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Precise Accuracy Evaluation of Transformation Parameters and Point Coordinates upon Helmert Transformation**

1. Introduction

Evaluation of point coordinate transformation accuracy is most frequently presented by the references in a general and approximated way. In assumption that a secondary system point coordinates may practically be considered errorless, the following formulas were given in [5]:

$$m_x = \sqrt{\frac{[v_x^2]}{n}}, \qquad m_y = \sqrt{\frac{[v_y^2]}{n}} \tag{1}$$

$$m_p = \sqrt{\frac{[v_x^2]+[v_y^2]}{n}} \tag{2}$$

where:

v_x, v_y – differences between common point coordinates in a secondary system before and after transformation,

n – number of common points.

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Formula below was given for evaluation of common point coordinate adjustment accuracy in [6]

$$m=\sqrt{\frac{1}{2n-4}\sum_{i=1}^{n}(V_{X_i}^2+V_{Y_i}^2)} \tag{3}$$

Identical formula, defined as mean square error of observation m_0 is also shown in publication reference [4].

Transformation error construed as a point position error is defined in the Technical Guidelines G-1.10 [8] by the following formula

$$\mu_t=\left[\sum(V_{xi}^2+V_{yi}^2)/n-2\right]^{\frac{1}{2}} \tag{4}$$

Extended form of accuracy evaluation, leading to definition of mean square errors of coordinates after transformation, is shown in reference publications [3] and [7].

Precise method for determination of transformation parameters with consideration of pseudo-observation dependent stochastic model was given in reference publication [1].

Issue pertaining to the effect of model selection for determination of Helmert transformation parameters on transformation results was reviewed by the author of this paper in the reference publication [2].

This paper presents a precise method for evaluation of transformation accuracy with consideration of two functional models for determining Helmert transformation coefficients, namely based on:

- error equations,
- conditional equations with unknowns, with a consideration of assumed stochastic models.

It should be noted that the transformation coefficient determination errors were taken into consideration in proposed methods for evaluation of accuracy. The way those errors are formed depends on:

- layout of common points,
- number of common points,
- mean square errors of common point coordinates.

In practice, a transformation parameter determination accuracy was evaluated by the way of description, relating to favourable and unfavourable arrays of common points.

The method presented in this publication allows to precisely evaluate transformation parameters accuracy and use it in the evaluation of coordinate accuracy after transformation.

2. Transformation Model I

2.1. Determination of Pseudo-observation Equations

If the transformation parameters are determined by assuming that

$$2n > 4,$$

where n – number of common points, then, generally speaking

$$\overline{X}_i \neq \overline{X}_i^t \quad \text{and} \quad \overline{Y}_i \neq \overline{Y}_i^t,$$

where:

$\overline{X}_i$, $\overline{Y}_i$ – secondary coordinates of common points prior to transformation (catalog coordinates),

$\overline{X}_i^t$, $\overline{Y}_i^t$ – secondary coordinates of common points after transformation,

only after introduction of corrections to pseudo-observations, i.e. will catalog coordinates of secondary common point coordinates lead to identical coordinates after transformation, therefore:

$$\begin{aligned} \overline{X}_i + v_{\overline{X}_i} &= \overline{X}_i^t \\ \overline{Y}_i + v_{\overline{Y}_i} &= \overline{Y}_i^t \end{aligned} \tag{5}$$

and

$$\begin{bmatrix} \overline{X}_i^t \\ \overline{Y}_i^t \end{bmatrix} = \begin{bmatrix} -\overline{y}_i & \overline{x}_i & 1 & 0 \\ \overline{x}_i & \overline{y}_i & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \tag{6}$$

where:

$\overline{x}_i$, $\overline{y}_i$ – common point primary coordinates,

a, b, c, d – defined transformation parameters.

If we substitute (5) with (6), assuming that:

$$\begin{aligned} a &= a_0 + \mathrm{d}a \\ b &= b_0 + \mathrm{d}b \\ c &= c_0 + \mathrm{d}c \\ d &= d_0 + \mathrm{d}d \end{aligned} \tag{7}$$

then, for n common points we may obtain a set of pseudo-observation correction equations

$$\begin{bmatrix} v_{\overline{X}_1} \\ v_{\overline{Y}_1} \\ \vdots \\ v_{\overline{X}_n} \\ v_{\overline{Y}_n} \end{bmatrix} = \begin{bmatrix} -\overline{y}_1 & \overline{x}_1 & 1 & 0 \\ \overline{x}_1 & \overline{y}_1 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ -\overline{y}_n & \overline{x}_n & 1 & 0 \\ \overline{x}_n & \overline{y}_n & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{d}a \\ \mathrm{d}b \\ \mathrm{d}c \\ \mathrm{d}d \end{bmatrix} + \begin{bmatrix} \overline{X}_{01}^{t} - \overline{X}_1 \\ \overline{Y}_{01}^{t} - \overline{Y}_1 \\ \vdots \\ \overline{X}_{0n}^{t} - \overline{X}_n \\ \overline{Y}_{0n}^{t} - \overline{Y}_n \end{bmatrix} \tag{8}$$

or a form of matrix equation

$$\mathbf{V} = \mathbf{AX} + \mathbf{D} \tag{9}$$

where: $\overline{X}_{0i}^{t}$, $\overline{Y}_{0i}^{t}$ – approximate coordinates transformed with formula (6) based on previously calculated a_0, b_0, c_0, d_0, with a use of coordinates of two common points.

