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Spatial-serial dependency in multivariate GARCH models and dynamic copulas: a simulation study

1. Introduction

Since the pioneering work of Embrechts et al. (1999), copula models have enjoyed steadily increasing popularity in finance. But there is also a lot of criticism concerning the use of copulas for modelling stochastic dependence. Mikosch (2006, p. 12) states in his famous paper with the title "Copulas: Tales and Facts" as point nine:

> Copulas completely fail in describing complex space-time dependence structures. Their focus is on spatial dependence and the related statistics (…) are aimed at iid data. It is contradictory that in risk management, where one observes a lot of dependence through time, copulas are applied most frequently.

With some exceptions (see f.e. Patton (2006)) copulas are applied to financial data after the serial dependency of this data has been filtered by adapting univariate GARCH-processes to each series (see f.e. Poon et al. (2004), Klein & Fischer (2004), Köck (2008)). Copulas will be fitted to the set of univariate filtered time series. Aside the fact that the model used for filtering could be misspecified there must still be serial dependency in the cross-relationship of the filtered time series. Otherwise, multivariate GARCH models could not be so successful

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for fitting financial data. On the other side, filtering the serial dependency by multivariate GARCH models and using a copula model for the remaining contemporaneous dependency of the filtered series is not useful. If the multivariate GARCH model is not misspecified the standardized residuals necessarily show independence. This means that the only useful copula for the filtered vector is the independence copula.

In the following we would like to investigate how these filtering strategies work. What we can see from a multivariate GARCH model is the conditional spatial dependency in the sense of Mikosch. Up to now, there is no method to identify the copula of the conditional multivariate distribution that is implicitly part of a multivariate GARCH model. In this article we show, that as long as the innovation process is a spherical one, an elliptic conditional copula like the Gaussian will be appropriate to recapture the dependence of a MGARCH model. Therefore, we use a simulation design to get an impression of the interplay between the serial dependency of multivariate GARCH models and the spatial dependency of copulas.

Within the framework of the first study we draw two series of random observations from a bivariate copula model with fixed cross dependence. We make both series serial dependent by transforming each series using two separate univariate GARCH models. Afterwards, we fit several multivariate GARCH models to the generated data. This design allows to compare the spatial dependence of the copula we start with and the kind of conditional spatial dependence in the fitted multivariate GARCH model.

In the second study we generate two serial and cross dependent times series from a bivariate GARCH model, fitting to each of the series a univariate GARCH model, filtering the two series separately and estimate several copula models to the two filtered series. This imitates the above mentioned strategy mostly used in literature and practice.

The simulation design needs a lot of specification. The considered multivariate GARCH models are the Constant Conditional Correlation (=CCC) GARCH model, the diagonal BEKK and the BEKK model. As copula families we alternatively specify the Gauss copula, the *t*-copula and the Clayton copula with normally or *t*-distributed margins.

Our paper is organized in the following way. After a short introduction into univariate and multivariate GARCH models and the concept of copulas in section 2, we give a brief description of the BEKK model and their relationship to elliptical distributions in the third section. The first simulation study that generates spatial dependent data from a copula and fits a multivariate GARCH model is presented in section 4. In the fifth section the way of simulation and fitting goes the other way around. Section six concludes.

2. Basic concepts

2.1. Multivariate GARCH models

Since there exist an abundant literature and investigations on univariate GARCH models, we just refer to Gourieroux (1997) or Jondeau and Rockinger (2007) just to mention a few. So we skip right away to the multivariate case and give a brief introduction to MGARCH models. For further studies we refer to Laurent, Bauwens (2006) and to McNeil, Frey and Embrechts (2005).

Following McNeil, Frey and Embrechts we start with the definition of a strict white noise process.

Definition 1 *$(X_t)_{t\in Z}$ is a strict multivariate white noise process (SWN) if it is a series of stochastic independent and identically distributed (iid) random vectors with finite covariance matrix.*

Let $\mathbf{1}_d$ be the *d*-dimensional identity matrix. A multivariate GARCH process can be defined as follows.

Definition 2 *Let $(Z_t)_{t\in Z}$ be a d-dimensional SWN with mean zero and covariance matrix $\mathbf{1}_d$. The process $(X_t)_{t\in Z}$ is said to be a multivariate GARCH process if it is strictly stationary and satisfies equations of the form*

$$X_t = \Sigma_t^{1/2} Z_t, \forall t \in Z$$

where $\Sigma_t^{1/2} \in \boldsymbol{R}^{dxd}$ is the Cholesky decomposition of a positive-definite matrix Σt, which is measurable with respect to $\Im_{t-1} = \sigma\left(\{X_s : s \le t-1\}\right)$, the filtration of the process up to time $t-1$.

In most applications the innovation process follows a multivariate normal distribution, but especially for modeling daily returns a multivariate *t*-distribution or other distributions with fatter tails than the normal one would be preferable, as long as it has zero mean and the covariance matrix takes the form $\mathbf{1}_d$.

There exist various different characterizations of multivariate GARCH models like the Vech-model of Kraft and Engle (1982) or the EGARCH of Nelson (1991). We just consider two different very popular GARCH models for our investigations: The CCC-GARCH model of Bollerslev because of its simplicity and in contrast the BEKK-model of Baba, Engle, Kraft and Kroner (1995), which involves many parameters to be estimated.

