := \begin{cases} -1, & \text{if} \quad r_l^{Mij} = r_l^{-Mij}, \\ 1, & \text{if} \quad r_l^{Mij} = r_l^{+Mij} \end{cases}$$

for $i, j = 1, \ldots, n$, $i \neq j$ (see (2.18), (3.23)). Hence, from Lemma 5.5 and assumption F_3, there follows

$$\begin{aligned} |r_l|^{M-} \leq \varepsilon(k,h) + \sum_{s \in I_1} \alpha_{ls} \left|r_s^M\right| + \sum_{i=1}^{n} \beta_{li} |r_l|^{Mi} + \sum_{i,j=1}^{n} \gamma_{lij} |r_l|^{Mij} + \\ + (p-q) L_1 \lambda \left(\|\varepsilon^\mu\| + q L_2 \left\| \widetilde{R}_\mu \right\| \right), \end{aligned} \tag{5.65}$$

where $\widetilde{R}^M_{\mu l} := r_l^M$. Combining the fact

$$\left\| \widetilde{R}_\mu \right\| = \| \, |\mathbf{r}| \, \| (\mu k) \tag{5.66}$$

(see (2.17), (2.9)) and formulas (5.65), (5.47), (5.49), we obtain inequality (5.56).

We now prove inequality (5.57). For simplicity, we introduce a notation

$$z^\mu := \frac{\varepsilon^*(k,h)}{L+K} \left\{ [1 + k(L+K)]^\mu - 1 \right\} + \frac{\overline{\varepsilon}(k,h)}{G}. \tag{5.67}$$

Note that if $A \in \partial Z_\mu$, then

$$\begin{aligned} y^A - y^M &= (1 - hG) z^\mu + h\overline{\varepsilon}(k,h) - z^\mu = \\ &= -hG \frac{\varepsilon^*(k,h)}{L+K} \left\{ [1 + k(L+K)]^\mu - 1 \right\} - hG \frac{\overline{\varepsilon}(k,h)}{G} + h\overline{\varepsilon}(k,h) = \\ &= -\frac{hG\varepsilon^*(k,h)}{L+K} \left\{ [1 + k(L+K)]^\mu - 1 \right\} \leq 0. \end{aligned} \tag{5.68}$$

Therefore, by (5.54), (5.67), (5.68) and Remark 5.1, there holds

$$\|\mathbf{y}\| (\mu k) = z^\mu \tag{5.69}$$

and

$$\begin{aligned} y^{M-} &= \frac{1}{k} \left(z^{\mu+1} - z^\mu \right) = \\ &= \frac{\varepsilon^*(k,h)}{k(L+K)} \left\{ [1 + k(L+K)]^{\mu+1} - [1 + k(L+K)]^\mu \right\} = \\ &= \frac{\varepsilon^*(k,h)}{k(L+K)} [1 + k(L+K)]^\mu \, [1 + k(L+K) - 1] = \\ &= \varepsilon^*(k,h) \, [1 + k(L+K)]^\mu . \end{aligned} \tag{5.70}$$

Then observe that

$$\begin{aligned} \varepsilon^*(k,h) \, [1+k(L+K)]^\mu = \left(\frac{\varepsilon^*(k,h)}{L+K} \left\{ [1+k(L+K)]^\mu - 1 \right\} + \frac{\overline{\varepsilon}(k,h)}{G} \right) (L+K) - \\ - \frac{\overline{\varepsilon}(k,h)}{G} (L+K) + \varepsilon^*(k,h). \end{aligned} \tag{5.71}$$

By (5.70), (5.71), (5.49), (5.67) and (5.69), we can write

$$\begin{aligned} y^{M-} &= z^{\mu}(L+K) - \frac{\overline{\varepsilon}(k,h)}{G}(L+K) + \widehat{\varepsilon}(k,h) + \frac{\overline{\varepsilon}(k,h)}{G}(L+K) = \\ &= \widehat{\varepsilon}(k,h) + Ly^{M} + K\|\mathbf{y}\|(\mu k). \end{aligned} \tag{5.72}$$

Note that (5.68) and assumption K_1 imply

$$\sum_{i=1}^{n} \beta_{li} y^{Mi} + \sum_{i,j=1}^{n} \gamma_{lij} y^{Mij} \leq 0. \tag{5.73}$$

A proof of this inequality is similar to that in [12] and is omitted. The above two relations and assumption F_3 give (5.57).

To verify inequalities (5.59), (5.60), we fix $l \in I_1$, $i \in \{1,\dots,n\}$, $\mu \in \{1,\dots,N_1\}$ and $M \in Z_{\mu 1 i}$.