Upon setting the weight matrix of secondary common point coordinates in the form shown as:

$$\mathbf{P} = \mathrm{diag}[m_{\overline{X}_1}^2 \quad m_{\overline{Y}_1}^2 \quad \ldots \quad m_{\overline{X}_n}^2 \quad m_{\overline{Y}_n}^2]^{-1}$$

we can determine

$$\mathbf{X} = -(\mathbf{A}^T\mathbf{PA})^{-1}\mathbf{A}^T\mathbf{PD} \tag{10}$$

2.2. Evaluation of Accuracy for Transformation Parameters Determination

Evaluation of accuracy may be commenced from determination of a standard error of unit weight m_0

$$m_0 = \sqrt{\frac{\mathbf{V}^T\mathbf{PV}}{r-s}} \tag{11}$$

where:

- **V** – correction matrix determined by the formula (8) upon solving normal equations and determination of matrix **X** with formula (10),
- r – number of pseudo-observations ($r = 2n$),
- s – number of transformation parameters (in given case $s = 4$).

Taking into consideration the presented explanations, we may find that

$$m_0 = \sqrt{\frac{\mathbf{V}^T\mathbf{PV}}{2n-4}} \tag{12}$$

Further calculations should be preceded by testing performed according to known procedures of mathematical statistics and leading to verification of zero--hypothesis H_0 pertaining to coefficient of variance, i.e. whether $\sigma_0^2 = \sigma_{0H}^2 (\sigma_{0H}^2 = 1$ – hypothetical values of variance coefficient), in combination with alternative hypothesis of $H_1 : \sigma_0^2 \neq \hat{\sigma}_{0H}^2$ considering $\hat{\sigma}_0^2$ as the coefficient $\sigma_0^2 = 1$ estimator. If it appears that $\sigma_{0H}^2 \notin \Omega(\hat{\sigma}_0^2)_{LP}$, then variance coefficient σ_0^2 does not belong to bilateral confidence range $\Omega(\hat{\sigma}_0^2)_{LP}$ and zero-hypothesis should be rejected. There might be various reasons for this situation, but it may prove rather inaccurate selection of weights.

However, it should be noted that this may not be the only reason for a significant deviation $\hat{\sigma}_0^2$ from 1, since:

- value of one or several coordinates may result in gross error,
- transformation model may not be selected in a precise way.

Gross errors of coordinates are relatively easily discovered based on values of corrections $v_{\overline{X}_i}$, $v_{\overline{Y}_i}$ and mean square errors of their determination. Coordinates with gross errors may be eliminated from procedure during recalculation of transformation parameters or in case of small number of common points we may determine a new diagonal matrix of weights by calculating the weights using the following formulas:

$$p_{\overline{X}_i} = \frac{1}{v_{\overline{X}_i}^2}, \quad p_{\overline{Y}_i} = \frac{1}{v_{\overline{Y}_i}^2} \tag{13}$$

based on corrections determined in the first iteration of calculations.

Such steps should bring a desired effect in the next or even perhaps subsequent iteration.

As mentioned above it is commonly practiced to assume **P** = **E** (**E** – unit matrix). This could be only done in a final resulting variant of calculations. The obtained results of determined corrections and error require verification. It should be indicated that error m_0 is expressed in named unit, when **P** = **E** is assumed. In this case its value does not indicate directly whether the assumption pertaining to weights was right; therefore we have to calculate a variance of standardized corrections given by formula

$$\mathrm{Var}(V) = \frac{1}{2n-4}[VV] \tag{14}$$

whereas subsequent standardized corrections are calculated by formulas below:

$$V_1 = \frac{v_1}{m_0}, \quad V_2 = \frac{v_2}{m_0}, \quad \ldots, \quad V_r = \frac{v_r}{m_0} \tag{15}$$

hence a new version of formula (14) will be

$$\mathrm{Var}(V) = \frac{1}{2n-4}\left[\frac{vv}{m_0 m_0}\right] \tag{16}$$

Theoretically speaking Var(V) = 1, but if Var(V) ≠ 1 in a significant way, then we may expect a theoretical result by substituting the weights calculated with formulas (13) for equalization.

Upon obtaining a satisfactory result in determination of m_0 we may set to work on further evaluation of accuracy.

Mean square errors of transformation parameters are calculated with formula

$$m_k = m_0 \sqrt{(\mathbf{A}^T\mathbf{PA})^{-1}_{kk}} \tag{17}$$

where:

$k = a, b, c, d,$

$(\mathbf{A}^T\mathbf{PA})^{-1}_{kk}$ – diagonal element of matrix $(\mathbf{A}^T\mathbf{PA})^{-1}$.

2.3. Determination of Coordinate Mean Square Errors after Transformation

In order to calculate mean square errors of point coordinates after transformation we will apply law of error propagation to function

$$\underbrace{\begin{bmatrix} \overline{X}_1^t \\ \overline{Y}_1^t \\ \vdots \\ \overline{X}_n^t \\ \overline{Y}_n^t \\ X_{n+1}^t \\ Y_{n+1}^t \\ \vdots \\ X_u^t \\ Y_u^t \end{bmatrix}}_{\mathbf{W}^t} = \underbrace{\begin{bmatrix} -\overline{y}_1 & \overline{x}_1 & 1 & 0 \\ \overline{x}_1 & \overline{y}_1 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ -\overline{y}_n & \overline{x}_n & 1 & 0 \\ \overline{x}_n & \overline{y}_n & 0 & 1 \\ -y_{n+1} & x_{n+1} & 1 & 0 \\ x_{n+1} & y_{n+1} & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ -y_u & x_u & 1 & 0 \\ x_u & y_u & 0 & 1 \end{bmatrix}}_{\mathbf{T}} \left(\underbrace{\begin{bmatrix} a_0 \\ b_0 \\ c_0 \\ d_0 \end{bmatrix}}_{\mathbf{X}_0} + \underbrace{\begin{bmatrix} \mathrm{d}a \\ \mathrm{d}b \\ \mathrm{d}c \\ \mathrm{d}d \end{bmatrix}}_{\mathbf{X}} \right) \tag{18}$$

or, generally described as

$$\mathbf{W}^t = \mathbf{T}(\mathbf{X}_0 + \mathbf{X}) = \mathbf{T}\mathbf{X}_0 + \mathbf{T}\mathbf{X} \tag{19}$$