Definition 3 *The process* $(X_t)_{t\in Z}$ *is a CCC-GARCH process if it is a process with the general structure given in Definition 2, such that the conditional covariance matrix is of the form* $\sum_t = D_t P_c D_t$, *where*

- P_c *is a constant positive-definite correlation matrix*
- D_t *is a diagonal volatility matrix with elements* $\sigma_{t,k}$ *satisfying:*

$$\sigma_{t,k}^2 = \alpha_{k0} + \sum_{i=1}^{p_k} \alpha_{ki} X_{t-i,k}^2 + \sum_{j=1}^{q_k} \beta_{kj} \sigma_{t-j,k}^2, \qquad k = 1, \dots d$$

where $\alpha_{k0} > 0$, $\alpha_{ki} \geq 0$, $i = 1, \dots, q_k \beta_{kj} \geq 0$, $j = 1, \dots, q_k$.

There are some advantages of this approach, like the reduced number of parameters. Another one is, if the conditional variances of D_t are all positive then $\sum_t$ is, too. The main problem of the CCC-GARCH model, however, lies in the assumption of constant correlations between e.g. two financial assets, which is rejected by empirical studies.

A different specification of $\sum_t$ leads to the very flexible and very popular BEKK model described by Engle and Kroner (1995) that overcomes the disadvantage of constant correlations.

Definition 4 *The process* $(X_t)_{t\in Z}$ *is a BEKK-GARCH process if it is a process with the general structure given in Definition 2, and if the conditional covariance matrix* $\sum_t$ *is given by:*

$$\sum_t = C + \sum_{i=1}^{p} A_i' X_{t-i} X_{t-i}' A_i + \sum_{j=1}^{q} B_j' \sum_{t-j} B_j \tag{1}$$

where C is a (d, d) positive definite and symmetric matrix and A_i *and* B_j *are some (d, d)-matrices.*

As long as C is positive definite $\sum_t$ also is. Though the BEKK model is one of the most flexible multivariate GARCH models its main disadvantage is the huge amount of parameters to specify. For a simple BEKK(1,1) model dealing with two time series, 11 parameters have to be estimated. To reduce this problem one constrains the model to the diagonal BEKK, where $A'_i s$ and $B'_i s$ are diagonal matrices or to the scalar BEKK model, where the $A's$ and B_s are simply scalars.

Another constraint which is often used in practice is to replace the matrix C by the long-run covariance matrix equal to the sample covariance matrix.

2.2. Copulas

In order to deal with the difficulties of defining joint distributions with arbitrary margins and to receive a new possibility to measure the dependency between time series, one draws back to the copula concept first introduced by Sklar (1959).

In the following we consider the bivariate case. Note that a *d*-dimensional generalization could be made. However, our simulation design does not suffer from the restriction to the bivariate case.

Definition 5 A *copula is a bivariate function* C: $[0,1] \times [0,1] \to [0,1]$ *with the following properties:*

- $C(0, v) = C(u,0) = 0$, *and* $C(u,1) = u$, $C(1,v) = v$
- $C(u, v)$ *ist two-increasing, that is:*

$$C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0, \forall u_1, u_2, v_1, v_2 \in [0,1]$$

where $u_1 \geq u_2$, $v_1 \geq v_2$.

The importance of copulas is summarized in the well-known theorem of Sklar:

Theorem 1 *Let* F_X *and* F_Y *be the marginal distributions of some real valued, continuous random variables X and Y and G the joint distribution function of (X, Y). Then there exists a copula C such that, for all* $(x,y) \in \boldsymbol{R}^2$:

$$G(x, y) = C\left(F_X(x), F_y(y)\right). \tag{2}$$

Moreover, if F_X *and* F_Y *are continuous, then C is unique.*
Conversely, if F_X *and* F_Y *are the distributions of X and Y, respectively, the function G defined by (2) is a joint distribution function with marginal distributions* F_X *and* F_Y.

The theorem says that we can decompose a bivariate cumulative distribution function into its marginal distributions and an unique copula if the marginal cumulative distribution functions are continuous. The second assertion is the more important one for our context. By defining two marginal distributions and taking one copula we are capable to create any bivariate cumulative distribution function. As indicated above the main target of the copula approach is to model dependencies beyond the correlation. A popular measure involving copulas is Kendall's τ (see Nelsen 2006) as a measure of concordance.

Theorem 2 *Let X and Y be continuous random variables whose copula is C. Then Kendall's tau is defined as:*

$$\tau(X,Y) = 4\int\int_{[0,1]^2} C(u,v)\,dC(u,v) - 1 = 4E\left[C(U,V)\right] - 1 \tag{3}$$

Note that τ (X, Y) is bounded between [–1, 1]. As there exists an abundant literature dealing with copula we just refer to Joe (1997) and Nelson (2006) for further details. We now present three copulas we are dealing with in our simulation: The Gaussian, the Student-t (*t*-copula for short) and the Clayton copula. The first two belong to the so called elliptical family and the third to the Archimedean family.

Definition 6 *The Gaussian copula is defined by the following cumulative distribution function:*

$$C_\rho(u, v =) \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{s^2 - 2\rho st + t^2}{2(1-\rho^2)}\right) dsdt,$$

∀ *where* $\psi(u)$ *is the cdf of a standard normal distribution.*

The parameter $\rho \in [-1,1]$ is Pearson's correlation coefficient. Kendall's τ can be achieved by the following formula:

$$\tau(C_\rho) = \frac{2}{\pi}\arcsin(\rho) \tag{4}$$

Note that (4) holds true also for the *t*-copula and for other members of the elliptical family (see Lindskog et al. (2004)).