From (5.48), (4.1), (2.19), (2.18) and assumption F_4, it is obvious that

$$\begin{aligned} \eta^{M}_{1il} &= \varphi_{l1i}\left(x^M, \widetilde{U}^{i(M)}, U^{Mi-}_{l}\right) - \varphi_{l1i}\left(x^M, \widetilde{v}^{i(M)}, v^{Mi-}_{l}\right) = \\ &= \sum_{s\in I_1} \delta_{l1is} r^{i(M)}_{s} + \rho_{l1i} r^{Mi-}_{l} = \\ &= \sum_{s\in I_1} \delta_{l1is} r^{i(M)}_{s} + h^{-1}\rho_{l1i}\left(r^{i(M)}_{l} - r^{M}_{l}\right) = \\ &= \sum_{s\in I_1} \delta_{l1is} r^{i(M)}_{s} + h^{-1}\rho_{l1i} r^{i(M)}_{l} - h^{-1}\rho_{l1i} r^{M}_{l}. \end{aligned} \tag{5.74}$$

After having grouped the expressions, these equalities are equivalent to

$$\rho_{l1i} r^{M}_{l} = h \sum_{\substack{s\in I_1\\ s\neq l}} \delta_{l1is} r^{i(M)}_{s} + \left(h\delta_{l1il} + \rho_{l1i}\right) r^{i(M)}_{l} - h\eta^{M}_{1il}. \tag{5.75}$$

Use of assumptions F_6, K_2 and definition (5.49) imply

$$\begin{aligned} \rho_{l1i}\left|r^{M}_{l}\right| &\leq h \sum_{\substack{s\in I_1\\ s\neq l}} \delta_{l1is}\left|r^{i(M)}_{s}\right| + \left(\rho_{l1i} + h\delta_{l1il}\right)\left|r^{i(M)}_{l}\right| + h\overline{\varepsilon}(k,h), \\ \left(\rho_{l1i} + h\delta_{l1il}\right)\left|r^{i(M)}_{l}\right| &\geq \rho_{l1i}\left|r^{M}_{l}\right| - h \sum_{\substack{s\in I_1\\ s\neq l}} \delta_{l1is}\left|r^{i(M)}_{s}\right| - h\overline{\varepsilon}(k,h), \\ \rho_{l1i}\left|r_l\right|^{Mi-} &\geq -\overline{\varepsilon}(k,h) - \sum_{s\in I_1} \delta_{l1is}\left|r^{i(M)}_{s}\right|, \end{aligned}$$

and hence immediately (5.59).

Next, we examine (5.60). It is clear that the difference quotient y^{Mi-} can be written in the equivalent form

$$y^{Mi-} = \begin{cases} Gy^{i(M)} - \overline{\varepsilon}(k,h), & \text{if} \quad i(M) \in Z^{0}_{\mu}, \\ 0, & \text{if} \quad i(M) \in \partial Z_{\mu}. \end{cases} \tag{5.76}$$

Since, if $i(M) \in \partial Z_\mu$, then $Gy^{i(M)} - \overline{\varepsilon}(k,h) \geq 0$ (see (5.50), (5.49) and Remark 5.1), by (5.76) we get

$$y^{Mi-} \leq Gy^{i(M)} - \overline{\varepsilon}(k,h). \tag{5.77}$$

Application of assumption F_6 and Remark 5.1 gives the estimate

$$Gy^{i(M)} - \overline{\varepsilon}(k,h) \leq -\overline{\varepsilon}(k,h)\rho_{l1i}^{-1} - \rho_{l1i}^{-1} \sum_{s \in I_1} \delta_{l1is} y^{i(M)}$$

and by (5.77), inequality (5.60).

Inequalities (5.61) and (5.62) are proven in the same manner.

The application of the maximum principle in [10] to system of inequalities (5.56)–(5.62) concludes the proof of Theorem 5.3. □

6. NUMERICAL RESULTS

To illustrate a little the class of problems which can be treated with our method we consider a system of differential equations of the form

$$\begin{cases} D_t u_1(t,x) = \operatorname{arctg}(D_x^2 u_1(t,x)) + u_2(t,x) + g_1(t,x), \\ D_x^2 u_2(t,x) + \cos(u_1(t,x)) - u_2(t,x) = g_2(t,x) \end{cases} \tag{6.1}$$

for $(t,x) \in [0,1] \times (0,1)$, with the initial-boundary conditions

$$\begin{cases} u_1(0,x) = \sin x, & x \in [0,1], \\ D_x u_1(t,0) - u_1(t,0) = \cos t - \sin t, & t \in (0,1], \\ D_x u_1(t,1) + u_1(t,1) = \cos(t+1) + \sin(t+1), & t \in (0,1], \\ u_2(t,0) - u_1(t,0) = \cos t - \sin t, & t \in [0,1], \\ u_2(t,1) - u_1(t,1) = \cos(t+1) - \sin(t+1), & t \in [0,1], \end{cases} \tag{6.2}$$

where $g_1(t,x) := \operatorname{arctg}(\sin(t+x))$, $g_2(t,x) := \cos(\sin(t+x)) - 2\cos(t+x)$. It is obvious that problem (6.1), (6.2) is a special case of (3.1), (3.2) with $n=1$, $\delta=1$, $T=1$, $E=(0,1)$ and $\Omega = [0,1] \times (0,1)$. Moreover, Assumption F and assumption (5.22) are fulfilled.