Coordinates W^t are a nonlinear function of variables X_i, Y_i, x_i, y_i. A linear form of formula (19), necessary to apply a law of error propagation, may be generally expressed as

$$\mathbf{W}^t = \mathbf{W}_0^t + \mathrm{d}\mathbf{W}^t \tag{20}$$

where

$$\mathrm{d}\mathbf{W}^t = \frac{\partial \mathbf{W}^t}{\partial \mathbf{W}} \mathrm{d}\mathbf{W} \tag{21}$$

or, in more detailed matrix form

$$\mathrm{d}\mathbf{W}^t = \begin{bmatrix} \dfrac{\partial \mathbf{W}^t}{\partial \overline{\mathbf{W}}_w} & \dfrac{\partial \mathbf{W}^t}{\partial \overline{\mathbf{W}}_p} & \dfrac{\partial \mathbf{W}^t}{\partial \mathbf{W}_p} \end{bmatrix} \begin{bmatrix} \mathrm{d}\overline{\mathbf{W}}_w \\ \mathrm{d}\overline{\mathbf{W}}_p \\ \mathrm{d}\mathbf{W}_p \end{bmatrix} \tag{22}$$

where $\mathbf{W}$ – matrix of common point coordinates $\overline{\mathbf{W}}_w$ (secondary system) and $\overline{\mathbf{W}}_p$ (primary system) as well as coordinates of remaining points $\mathbf{W}_p$ in a primary system.

Before reaching form (22) we will introduce a total differential $\mathrm{d}\mathbf{W}^t$ taking form that results from formula (19)

$$\mathrm{d}\mathbf{W}^t = \mathrm{d}\mathbf{T}\,\mathbf{X}_0 + \mathrm{d}\mathbf{T}\,\mathbf{X} + \mathbf{T}\,\mathrm{d}\mathbf{X} = \mathrm{d}\mathbf{T}(\mathbf{X}_0 + \mathbf{X}) + \mathbf{T}\,\mathrm{d}\mathbf{X} = \mathrm{d}\mathbf{T}\,\hat{\mathbf{X}} + \mathbf{T}\,\mathrm{d}\mathbf{X} \tag{23}$$

In this case:

- $\mathrm{d}\mathbf{T}$ – is a total differential of matrix $\mathbf{T}$ in relation to variables of the set $\{\overline{x}_1, \overline{y}_1 \ldots \overline{x}_n, \overline{y}_n, x_{n+1}, y_{n+1} \ldots x_u, y_u\} \in \omega$;
- $\mathrm{d}\mathbf{X}$ – is a total differential of matrix $\mathbf{X}$ in relation to variables of the set $\{\overline{X}_1, \overline{Y}_1 \ldots \overline{X}_n, \overline{Y}_n, \overline{x}_1, \overline{y}_1 \ldots \overline{x}_u, \overline{y}_u\} \in \overline{W}$;
- $\hat{\mathbf{X}}$ – estimated values of parameters.

By differentiating matrix **T**, and then calculating product $\mathrm{d}\mathbf{T}\,\hat{\mathbf{X}}$ we will receive

$$\mathrm{d}\mathbf{T}\,\hat{\mathbf{X}} = \begin{bmatrix} -\mathrm{d}\bar{y}_1 & \mathrm{d}\bar{x}_1 & 0 & 0 \\ \mathrm{d}\bar{x}_1 & \mathrm{d}\bar{y}_1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ -\mathrm{d}\bar{y}_n & \mathrm{d}\bar{x}_n & 0 & 0 \\ \mathrm{d}\bar{x}_n & \mathrm{d}\bar{y}_n & 0 & 0 \\ -\mathrm{d}y_{n+1} & \mathrm{d}x_{n+1} & 0 & 0 \\ \mathrm{d}x_{n+1} & \mathrm{d}y_{n+1} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ -\mathrm{d}y_u & \mathrm{d}x_u & 0 & 0 \\ \mathrm{d}x_u & \mathrm{d}y_u & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -a\mathrm{d}\bar{y}_1 + b\mathrm{d}\bar{x}_1 \\ a\mathrm{d}\bar{x}_1 + b\mathrm{d}\bar{y}_1 \\ \vdots \\ -a\mathrm{d}\bar{y}_n + b\mathrm{d}\bar{x}_n \\ a\mathrm{d}\bar{x}_n + b\mathrm{d}\bar{y}_n \\ -a\mathrm{d}y_{n+1} + b\mathrm{d}x_{n+1} \\ a\mathrm{d}x_{n+1} + b\mathrm{d}y_{n+1} \\ \vdots \\ -a\mathrm{d}y_u + b\mathrm{d}x_u \\ a\mathrm{d}x_u + b\mathrm{d}y_u \end{bmatrix} \tag{24}$$

Due to (22) this product will be presented as an extended block matrix

$$\mathrm{d}\mathbf{T}\,\hat{\mathbf{X}} = \begin{bmatrix} 0 & \mathbf{J}_{12} & 0 \\ 0 & 0 & \mathbf{J}_{23} \end{bmatrix} \begin{bmatrix} \mathrm{d}\overline{\mathbf{W}}_w \\ \mathrm{d}\overline{\mathbf{W}}_p \\ \mathrm{d}\mathbf{W}_p \end{bmatrix} \tag{25}$$

where

$$\mathbf{J}_{12} = \begin{bmatrix} b & -a & 0 & 0 & \cdots & 0 & 0 \\ a & b & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & b & -a & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & b & -a \\ 0 & 0 & 0 & 0 & \cdots & a & b \end{bmatrix}_{2n\times 2n} \tag{26}$$

Matrix $\mathbf{J}_{23}$ with dimensions $(2u - 2n) \times (2u - 2n)$ is created in analogical manner as matrix $\mathbf{J}_{12}$.