Definition 7 *The t-copula is defined by*:

$$C_{\rho,v} = \int_{-\infty}^{t_v^{-1}(u)} \int_{-\infty}^{t_v^{-1}(\upsilon)} \frac{\Gamma\left(\frac{v+2}{v}\right)}{\Gamma\left(\frac{v}{2}\right)\pi v\sqrt{1-\rho^2}} \left(1 + \frac{\varphi' R^{-1}\varphi}{v}\right)^{-\frac{v+2}{2}} d\varphi,$$

where $\phi = \left(t_v^{-1}(u), t_v^{-1}(\upsilon)\right)$, *and R is the correlation matrix with correlation coefficient* ρ *and v are the degrees of freedom,* ∀ *and* $t_v(u)$ *is the cdf of a t-distributation with v do F.*

Among the class of non-elliptical copulas, archimedean copulas enjoy great popularity.

Theorem 3 *Let* ϕ *be a continuous, strictly decreasing function from* $[0,1]$ *to* $[0, \infty)$ *such that* $\phi(1)$ 0 *and let* ϕ^{-1} *be the inverse of* ϕ*. Then, the function from* $[0,1]^2 \to [0,1]$ *given by:*

$$C(u,\upsilon) = \phi^{-1}\left(\phi(u) + \phi(\upsilon)\right)$$

is a copula if and only if ϕ ***is convex.***
Moreover, if ϕ^{-1} *is twice continuous differentiable Kendall's tau is given by*

$$\tau(C) = 1 + 4\int_0^1 \frac{\phi(t)}{\phi'(t)}dt$$

The function ϕ *is called the generator of the copula.*

Definition 8 *For* $\phi(t) = (t^{-\theta} - 1)/\theta$, *with* $\theta \in [-1, \infty)\backslash\{0\}$, *the Clayton copula is obtained:*

$$C_\phi = \max\left(\left(u^{-\theta} + \upsilon^{-\theta} - 1\right)^{-1/\theta}, 0\right)$$

Kendall's tau is given by

$$\tau(C_\theta) = \frac{\theta}{\theta + 2}$$

2.3. Connecting copulas and GARCH models

Hu (2005), Jondeau and Rockinger (2006) and Patton (2000a,b) suggest to replace the unconditional margins of a copula with conditional margins coming from univariate GARCH models. This leads to a special case of the so-called copula based multivariate dynamic (CMD) model.

Starting with Sklar's theorem for conditional distributions Chen characterizes a CMD model by

$$F_{X_t \cdot Y_t}(x, y, | \boldsymbol{F}_{t-1}; \lambda) = C\left(F_{X_t}(x \,|\, \boldsymbol{F}_{t-1}; \gamma_X), F_{Y_t}(y \,|\, \boldsymbol{F}_{t-1}; \theta)\right),$$

Where $\lambda := (\gamma'_X, \gamma'_Y, \theta)'$ is the parameter vector. γ_X and γ_Y contain f.e. the GARCH parameters of the marginal distributions and θ is the dependence parameter. Due to Chen (2007) the key feature of a CMD model is the separability of γ_X and γ_Y. This separability insures that the parameters of the conditional marginal distribtuions can be estimated separately before estimating the copula parameter in a second step.

The above introduced CCC model and the dynamic conditional correlation (=DCC) model of Engle (2002) and Tse and Tsui (2002) are special cases of the CMD model where a normal copula and GARCH margins with normal innovations are considered.

Other models, like the VEC model of Bollerslev, Engle and Wooldridge (1988) and the BEKK model, don't have separable parameters and are not members of the class of CMD models.

3. BEKK models and elliptical Distributions

In this section we show, that the unconditional distribution of multivariate GARCH models belongs to the elliptical family when the Innovation process is spherical distributed. We further show under which circumstances a BEKK model can be decomposed into its margins and common dependency part. First following McNeil, Frey and Embrechts we give the basic definitions and properties of elliptical distributions.

Definition 9 *A random vector* $X = (X_1,\dots, X_d)$ *is spherically distributed if for every orthogonal map* $U \in \mathbf{R}^{dxd}$*:*

$$UX \overset{d}{\sim} X$$

and we write $X \sim S_d(\psi)$ *with* ψ *the generator function of the distribution.*

It is known that the multivariate normal distribution and the multivariate t-distribution belong to the spherical family.

Definition 10 *X has an elliptical distribution if:*

$$X \overset{d}{\sim} \mu + AY$$

with $Y \sim S_d(\psi)$ *and A some non-stochastic* $d \times k$ *matrix with* $\Sigma = AA'$ *and we write* $X \sim E_d(\mu, \Sigma, \psi)$.

It follows from this definition that $\Sigma^{-1/2}(X-\mu) \sim S_d(\psi)$. Elliptical distributed rv have some interesting features as following corollary shows:

Corollary 1 *If X has an elliptical distribution then*:

- *The margins are also elliptically distributed with* $X_i \sim E_1(\mu_i, \Sigma_{ii}, \psi)\ \forall i$.
- *The conditional distribution* $X_t|\Im_{t-1}$ *is also elliptic but with a possibly different generator.*
- *Let* $(X_1, X_2) \sim E_2(\mu, \Sigma, \psi)$ *then the rank correlation* τ *is given for all members of the elliptic family by the formula:*

$$\tau(X_1, X_2) = 2/\pi \arcsin(\rho).$$

With those results we are able to show that MGARCH models like the BEKK models are conditionally elliptically distributed.