Observe that the analytical solution of (6.1), (6.2) is given explicitly by $u_1(t,x) = \sin(t+x)$, $u_2(t,x) = \cos(t+x)$. It will be compared with numerical results.

Difference scheme (4.1) corresponding to the above differential problem has the form

$$\begin{cases} a_1^M = \sin x^m, & \mu = 0, & m \in [0,N], \\ a_1^{M-} = \operatorname{arctg} a_1^{-M11} + a_2^M + g_1(x^M), & \mu \in [0, N_1 - 1], & m \in [1, N-1], \\ a_2^{-M11} + \cos a_1^M - a_2^M = g_2(x^M), & \mu \in [0, N_1], & m \in [1, N-1], \\ a_1^{M1-} - a_1^{1(M)} = \cos t^\mu - \sin t^\mu, & \mu \in [1, N_1], & m = 0, \\ a_1^{-M1} + a_1^{-1(M)} = \cos(t^\mu + 1) + \sin(t^\mu + 1), & \mu \in [1, N_1], & m = N, \\ a_2^M - a_1^M = \cos t^\mu - \sin t^\mu, & \mu \in [0, N_1], & m = 0, \\ a_2^M - a_1^M = \cos(t^\mu + 1) - \sin(t^\mu + 1), & \mu \in [0, N_1], & m = N, \end{cases} \tag{6.3}$$

where $M = (\mu, m) \in \mathbf{Z}^2$ (see Section 2).

Let $N_1 = 10^4$ and $N = 0.5 \cdot 10^2$. Then $k = 10^{-4}$ and $h = 2 \cdot 10^{-2}$. Assumption K holds for such the steps. Therefore, by Theorem 5.1, scheme (6.3) has exactly one solution $v = (v_1, v_2) \in F(Z, \mathbf{R}^2)$ and, by Theorem 5.3, the numerical method is convergent.

Let $r = (r_1, r_2) \in F(Z, \mathbf{R}^2)$ be the error of difference method (6.3), where $r_1 := U_1 - v_1$, $r_2 := U_2 - v_2$, $U_1^M := u_1(x^M)$, $U_2^M := u_2(x^M)$. Moreover, let ε^1_{max}, ε^2_{max} be the largest and ε^1_{mean}, ε^2_{mean} mean value of the errors $|r_1|$, $|r_2|$, respectively, at the moment t^μ.

Table 1. Table of errors of the difference method

t^μ	ε^1_{max}	ε^1_{mean}	ε^2_{max}	ε^2_{mean}
0.1	$6.03 \cdot 10^{-3}$	$1.70 \cdot 10^{-3}$	$6.03 \cdot 10^{-3}$	$2.68 \cdot 10^{-3}$
0.2	$7.98 \cdot 10^{-3}$	$3.13 \cdot 10^{-3}$	$7.98 \cdot 10^{-3}$	$3.95 \cdot 10^{-3}$
0.3	$9.41 \cdot 10^{-3}$	$4.58 \cdot 10^{-3}$	$9.41 \cdot 10^{-3}$	$5.19 \cdot 10^{-3}$
0.4	$1.06 \cdot 10^{-2}$	$6.06 \cdot 10^{-3}$	$1.06 \cdot 10^{-2}$	$6.43 \cdot 10^{-3}$
0.5	$1.17 \cdot 10^{-2}$	$7.56 \cdot 10^{-3}$	$1.17 \cdot 10^{-2}$	$7.68 \cdot 10^{-3}$
0.6	$1.28 \cdot 10^{-2}$	$9.09 \cdot 10^{-3}$	$1.28 \cdot 10^{-2}$	$8.92 \cdot 10^{-3}$
0.7	$1.39 \cdot 10^{-2}$	$1.06 \cdot 10^{-2}$	$1.39 \cdot 10^{-2}$	$1.01 \cdot 10^{-2}$
0.8	$1.49 \cdot 10^{-2}$	$1.22 \cdot 10^{-2}$	$1.49 \cdot 10^{-2}$	$1.13 \cdot 10^{-2}$
0.9	$1.58 \cdot 10^{-2}$	$1.38 \cdot 10^{-2}$	$1.58 \cdot 10^{-2}$	$1.26 \cdot 10^{-2}$
1.0	$1.67 \cdot 10^{-2}$	$1.54 \cdot 10^{-2}$	$1.67 \cdot 10^{-2}$	$1.38 \cdot 10^{-2}$

The table of errors (Tab. 1) is typical of difference methods. The computation was performed on a PC computer.

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