We will now proceed a second component of formula (23), i.e. product of matrix **T** d**X**. Therefore formula (10) is shown as

$$\mathbf{X} = -(\mathbf{A}^T\mathbf{P}\mathbf{A})^{-1}\mathbf{A}^T\mathbf{P}(\overline{\mathbf{W}}^t_{0w} - \overline{\mathbf{W}}_w) \tag{27}$$

taking into consideration a form of matrix **D** – consisting free words in equation (9) and in matrix (8).

A total differential of matrix **X,** expressed by formula (27) will be written as

$$\mathrm{d}\mathbf{X} = \mathbf{G}\,\mathrm{d}\overline{\mathbf{W}}_w \tag{28}$$

where

$$\mathbf{G} = (\mathbf{A}^T\mathbf{P}\mathbf{A})^{-1}\mathbf{A}^T\mathbf{P} \tag{29}$$

therefore

$$\mathbf{T}\,\mathrm{d}\mathbf{X} = \mathbf{T}\mathbf{G}\,\mathrm{d}\overline{\mathbf{W}}_w = \begin{bmatrix}\mathbf{T}_1\\ \mathbf{T}_2\end{bmatrix}\mathbf{G}\,\mathrm{d}\overline{\mathbf{W}}_w \tag{30}$$

$\mathbf{T}_1$ – submatrix of matrix **T** with components of coordinates of common points in a primary system, $\mathbf{T}_1 = [\mathbf{t}_1]_{2n \times 4}$,

$\mathbf{T}_2$ – submatrix of matrix **T** with components of coordinates of common points in a primary system, $\mathbf{T}_2 = [\mathbf{t}_2]_{(2u - 2n) \times 4.}$

In attempt to approach form (22), the product (30) may be presented as

$$\mathbf{T}\,\mathrm{d}\mathbf{X} = \begin{bmatrix}\mathbf{F}_{11} & 0 & 0\\ \mathbf{F}_{21} & 0 & 0\end{bmatrix}\begin{bmatrix}\mathrm{d}\overline{\mathbf{W}}_w\\ \mathrm{d}\overline{\mathbf{W}}_p\\ \mathrm{d}\mathbf{W}_p\end{bmatrix} \tag{31}$$

where

$$\mathbf{F}_{11} = \mathbf{T}_1\mathbf{G}, \quad \mathbf{F}_{21} = \mathbf{T}_2\mathbf{G} \tag{32}$$

Substituting (25) and (31) to (23) we receive

$$\mathrm{d}\mathbf{W}^t = \begin{bmatrix}\mathrm{d}\overline{\mathbf{W}}^t\\ \mathrm{d}\mathbf{W}_p^t\end{bmatrix} = \begin{bmatrix}\mathbf{F}_{11} & \mathbf{J}_{12} & 0\\ \mathbf{F}_{21} & 0 & \mathbf{J}_{23}\end{bmatrix}\begin{bmatrix}\mathrm{d}\overline{\mathbf{W}}_w\\ \mathrm{d}\overline{\mathbf{W}}_p\\ \mathrm{d}\mathbf{W}_p\end{bmatrix} \tag{33}$$

Variance-covariance matrix of coordinates W^t will be determined by applying a law of covariance to function (33), hence

$$\mathbf{C}_{W^t} = m_0^2\begin{bmatrix}\mathbf{F}_{11} & \mathbf{J}_{12} & 0\\ \mathbf{F}_{21} & 0 & \mathbf{J}_{23}\end{bmatrix}\begin{bmatrix}\mathbf{P}_{\overline{W}_w}^{-1} & 0 & 0\\ 0 & \mathbf{P}_{\overline{W}_p}^{-1} & 0\\ 0 & 0 & \mathbf{P}_{W_p}^{-1}\end{bmatrix}\begin{bmatrix}\mathbf{F}_{11}^T & \mathbf{F}_{21}^T\\ \mathbf{J}_{12}^T & 0\\ 0 & \mathbf{J}_{23}^T\end{bmatrix} \tag{34}$$

and upon multiplying and consideration of relations (29) and (32) we will receive

$$\mathbf{C}_{W^t} = \begin{bmatrix} \mathbf{C}_{\overline{W}^t} & \mathbf{C}_{\overline{W}^t W_p^t} \\ \mathbf{C}_{W_p^t \overline{W}^t} & \mathbf{C}_{W_p^t} \end{bmatrix} =$$
$$= m_0^2 \begin{bmatrix} \mathbf{T}_1 (\mathbf{A}^T \mathbf{P}_{\overline{W}_w} \mathbf{A})^{-1} \mathbf{T}_1^T + \mathbf{J}_{12} \mathbf{P}_{\overline{W}_p}^{-1} \mathbf{J}_{12}^T & \mathbf{T}_1 (\mathbf{A}^T \mathbf{P}_{\overline{W}_w} \mathbf{A})^{-1} \mathbf{T}_2^T \\ \mathbf{T}_2 (\mathbf{A}^T \mathbf{P}_{\overline{W}_w} \mathbf{A})^{-1} \mathbf{T}_1^T & \mathbf{T}_2 (\mathbf{A}^T \mathbf{P}_{\overline{W}_w} \mathbf{A})^{-1} \mathbf{T}_2^T + \mathbf{J}_{23} \mathbf{P}_{W_p}^{-1} \mathbf{J}_{23}^T \end{bmatrix} \tag{35}$$

Taking diagonal elements from matrix (35) we will calculate:

– mean square errors of common point coordinates after transformation

$$m_{\overline{W}_i^t} = m_0 \sqrt{\left\{ \mathbf{T}_1 \left(\mathbf{A}^T \mathbf{P}_{\overline{W}_w} \mathbf{A} \right)^{-1} \mathbf{T}_1^T + \mathbf{J}_{12} \mathbf{P}_{\overline{W}_p}^{-1} \mathbf{J}_{12}^T \right\}_{ii}} \tag{36}$$
$$i = 1,\ 2,\ \ldots,\ 2n$$