Theorem 4 *Let* X_t *follow a MGARCH model, then* $X_t|\Im_{t-1}$ *is elliptically distributed if and only if* Z_t *is spherically distributed.*

Proof:
Write $X_t = \mu(\theta)_t + \Sigma_t^{1/2}(\theta)Z_t$. Without any loss of generality we set $\mu_t \equiv 0$ and check the conditions for elliptically distributed rv:

- Σ_t is measurable with respect to $\Im_{t-1}$ and non stochastic.
- Σ_t is positive definite a.s. $\forall t \in \mathbf{Z}$ and thus the Cholesky-decomposition is possible. We now set $\Sigma_t^{1/2} = A$.
- $Z_t \sim S_d(\psi) \forall t \in \mathbf{Z}$ and thus $X_t|\Im_{t-1} \sim E_d(0, \Sigma_t, \psi)$.

The way back is straightforward, as $X_t \sim E_d$. Then according to the definition of an elliptical distributed rv Z_t must be a spherical rv.

□

The theorem also shows, that the generator ψ is the same for the innovation process and the conditional cdf of X_t. For example let Z_t be iid and $Z_t \sim N(0,1_d)$, $\forall \in \mathbf{Z}$ then $X_t|\Im_{t-1}$ is also normally distributed and so the margins.

We now use these properties to examine the class of BEKK models. Though they are flexible and popular MGARCH models, their main disadvantage lies in the huge amount of parameters to estimate. Thus in practice the margins of a (multivariate) times series are estimated in a first step, and then, in the second step, the dependency parameters will be estimated. This two step estimation leads to reasonable results for example for CCC or DCC models. So, if the true process follows a BEKK model, this procedure will be inaccurate as the BEKK model can not be separated into its margins and a dependency function, like the CCC or DCC models. But if one does nonetheless, the error that occurs might be appraised.

In the following discussion we just refer to the BEKK(1,1) model for notation simplicity. For a BEKK(1,1) we have the following variance and covariance equations:

$$\sigma_{t,1}^2 = c_{11}^2 + a_{11}^2 X_{t-1,1}^2 + b_{11}^2 \sigma_{t-1,1}^2 + 2a_{11}a_{12}X_{t-1,1}X_{t-1,2} +$$

$$+a_{12}^2 X_{t-1,2}^2 + 2b_{11}b_{12}\sigma_{t-1,12} + b_{12}^2\sigma_{t-1,2}$$

$$\sigma_{t,2}^2 = c_{22}^2 + a_{22}^2 X_{t-1,2}^2 + b_{22}^2 \sigma_{t-1,2}^2 + 2a_{22}a_{21}X_{t-1,1}X_{t-1,2} +$$

$$+a_{21}^2 X_{t-1,1}^2 + 2b_{22}b_{21}\sigma_{t-1,12} + b_{21}^2\sigma_{t-1,1}$$

$$\sigma_{t,12} = c_{12}^2 + \left(a_{11}a_{22} + a_{12}a_{21}\right) X_{t-1,1}X_{t-1,2} + a_{11}a_{21}X_{t-1,1}^2 +$$

$$+a_{22}a_{21}X_{t-1,12}^2 + \left(b_{11}b_{22} + b_{12}b_{21}\right)\sigma_{t-1,12} + b_{11}b_{21}\sigma_{t-1,1}^2 + b_{22}b_{12}\sigma_{t-1,2}^2 .$$

Considering the above mentioned properties it is essential that $X_{t,1}|\Im_{t-1} \sim E\left(0,\sigma_{t,1}^2\right)$ and $X_{t,2}|\Im_{t-1} \sim E\left(0,\sigma_{t,2}^2\right)$. If one tries to approximate the true marginal distribution denoted by E^0 with a standard univariate GARCH(1,1) model the question arises 'how far' the elliptic distribution is away from E^0.

To answer this question we introduce the so-called Kullback-Leibler information (relative entropy) for measures.

Definition 11

$$D\left(P \parallel Q\right) = \int p \log \frac{p}{q} dx$$

where p and q are the densities of the probability measures P and Q, respectively.

The Kullback-Leibler information has the interesting property of being equal to zero if p equals q a.s. and for every other measure being strictly positive. Let P denote the distribution function of an estimated univariate GARCH(1,1) model and Q the distribution function of the true marginal model. We then have following assertion.

Theorem 5 *If X_t follows a diagonal BEKK(1,1) process, the margins can be consistently estimated by an univariate GARCH(1,1) model, with the following parameter equations for the i-th model:*

$$\alpha_{0i} = c_{ii}^2$$

$$\alpha_{1i} = a_{ii}^2$$

$$\beta_{1i} = b_{1,ii}^2$$

Furthermore the correlation ρ_t will be time-varying with the formula:

$$\rho_t = \frac{\sigma_{t,12}}{\sqrt{\sigma_{t,1}^2}\sqrt{\sigma_{t,2}^2}},$$

where

$$\sigma_{t,12} = c_{12} + a_{11}a_{22}X_{t-1,1}X_{t-1,2} + b_{11}b_{22}\sigma_{t-1,12}\,.$$

Proof: The density function of an elliptical distribution is given by:

$$f(x) = \frac{1}{|\Sigma|^{1/2}} g\left((x-\mu)'\Sigma^{-1}(x-\mu)\right)$$

In our case this formula reduces to:

$$f(x) = \frac{1}{\sigma_{t,i}} g\left(x_t^2 \sigma_{t,i}^{-2}\right)$$

where *i* is the *i*-th marginal GARCH model of the diagonal BEKK(1,1) model. The Kullback-Leibler information is then:

$$D(P\|Q) = \int p\,log\frac{p}{q}dx = \int p\,log\left(\frac{\sigma_{t,i}}{\sigma_t}\cdot\frac{g\left(x_t^2/\sigma_t^2\right)}{g\left(x_t^2/\sigma_t^2\right)}\right)dx$$

where p is the density of an elliptical distributed rv, with generator function g and variance σ_t^2, which is characterized by an univariate GARCH model. Thus Q

is the unknown conditional distribution of a diagonal BEKK model and P is the conditional distribution of an univariate GARCH model. The only expression of interest now is the log expression, since if it equals 1, both densities are the same apart from a constant and $D(P \parallel Q) = 0$ i.e. both distribution functions agree. We get:

$$log \frac{\sigma_{t,i}}{\sigma_t} + log\left(\frac{g\left(x_t^2 / \sigma_t^2\right)}{g\left(x_t^2 / \sigma_{t,i}^2\right)}\right)$$

The second expression equals zero only if $\sigma_{t,i}^2 = \sigma_t^2$ as both densities have the same density generator, and consequently $\sigma_{t,i} = \sigma_t$, as $\sigma_{t,i}$ has to be positive. We now get:

$$c_{ii}^2 + a_{ii}^2 X_{t-1,i}^2 + b_{ii}^2 \sigma_{t-1,1}^2 = \alpha_0 + \alpha_1 X_{t-1,i}^2 + \beta_1 \sigma_{t-1}^2$$

Comparison of the two sides leads to the result. The consistency follows from the properties of (Q)ML estimation of the parameters.

The second result is trivial since $a_{12} = a_{21} = b_{12} = b_{21} = 0$.

□

This result is quite general for all density generator g of a spherically distributed rv.

One of the main disadvantages of the BEKK specification is that one has to estimate all unknown parameters simultaneously. This may be a source of bias, see Baur (2007). Thus some two step procedure would be grateful. But unfortunately this can be done only for the diagonal or scalar BEKK models. With the results obtained so far, we propose for the following estimation method for diagonal BEKK models:

- Estimate the margins with an univariate GARCH(1,1)-model. With the formula given in theorem 5 we get the parameter for the coefficients.
- In a second step, estimate the conditional Kendall's tau resp. the correlation coefficient from the data and get the parameter $c_{0,12}$.

4. First simulation study

We now present the general setup for our Monte-Carlo-Simulation. In a first simulation study we investigate what happens, when two univariate

GARCH(1,1) models with different innovations processes are simulated and (contemprarily) connected via different copulas, and then a bivariate CCC--GARCH(1,1) model or a BEKK(1,1) model are estimated also with different innovation processes.

In a first step a bivariate 2 x 2500-matrix of random numbers generated from one of the three copulas is simulated. Afterwards the two 1 x 2500 vectors with the random numbers are transformed with the quantile function of a standard normal, respectively of a $t(5)$-distribution.

Then, an univariate GARCH(1,1) with normal $N(0,1)$ or $t(5)$-innovations is adapted. Finally both time series are merged again.

We repeat this procedure 1000 times. Then at first a bivariate CCC-GARCH, a diagonal BEKK (=Diag-BEKK) and a full BEKK[1] model is estimated. We look if all three models are capable to re-find the dependency induced by the different copulas and we look at, how sensitive the bivariate GARCH models react if, for instance, the true innovation process is $t(5)$ distributed and we estimate nevertheless with normal distributed innovations, i.e. the classical QML method like in Bollerslev and Wooldridge (1992), for instance. To make the results comparable we let the coefficients of the univariate GARCH (1,1) models constant, i.e. $\alpha_0 = 0.05$, $\alpha_1 = 0.1$, $\beta = 80$. We choose different parameters for the three copulas; for Gauss $\rho = 0.1, 0.4, 0.7$, the t $\rho = 0.1, 0.4, 0.7$, $df = 5$ and the Clayton $\theta = 0.1362, 0.7099, 1.9497$ which corresponds approximately to the Kendall's tau of a Gaussian/t-copula with $\rho = 0.1, 0.4$ respectively $\rho = 0.7$. The results are summarized in the next section.

We refer to the three different simulation designs as:

- case 1: $N(0,1)$ distributed univariate innovations and $N(0,1)$ bivariate innovations
- case 2: $t(df = 5)$ distributed univariate innovations and $N(0,1)$ bivariate innovateons
- case 3: $t(5)$ distributed univariate innovations and $t(5)$ bivariate innovations

The results are summarized in table 1. The table contains the estimated value of the linear correlation coefficient ρ, the value of the maximum log-likelihood value LL.