– mean square errors after transformation of all remaining points of primary system

$$m_{W_j^t} = m_0 \sqrt{\left\{ \mathbf{T}_2 \left(\mathbf{A}^T \mathbf{P}_{\overline{W}_w} \mathbf{A} \right)^{-1} \mathbf{T}_2^T + \mathbf{J}_{23} \mathbf{P}_{W_p}^{-1} \mathbf{J}_{23}^T \right\}_{jj}} \tag{37}$$
$$j = (2n+1),\ \ldots,\ 2u$$

3. Transformation Model II

3.1. Determination of Transformation Parameters by Using Conditional Equations with Unknowns

Transformation model II is the most precise model used to determine parameters, since during the calculation process:

– we determine corrections to primary coordinates and secondary common points;
– we consider weights of the above mentioned coordinates, identified with pseudo-observations.

As a result of this, we may write two conditional equations with unknowns for each common point:

$$\begin{aligned} \overline{X}_i + v_{\overline{X}_i} &= (c_0 + \mathrm{d}c) + (b_0 + \mathrm{d}b)(\overline{x}_i + v_{\overline{x}_i}) - (a_0 + \mathrm{d}a)(\overline{y}_i + v_{\overline{y}_i}) \\ \overline{Y}_i + v_{\overline{Y}_i} &= (d_0 + \mathrm{d}d) + (a_0 + \mathrm{d}a)(\overline{x}_i + v_{\overline{x}_i}) + (b_0 + \mathrm{d}b)(\overline{y}_i + v_{\overline{y}_i}) \end{aligned} \tag{38}$$

where upon rejection of words $\mathrm{d}b\,v_{\bar{x}_i}$, $\mathrm{d}a\,v_{\bar{y}_i}$, $\mathrm{d}a\,v_{\bar{x}_i}$, $\mathrm{d}b\,v_{\bar{y}_i}$, due to their negligible values in relations to the other components, the above equations are forming a set written as the following matrix

$$\mathbf{E}\mathbf{V}_{\overline{W}_w} + \mathbf{B}_p \mathbf{V}_{\overline{W}_p} + \mathbf{A}\mathbf{X} + \mathbf{L} = 0 \tag{39}$$

where:

$$\mathbf{B}_p = \begin{bmatrix} -b_0 & a_0 & \cdots & 0 & 0 \\ -a_0 & -b_0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -b_0 & a_0 \\ 0 & 0 & \cdots & -a_0 & -b_0 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} \bar{y}_1 & -\bar{x}_1 & -1 & 0 \\ -\bar{x}_1 & -\bar{y}_1 & 0 & -1 \\ \vdots & \vdots & \vdots & \vdots \\ \bar{y}_n & -\bar{x}_n & -1 & 0 \\ -\bar{x}_n & -\bar{y}_n & 0 & -1 \end{bmatrix} \tag{40}$$

$$\mathbf{L} = \begin{bmatrix} \overline{X}_1 - (c_0 + b_0 \bar{x}_1 - a_0 \bar{y}_1) \\ \overline{Y}_1 - (d_0 + a_0 \bar{x}_1 + b_0 \bar{y}_1) \\ \vdots \\ \overline{X}_n - (c_0 + b_0 \bar{x}_n - a_0 \bar{y}_n) \\ \overline{Y}_n - (d_0 + a_0 \bar{x}_n + b_0 \bar{y}_n) \end{bmatrix}$$

$$\mathbf{V}_{\overline{W}_w} = [v_{\overline{X}_1} \quad v_{\overline{Y}_1} \quad \ldots \quad v_{\overline{X}_n} \quad v_{\overline{Y}_n}]^T \tag{41}$$

$$\mathbf{V}_{\overline{W}_p} = [v_{\bar{x}_1} \quad v_{\bar{y}_1} \quad \ldots \quad v_{\bar{x}_n} \quad v_{\bar{y}_n}]^T \tag{42}$$

$$\mathbf{X} = \left[\mathrm{d}a \quad \mathrm{d}b \quad \mathrm{d}c \quad \mathrm{d}d\right]^T \tag{43}$$

and **E** is a unit matrix.

As a result of precise solution of set (39) by using „the least squares method" we will receive:

$$\mathbf{X} = -\left[\mathbf{A}^T \left(\mathbf{P}_{\overline{W}_w}^{-1} + \mathbf{B}_p \mathbf{P}_{\overline{W}_p}^{-1} \mathbf{B}_p^T\right)^{-1} \mathbf{A}\right]^{-1} \mathbf{A}^T \left(\mathbf{P}_{\overline{W}_w}^{-1} + \mathbf{B}_p \mathbf{P}_{\overline{W}_p}^{-1} \mathbf{B}_p^T\right)^{-1} \mathbf{L} \tag{44}$$

$$\mathbf{K} = (\mathbf{P}_{\overline{W}_w}^{-1} + \mathbf{B}_p \mathbf{P}_{\overline{W}_p}^{-1} \mathbf{B}_p^T)^{-1} (\mathbf{A}\mathbf{X} + \mathbf{L}) \tag{45}$$

$$\begin{bmatrix} \mathbf{V}_{\overline{W}_w} \\ \mathbf{V}_{\overline{W}_p} \end{bmatrix} = -\begin{bmatrix} \mathbf{P}_{\overline{W}_w} & 0 \\ 0 & \mathbf{P}_{\overline{W}_p} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{E} \\ \mathbf{B}_p^T \end{bmatrix} \mathbf{K} \tag{46}$$

$$\begin{aligned} \mathbf{P}_{\overline{W}_w} &= \mathbf{C}_{\overline{W}_w}^{-1} \\ \mathbf{P}_{\overline{W}_p} &= \mathbf{C}_{\overline{W}_p}^{-1} \end{aligned} \tag{47}$$

where:

$\mathbf{K}$ – matrix of correlates,

$\mathbf{P}_{\overline{W}_w}$, $\mathbf{P}_{\overline{W}_p}$ – weight matrices of common point coordinates $\overline{X}$, $\overline{Y}$ in secondary system and $\overline{x}$, $\overline{y}$ in primary system,

$\mathbf{C}_{\overline{W}_w}$, $\mathbf{C}_{\overline{W}_p}$ – variance-covariance matrices of common point coordinates.