As it can be easily seen, the CCC, the Diag-BEKK and the BEKK models estimate the constant correlations induced by the Gaussian copula and the t-copula very well. As a first result it can be seen, that misspecification of the

[1] from now on just BEKK

univariate GARCH models doesn't effect the estimation of the correlation between the two time series, which is a quite reasonable result, as it stands in a line with the results of Bollerslev and Wooldridge (1992) for dynamic models for conditional means and covariances. As one can see, as long as the copulas belong to the elliptical family, all bivariate GARCH models specify the correlation and the degrees of freedom quite well, where the Kendall's tau induced by the Clayton copula is estimated worse, if the innovation processes change from the classical normal-normal approach to case 2 and 3 with $\theta = 0.7099$ or $\theta = 1.9497$. The CCC nearly re-find the Kendall's tau induced by the Clayton Copula even though this CMD approach generates a process with more probability mass in the lower tails. Remember that we choose the parameter of the Clayton copula in such a way, that it is equal to the Kendall's tau of a Gaussian respectively *t*-copula with correlation parameter $\rho = 0.1$, 0.4 resp. 0.7. As it should be expected the CCC model, as it is nested in the CMD context performs throughout the simulation quite well and even under misspecification (case 2) re-find the correlation no matter which copula was chosen. Note that even the degrees of freedom of the univariate innovation process in case 3 were correctly specified, except the data came from a Clayton-copula-GARCH model. On the other side the both BEKK models were not able to specify the degrees of freedom correctly and they remain about 11.1 even when models were simulated with $\rho = 0.01$ or $\rho = 0.99$ and it seems as the models were getting more extensive from case 2 to case 3 for the gaussian copula-GARCH simulation, the BEKK model failed completely to re-find correlation resp. Kendall's tau. This failure may due to numerical instabilities when the both BEKK models estimate all parameters simultaneously and didn't disappear even after some additional simulations. Interestingly, when the data came from a *t*-copula-GARCH simulation, the error didn't occur. In the Clayton-copula-GARCH context there is also a considerable underestimation of the simulated correlation resp. Kendall's tau, but this error is much smaller than in the gaussian copula-GARCH simulation. Also quite obvious is the decrease of the *LL* of the both BEKK models from case 1 to case 2, which is at first sight not surprising as case 2 is the misspecified one. But in scenario 3 the *LL*s are much lower than in scenario 2, when the parameters of the copulas are high ($\rho = 0.7$ resp. $\theta = 1.9497$). Another quite astonishing result is the dramatic decrease of the *LL* from case 2 to 3 for a CCC model, even though the Kendall's τ and the *df* of the *t*-distributed innovations were correctly specified. This might due to the numerical problems a minimization routine is facing when the negative of a log-likelihood of an extensive GARCH model has to be evaluated, estimating all parameters simultaneously.

Table 1

Fitting a multivariate GARCH model to data from a copula with unvariate GARCH margins

Innovations N(0,1)&N(0,1)						
GARCH	**CCC**		**Diag–BEKK**		**BEKK**	
Copula	ρ	**LL**	ρ	**LL**	ρ	**LL**
G:ρ = 0.1	0.102	403.9	0.102	404.0	0.102	403.3
t:ρ = 0.1	0.103	406.2	0.100	405.6	0.100	406.4
C:θ = 0.1362	0.119	688.7	0.118	688.2	0.118	688.2
G:ρ = 0.4	0.401	608.4	0.401	608.4	0.401	607.9
t:ρ = 0.4	0.396	606.0	0.396	605.4	0.396	606.1
C:θ = 0.7099	0.441	848.3	0.441	847.9	0.441	847.9
G:ρ = 0.7	0.700	1233.1	0.700	1232.7	0.700	1233.1
t:ρ = 0.7	0.695	1215.3	0.695	1215.4	0.694	1214.8
C:θ = 1.9497	0.716	1419.1	0.716	1414.5	0.716	1414.5
Innovations t(df=5)&N(0,1)						
	CCC		**Diag–BEKK**		**BEKK**	
	ρ	**LL**	ρ	**LL**	ρ	**LL**
G:ρ = 0.1	0.098	–868.0	0.097	–868.6	0.097	–869.3
t: = 0.1	0.102	–863.1	0.101	–864.7	0.101	–862.6
C:θ = 0.1362	0.127	–314.9	0.127	–315.6	0.127	–315.7
G:ρ = 0.4	0.390	–671.1	0.390	–672.5	0.390	–671.1
t:ρ = 0.4	0.402	–663.6	0.402	–665.1	0.401	–663.25
C:θ = 0.7099	0.464	–230.0	0.465	–230.9	0.465	–230.8
G:ρ = 0.7	0.689	–76.0	0.688	–76.0	0.688	–77.1
t:ρ = 0.7	0.701	–38.0	0.701	–38.9	0.701	–37.6
C:θ = 1.9497	0.723	299.15	0.724	298.8	0.724	298.8
Innovations t(df=5)& t(df=5)						
	CCC		**Diag–BEKK**		**BEKK**	
	ρ/*df*	**LL**	ρ/*df*	**LL**	ρ/*df*	**LL**
G:ρ = 0.1	0.097/5.0	–4998.3	0.089/11.1	–692.8	0.090/11.1	–693.4
t:ρ = 0.1	0.100/5.2	–4961.7	0.096/11.1	–636.7	0.095/11.1	–636.1
C:θ = 0.1362	0.126/9.4	–5398.4	0.109/11.1	–266.0	0.109/11.1	–266.0
G:ρ = 0.4	0.389/5.0	–4963.2	0.380/11.1	–507.3	0.378/11.1	–507.1
t:ρ = 0.4	0.401/5.2	–4904.9	0.399/11.1	–453.3	0.399/11.1	–452.9
C:θ = 0.7099	0.466/6.4	–5110.1	0.401/11.1	–168.4	0.402/11.1	–168.1
G:ρ = 0.7	0.688/5.0	–4945.9	–0.485/11.1	–1676.1	–0.475/11.1	–1676.1
t:ρ = 0.7	0.700/5.3	–4906.8	0.705/11.1	–214.8	0.720/11.1	–245.8
C:θ = 1.9497	0.723/6.2	–5074.2	0.365/11.1	–634.22	0.360/11.1	–620.4

5. Simulation design

At first we simulate a bivariate CCC-GARCH(1,1) process of length 2500 with either normally or t(5) distributed innovations, then split the 2 x 2500 matrix into two time series vectors and fit for each vector an univariate GARCH(1,1) process with also normally and *t*-distributed innovations, respectively. Afterwards we estimate the parameters of our three copulas one by one and look, if the copula-GARCH models are able to re-find the constant correlation of the CCC-GARCH(1,1) model.