3.2. Evaluation of Transformation Parameters Determination Accuracy

A mean square unit error m_0 is calculated from formula

$$m_0 = \sqrt{\frac{\mathbf{V}_{\overline{W}_w}^T \mathbf{P}_{\overline{W}_w} \mathbf{V}_{\overline{W}_w} + \mathbf{V}_{\overline{W}_p}^T \mathbf{P}_{\overline{W}_p} \mathbf{V}_{\overline{W}_p}}{r-4}} \tag{48}$$

where:

r – number of conditions, $r = 2n$,

n – number of common points,

and a mean square error of parameters $k = a, b, c, d$ may be determined from formula

$$m_k = m_0 \sqrt{\{\mathbf{A}^T (\mathbf{P}_{\overline{W}_w}^{-1} + \mathbf{B}_p \mathbf{P}_{\overline{W}_p}^{-1} \mathbf{B}_p^T)^{-1} \mathbf{A}\}_{kk}^{-1}} \tag{49}$$

The term under square root in formula (49) is a diagonal element of matrix

$$[\mathbf{A}^T (\mathbf{P}_{\overline{W}_w}^{-1} + \mathbf{B}_p \mathbf{P}_{\overline{W}_p}^{-1} \mathbf{B}_p^T)^{-1} \mathbf{A}]^{-1}.$$

3.3. Determination of Coordinate Mean Square Errors after Transformation

Similarly to model I, the transformed coordinates are calculated by applying formula (18). At the same time these formulas show functions that are the subject of accuracy evaluation. In the initial stage, derivation of formulas needed for this evaluation is identical to procedure for model I. For that reason, at this point we will repeat formula (23)

$$\mathrm{d}\mathbf{W}^t = \mathrm{d}\mathbf{T}\,\hat{\mathbf{X}} + \mathbf{T}\,\mathrm{d}\mathbf{X} \tag{50}$$

as well as the matrix product

$$\mathrm{d}\mathbf{T}\,\hat{\mathbf{X}} = \begin{bmatrix} 0 & \mathbf{J}_{12} & 0 \\ 0 & 0 & \mathbf{J}_{23} \end{bmatrix} \begin{bmatrix} \mathrm{d}\overline{\mathbf{W}}_w \\ \mathrm{d}\overline{\mathbf{W}}_p \\ \mathrm{d}\mathbf{W}_p \end{bmatrix} \tag{51}$$

It should be noted that in model II

$$\mathbf{J}_{12} = -\mathbf{B}_p \tag{52}$$

as well as

$$\mathbf{J}_{23} = -\mathbf{B}_u \tag{53}$$

where $\mathbf{B}_u$ – matrix created in analogical manner as $\mathbf{B}_p$, but having dimensions $(2u-2n)\times(2u-2n)$.

In order to determine a detailed form of product $\mathbf{T}\,\mathrm{d}\mathbf{X}$ existing in formula (50) we will write formula (44) as a following form:

$$\mathbf{X} = -\mathbf{U}\mathbf{L} \tag{54}$$

where:

$$\mathbf{U} = \left[\mathbf{A}^T\left(\mathbf{P}^{-1}_{\overline{W}_w} + \mathbf{B}_p\mathbf{P}^{-1}_{\overline{W}_p}\mathbf{B}^T_p\right)^{-1}\mathbf{A}\right]^{-1}\mathbf{A}^T\left(\mathbf{P}^{-1}_{\overline{W}_w} + \mathbf{B}_p\mathbf{P}^{-1}_{\overline{W}_p}\mathbf{B}^T_p\right)^{-1} \tag{55}$$

$$\mathbf{L} = \mathbf{B}\overline{\mathbf{W}} - \mathbf{C} \tag{56}$$

$$\mathbf{B} = [\mathbf{E} \quad \mathbf{B}_p\,] \tag{57}$$

$$\mathbf{C} = [c_0 \quad d_0 \quad \ldots \quad c_0 \quad d_0\,]^T_{(1\times 2n)} \tag{58}$$

$$\overline{\mathbf{W}} = [\overline{X}_1 \quad \overline{Y}_1 \quad \ldots \quad \overline{X}_n \quad \overline{Y}_n \quad \overline{x}_1 \quad \overline{y}_1 \quad \ldots \quad \overline{x}_n \quad \overline{y}_n\,]^T \tag{59}$$

Substituting (56) to (54) we receive

$$\mathbf{X} = -\mathbf{U}(\mathbf{B}\overline{\mathbf{W}} - \mathbf{C}) \tag{60}$$

hence

$$\mathrm{d}\mathbf{X} = -\mathbf{U}\mathbf{B}\,\mathrm{d}\overline{\mathbf{W}} \tag{61}$$

and

$$\mathbf{T}\,\mathrm{d}\mathbf{X} = -\mathbf{TUB}\,\mathrm{d}\overline{\mathbf{W}} \tag{62}$$