We do this also for a Diag-BEKK(1,1) and a BEKK(1,1) model. As in step one we repeat the procedure 1000 times. As the CCC-GARCH induces constant correlation we again choose three different correlation coefficients

- *Model 1:* $\rho = 0.1$
- *Model 2:* $\rho = 0.4$
- *Model 3:* $\rho = 0.7$

For the two BEKK models the situation is slightly different, as the correlation can only be determined indirectly via different parameter models. We refer to the following parameters as model 1 to 3. For the Diag-BEKK we have:

- *Model 4:* $C = \begin{pmatrix} 1.1 & 0.3 \\ 0 & 0.09 \end{pmatrix}, A = \begin{pmatrix} 0.25 & 0 \\ 0 & 0.05 \end{pmatrix}, B = \begin{pmatrix} 0.9 & 0 \\ 0 & 0.05 \end{pmatrix}$

- *Model 5:* $C = \begin{pmatrix} 0.6 & 0.3 \\ 0 & 0.4 \end{pmatrix}, A = \begin{pmatrix} 0.15 & 0 \\ 0 & 0.01 \end{pmatrix}, B = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.1 \end{pmatrix}$

- *Model 6:* $C = \begin{pmatrix} 0.1 & 0.3 \\ 0 & 0.05 \end{pmatrix}, A = \begin{pmatrix} 0.25 & 0.3 \\ 0 & 0.05 \end{pmatrix}, B = \begin{pmatrix} 0.7 & 0 \\ 0 & 0.05 \end{pmatrix}$

And for the BEKK(1,1) models we have in analogy to Hafner & Herwartz (2008):

- *Model 7:* $C = \begin{pmatrix} 1.1 & 0.3 \\ 0 & 0.9 \end{pmatrix}, A = \begin{pmatrix} 0.25 & -0.05 \\ 0.05 & 0.25 \end{pmatrix}, B = \begin{pmatrix} 0.9 & -0.05 \\ 0.05 & 0.9 \end{pmatrix}$

- *Model 8:* $C = \begin{pmatrix} 0.6 & 0.3 \\ 0 & 0.4 \end{pmatrix}, A = \begin{pmatrix} 0.15 & -0.01 \\ 0.01 & 0.15 \end{pmatrix}, B = \begin{pmatrix} 0.5 & -0.1 \\ 0.1 & 0.5 \end{pmatrix}$

- *Model 9:* $C = \begin{pmatrix} 0.1 & 0.3 \\ 0 & 0.05 \end{pmatrix}, A = \begin{pmatrix} 0.25 & -0.05 \\ 0.05 & 0.25 \end{pmatrix}, B = \begin{pmatrix} 0.9 & -0.05 \\ 0.05 & 0.9 \end{pmatrix}$

The different models induce different correlations resp. rank correlations. As mentioned above the Clayton copula is unable to measure correlation, we just refer to Kendall's tau and convert the coeficient of all three copulas to Kendall's tau. We get following taus for our simulation scenarios (table 2 and 3):

Table 2

Kendalls τ values for the simulation scenarios

	Innovations N(0,1)								
	M1	**M2**	**M3**	**M4**	**M5**	**M6**	**M7**	**M8**	**M9**
τ	0.07	0.28	0.52	0.077	0.36	0.49	0.07	0.40	0.49

	Innovationst(df=5)								
	M1	**M2**	**M3**	**M4**	**M5**	**M6**	**M7**	**M8**	**M9**
τ	0.70	0.28	0.51	0.42	0.44	0.88	0.07	0.40	0.49

Table 3

Fitting a copula with univariate GARCH margins to data steming from a multivariate GARCH model

Innovations N(0,1)&N(0,1)									
Copula/ GARCH	**CCCC-GARCH (1.1)**			**Diag-BEKK(1.1)**			**Full-BEKK(1.1)**		
Model	**M1**	**M2**	**M3**	**M4**	**M5**	**M6**	**M7**	**M8**	**M9**
Gauss: τ	0.069	0.261	0.493	0.078	0.359	0.486	0.071	0.401	0.487
sd	0.019	0.016	0.008	0.019	0.012	0.009	0.019	0.011	0.009
LL	–0.5	–107.9	–537.1	–13.4	–313.6	–611.6	–0.5	–320.4	–530.0
t: τ	0.062	0.260	0.492	0.076	0.358	0.485	0.066	0.399	0.488
sd	0.020	0.016	0.009	0.020	0.013	0.009	0.009	0.012	0.020
df	29.69	29.73	29.87	29.80	29.89	29.80	29.05	29.84	26.94
LL	1.2	–160.3	–728.6	–17.6	–422.4	–815.8	2.0	–428.9	–715.4
Clayton: τ:	0.045	0.195	0.382	0.048	0.271	0.376	0.048	0.304	0.380
sd	0.024	0.031	0.043	0.024	0.035	0.042	0.024	0.037	0.042
LL	–0.5	–107.9	–537.1	–13.4	–313.6	–611.6	–0.5	–320.4	–530.0