Formula (62) will be presented in a new more detailed form

$$\mathbf{T}\,\mathrm{d}\mathbf{X} = -\begin{bmatrix}\mathbf{T}_1\\ \mathbf{T}_2\end{bmatrix}\mathbf{U}[\mathbf{E} \quad \mathbf{B}_p]\begin{bmatrix}\mathrm{d}\overline{\mathbf{W}}_w\\ \mathrm{d}\overline{\mathbf{W}}_p\end{bmatrix} = -\begin{bmatrix}\mathbf{T}_1\mathbf{U} & \mathbf{T}_1\mathbf{UB}_p\\ \mathbf{T}_2\mathbf{U} & \mathbf{T}_2\mathbf{UB}_p\end{bmatrix}\begin{bmatrix}\mathrm{d}\overline{\mathbf{W}}_w\\ \mathrm{d}\overline{\mathbf{W}}_p\end{bmatrix} \tag{63}$$

and upon adjusting it to form (22), we may write

$$\mathbf{T}\,\mathrm{d}\mathbf{X} = -\begin{bmatrix}\mathbf{S}_{11} & \mathbf{S}_{12} & 0\\ \mathbf{S}_{21} & \mathbf{S}_{22} & 0\end{bmatrix}\begin{bmatrix}\mathrm{d}\overline{\mathbf{W}}_w\\ \mathrm{d}\overline{\mathbf{W}}_p\\ \mathrm{d}\mathbf{W}_p\end{bmatrix} \tag{64}$$

where

$$\mathbf{S}_{11} = \mathbf{T}_1\mathbf{U},\ \mathbf{S}_{12} = \mathbf{T}_1\mathbf{UB}_p,\ \mathbf{S}_{21} = \mathbf{T}_2\mathbf{U},\ \mathbf{S}_{22} = \mathbf{T}_2\mathbf{UB} \tag{65}$$

Substituting (51) and (64) to (50) we receive

$$\mathrm{d}\mathbf{W}^t = -\begin{bmatrix}\mathbf{S}_{11} & \mathbf{R}_{12} & 0\\ \mathbf{S}_{21} & \mathbf{S}_{22} & -\mathbf{J}_{23}\end{bmatrix}\begin{bmatrix}\mathrm{d}\overline{\mathbf{W}}_w\\ \mathrm{d}\overline{\mathbf{W}}_p\\ \mathrm{d}\mathbf{W}_p\end{bmatrix} \tag{66}$$

where

$$\mathbf{R}_{12} = \mathbf{S}_{12} - \mathbf{J}_{12} \tag{67}$$

Formula (66) within context of formula (20) renders a linear form of solution to transformation of coordinates from a primary to secondary system. By applying a law of covariance to this form of equation we will receive a variance-covariance matrix of transformed coordinates:

$$\mathbf{C}_{W^t} = m_0^2\begin{bmatrix}\mathbf{S}_{11} & \mathbf{R}_{12} & 0\\ \mathbf{S}_{21} & \mathbf{S}_{22} & -\mathbf{J}_{23}\end{bmatrix}\begin{bmatrix}\mathbf{P}_{\overline{W}_w}^{-1} & 0 & 0\\ 0 & \mathbf{P}_{\overline{W}_p}^{-1} & 0\\ 0 & 0 & \mathbf{P}_{W_p}^{-1}\end{bmatrix}\begin{bmatrix}\mathbf{S}_{11}^T & \mathbf{S}_{21}^T\\ \mathbf{R}_{12}^T & \mathbf{S}_{22}^T\\ 0 & -\mathbf{J}_{23}^T\end{bmatrix} \tag{68}$$

Upon matrix multiplication in (68) we will receive

$$\mathbf{C}_{W^t} = m_0^2\begin{bmatrix}\mathbf{S}_{11}\mathbf{P}_{\overline{W}_w}^{-1}\mathbf{S}_{11}^T + \mathbf{R}_{12}\mathbf{P}_{\overline{W}_p}^{-1}\mathbf{R}_{12}^T & \mathbf{S}_{11}\mathbf{P}_{\overline{W}_w}^{-1}\mathbf{S}_{21}^T + \mathbf{R}_{12}\mathbf{P}_{\overline{W}_p}^{-1}\mathbf{S}_{22}^T\\ \mathbf{S}_{21}\mathbf{P}_{\overline{W}_w}^{-1}\mathbf{S}_{11}^T + \mathbf{S}_{22}\mathbf{P}_{\overline{W}_p}^{-1}\mathbf{R}_{12}^T & \mathbf{S}_{21}\mathbf{P}_{\overline{W}_w}^{-1}\mathbf{S}_{21}^T + \mathbf{S}_{22}\mathbf{P}_{\overline{W}_p}^{-1}\mathbf{S}_{22}^T + \mathbf{J}_{23}\mathbf{P}_{W_p}^{-1}\mathbf{J}_{23}^T\end{bmatrix} \tag{69}$$

or in block matrix form

$$\mathbf{C}_{W^t} = \begin{bmatrix} \mathbf{C}_{\overline{W}_w} & \mathbf{W}_{\overline{W}_w W_p} \\ \mathbf{W}_{W_p \overline{W}_w} & \mathbf{C}_{W_p} \end{bmatrix} \tag{70}$$

Taking into consideration relations between matrices mentioned earlier in (69), and also relations between (69) and (70) we may write

$$\begin{aligned} \mathbf{C}_{\overline{W}_w} = m_0^2 \Bigg[& \mathbf{T}_1 \left[\mathbf{A}^T \left(\mathbf{P}^{-1}_{\overline{W}_w} + \mathbf{B}_p \mathbf{P}^{-1}_{\overline{W}_p} \mathbf{B}_p^T \right)^{-1} \mathbf{A} \right]^{-1} \mathbf{T}_1^T + \mathbf{T}_1 \mathbf{U} \mathbf{B}_p \mathbf{P}^{-1}_{\overline{W}_p} \mathbf{B}_p^T + \\ & + \left(\mathbf{T}_1 \mathbf{U} \mathbf{B}_p \mathbf{P}^{-1}_{\overline{W}_p} \mathbf{B}_p^T \right)^T + \mathbf{B}_p \mathbf{P}^{-1}_{\overline{W}_p} \mathbf{B}_p^T \Bigg] \end{aligned} \tag{71}$$

$$\mathbf{C}_{W_p} = m_0^2 \left[\mathbf{T}_2 \left[\mathbf{A}^T \left(\mathbf{P}^{-1}_{\overline{W}_w} + \mathbf{B}_p \mathbf{P}^{-1}_{\overline{W}_p} \mathbf{B}_p^T \right)^{-1} \mathbf{A} \right]^{-1} \mathbf{T}_2^T + \mathbf{B}_u \mathbf{P}^{-1}_{W_p} \mathbf{B}_u^T \right] \tag{72}$$

$$\mathbf{C}_{\overline{W}_w W_p} = m_0^2 \left[\mathbf{T}_1 \left[\mathbf{A}^T \left(\mathbf{P}^{-1}_{\overline{W}_w} + \mathbf{B}_p \mathbf{P}^{-1}_{\overline{W}_p} \mathbf{B}_p^T \right)^{-1} \mathbf{A} \right]^{-1} \mathbf{T}_2^T + \mathbf{B}_p \mathbf{P}^{-1}_{\overline{W}_p} \mathbf{B}_p^T \mathbf{U}^T \mathbf{T}_2^T \right] \tag{73}$$

Mean square errors of common point coordinates after transformation can be determined based on diagonal elements existing in matrix (71)

$$\begin{aligned} & m_{\overline{W}_i^t} = m_0 \sqrt{\{ \mathbf{T}_1 [\mathbf{A}^T (\mathbf{P}^{-1}_{\overline{W}_w} + \mathbf{H}_p)^{-1} \mathbf{A}]^{-1} \mathbf{T}_1^T + \mathbf{T}_1 \mathbf{U} \mathbf{H}_p + (\mathbf{T}_1 \mathbf{U} \mathbf{H}_p)^T + \mathbf{H}_p \}_{ii}} \\ & i = 1,\, 2,\, \ldots,\, 2n \end{aligned} \tag{74}$$

and mean square errors of other remaining points will be determined by using diagonal elements from formula (72)

$$\begin{aligned} & m_{W_j^t} = m_0 \sqrt{\{ \mathbf{T}_2 [\mathbf{A}^T (\mathbf{P}^{-1}_{\overline{W}_w} + \mathbf{H}_p)^{-1} \mathbf{A}]^{-1} \mathbf{T}_2^T + \mathbf{H}_u \}_{jj}} \\ & j = (2n+1),\, \ldots,\, 2u \end{aligned} \tag{75}$$

where:

$$\mathbf{H}_p = \mathbf{B}_p \mathbf{P}^{-1}_{\overline{W}_p} \mathbf{B}_p^T$$

$$\mathbf{H}_u = \mathbf{B}_u \mathbf{P}^{-1}_{W_p} \mathbf{B}_u^T$$

4. Final Conclusions

1) In case number of common points is more than 2, Helmert transformation parameters should be determined based on functional model created with error equations (model I) or conditional equations with unknowns (model II) while considering the weights of pseudo-observations, i.e. common point coordinates.
2) From the point of view of assumptions applied during both model creations, these models are not equivalent. In model I the randomness may be related only to pseudo-observations which are coordinates of common points in a secondary system, while in model II that randomness is related to common point coordinates in both primary and secondary systems. It could therefore be appreciated that model II is more precise than model I.
3) Due to the lack of data pertaining to the coordinate variance matrix, determination of transformation parameters is most commonly conducted by assumption of identical weights (**P** = **E**). In this case variances – Var (V) of standardized corrections should be calculated, and a result of calculations compared with theoretical assumption Var (V) = 1. Evaluation of this comparison should be conducted based on principles of Statistical hypothesis verification. Negative result could, among the other things, prove inaccuracy of stochastic model or wrongly selected weights. In this case it is necessary to conduct another equalization, assuming the weights are reversely proportional to the squares of coordinate corrections.
4) Proposed method of accuracy evaluation allows to determine mean square errors of transformation parameters, and also mean square errors of transformed coordinates of all the points. This provides basis for drawing the detailed conclusions pertaining to calculation results, as well as allows to render a decision concerning further action leading to transformation process finalization.
5) Proposed method for accuracy evaluation of point transformation parameters could also be applied to other transformation methods.

References

[1] Bałut A., Czaja J., Pokrzywa A.: *Uzgodnienie współrzędnych punktów dwóch sieci wyrównanych niezależnie na drodze transformacji z uwzględnieniem obserwacji zależnych.* Geodezja i Kartografia. Kwartalnik Naukowy, T. XXVIII, z. 4, 1979.

[2] Beluch J.: *Wpływ wyboru modelu wyznaczania parametrów transformacji Helmerta na wyniki transformacji*. Acta Scientarum Polonarum. Geodesia et Descriptio Terrarum No. 7(1), 2008.

[3] Gaździcki J.: *Transformacja Helmerta przy różnej dokładności położenia punktów dostosowania.* Prace Instytutu Geodezji i Kartografii, t. X, z. 1 (21), 1963.

[4] Kluss T.: *Transformacje wykonywane na podstawie punktów łącznych*. Prace Instytutu Geodezji i Kartografii, t. XI, z. 1/23, 1964.

[5] Lazzarini T.: *Wykłady geodezji II*. PWN, Warszawa 1983.

[6] Lazzarini T., Hermanowski A., Gaździcki J., Dobrzycka M., Laudyn I.: *Geodezja. Geodezyjna osnowa szczegółowa.* PPWK, Warszawa – Wrocław 1990.

[7] Osada E.: *Analiza, wyrównanie i modelowanie geo-danych*. Wydawnictwo Akademii Rolniczej, Wrocław 1998.

[8] Wytyczne techniczne G-1.10: *Formuły odwzorowawcze i parametry układów współrzędnych*. Wydanie 2 zmienione, GUGiK, Warszawa 2001.