Innovations t(df=5)&N(0.1)									
Copula/ GARCH	**CCCC-GARCH (1.1)**			**Diag-BEKK(1.1)**			**Full-BEKK(1.1)**		
Model	**M1**	**M2**	**M3**	**M4**	**M5**	**M6**	**M7**	**M8**	**M9**
Gauss: τ	0.070	0.270	0.502	0.408	0.432	0.880	0.071	0.413	0.493
sd	0.019	0.015	0.008	0.011	0.010	0.0	0.019	0.011	0.008
LL	5.3	–156.2	–474.0	–557.1	–630.9	–4200.5	1.1	–479.6	–753.0
t: τ	0.065	0.271	0.504	0.412	0.434	0.881	0.067	0.414	0.494
sd	0.020	0.016	0.008	0.012	0.011	0.001	0.020	0.011	0.009
df	29.61	28.95	26.89	9.61	25.680	18.554	29.788	28.091	27.106
LL	2.2	–155.1	–469.3	–577.3	–633.5	–4218.3	–1.0	–483.6	–756.3
Clayton: τ:	0.047	0.203	0.391	0.322	0.332	0.816	0.047	0.315	0.382
sd	0.024	0.032	0.044	0.038	0.039	0.175	0.024	0.038	0.043
LL	–0.1	–93.1	–337.0	–436.1	–471.7	–3429.4	–1.3	–343.9	–543.3

Table 3 cont.

Innovations t(df=5)&t(df=5)									
Copula/ GARCH	CCCC-GARCH (1.1)			Diag-BEKK(1.1)			Full-BEKK(1.1)		
Model	M1	M2	M3	M4	M5	M6	M7	M8	M9
Gauss: τ	0.071	0.271	0.503	0.407	0.432	0.880	0.070	0.413	0.493
sd	0.008	0.015	0.019	0.011	0.010	0.001	0.019	0.011	0.008
LL	0.8	–140.5	–476.8	–554.9	–631.3	–4195.2	0.1	–489.4	–756.8
t: τ	0.504	0.272	0.067	0.411	0.434	0.881	0.067	0.414	0.494
sd	002	0.016	0.008	0.012	0.011	0.001	0.020	0.011	0.009
df	29.680	28.931	26.876	9.788	26.036	18.312	29.779	28.106	27.474
LL	–1.0	–139.2	–471.7	–574.6	–633.7	–4214.0	0.2	–492.0	–763.0
Clayton: τ :	0.048	0.204	0.392	0.322	0.332	0.816	0.046	0.315	0.383
Sd	0.024	0.032	0.044	0.038	0.039	0.175	0.020	0.011	0.009
LL	–1.2	–102.0	–338.8	–434.3	–472.2	–3427.2	–1.5	–347.2	–544.1

We now present the results of the simulation. In all simulation circumstances it can be seen that a CMD model with Gaussian or *t*-copula re-finds the (rank) correlation of the simulated data regardless whether a CCC, Diag-BEKK or BEKK model is used. The CMD model with a Clayton copula in contrast isn't able to capture the assumed Kendall's tau. This underlines the importance of a correct copula specification within a model. As shown by Chen (2007) moment-based specification tests for copulas are able to detect misspecification of a selected copula model.

As the *t*-copula in the most cases has a huge number of degrees of freedom, it differs not really from a Gaussian copula. The degrees of freedom reduce when the simulated correlation resp. Kendall's tau is increased. Through all the different simulation designs it can be seen that a QML estimation does not perform worse than the estimation with the correct innovation process, when one is interested in the dependency between time series. The asymptotic properties shown be White (1994) and Bollerslev, Wooldridge (1992) for the QML resp. a two step QML (2SQML) estimation of GARCH can be re-find even in relatively small sample sizes as shown by our work. Of course further studies have to be made to investigate the differences and the advantages of CMD-models with different error distributions, but in our study the misspecified ones perform well against the background of the additional computational burden a complicate error distribution implicates. The copula-GARCH models seems also suitable for BEKK and Diag-BEKK models even though these are in a different class of GARCH models. For the Diag-BEKK this is the result of the section 3, where the estimation method used in this simulation study is proposed. For the general BEKK model this is due to the assertion of theorem 4, that because of the spherical error distribution, we stay in an elliptic world, so that the both elliptic copulas perform very well in re-finding the correlation induced by the BEKK model.

6. Conclusion

In our study it can be seen, that CMD models are flexible tools for investigating different times series with GARCH structure for the squared residuals. This holds even under misspecification and in situations where GARCH models are simulated which are direct generalization of the univariate ones like the simulated BEKK and Diag-BEKK. We proved that as long as the error distribution of a GARCH model belongs to the spherical distribution family, the conditional distribution of such a model follows an elliptical distribution. Moreover a two step estimation procedure for diagonal BEKK models is established.

In praxis one may be interested in the dependency of different time series and therefore have an eye for the Kendall‘s tau or correlation between the two series. In this situation a CMD model can be a helpful tool to investigate the dependency structure and estimate coevally less parameters than in a BEKK model. The other way around instead shows that when data where simulated from a CMD model the only GARCH model which seems to fit the data in all circumstances were the CCC-GARCH, which isn‘t surprising as it is a subclass of the CMD models. Moreover the importance of a correct copula specification is essential for the application of a CMD model as can be seen in our study. Thus, copula (mis-)specification should play a key role before the adaption of a CMD model.

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