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$[r, s, t]$-COLOURINGS OF PATHS

Abstract. The concept of $[r, s, t]$-colourings was recently introduced by Hackmann, Kemnitz and Marangio [3] as follows: Given non-negative integers r,s and t, an $[r, s, t]$-colouring of a graph $G = (V(G), E(G))$ is a mapping c from $V(G) \cup E(G)$ to the colour set $\{1, 2, \ldots, k\}$ such that $|c(v_i) - c(v_j)| \geq r$ for every two adjacent vertices v_i, v_j, $|c(e_i) - c(e_j)| \geq s$ for every two adjacent edges e_i, e_j, and $|c(v_i) - c(e_j)| \geq t$ for all pairs of incident vertices and edges, respectively. The $[r, s, t]$-chromatic number $\chi_{r,s,t}(G)$ of G is defined to be the minimum k such that G admits an $[r, s, t]$-colouring.
In this paper, we determine the $[r, s, t]$-chromatic number for paths.

Keywords: Total colouring, paths.

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1. INTRODUCTION

We use [5] for terminology and notation not defined here and consider finite and simple graphs only.

The concept of $[r, s, t]$-colourings is a generalization of the classical colourings: vertex colouring, edge colouring and total colouring. It was recently introduced by Hackmann, Kemnitz and Marangio [3].

Given non-negative integers r, s and t, an $[r, s, t]$-colouring of a graph $G = (V(G), E(G))$ is a mapping c from $V(G) \cup E(G)$ to the colour set $\{1, 2, \ldots, k\}$ such that $|c(v_i) - c(v_j)| \geq r$ for every two adjacent vertices v_i, v_j, $|c(e_i) - c(e_j)| \geq s$ for every two adjacent edges e_i, e_j, and $|c(v_i) - c(e_j)| \geq t$ for all pairs of incident vertices and edges, respectively. The $[r, s, t]$-chromatic number $\chi_{r,s,t}(G)$ of G is defined to be the minimum k such that G admits an $[r, s, t]$-colouring.

Obviously, a $[1, 0, 0]$-colouring is a classical vertex colouring, a $[0, 1, 0]$-colouring is a classical edge colouring, and a $[1, 1, 1]$-colouring is a classical total colouring. We refer the reader to [2] and [6].

Hence, there are several different applications of such $[r, s, t]$-colouring. The following example is given in [3].

Assume that in a soccer tournament there are four teams in an elimination round such that each team plays one match against each other team. During this round each team should get the possibility of a training day. Since there is only one training field, different training days must be assigned to the teams. Furthermore, a training day of a team should be different from a playing day and no team should play two successive days.

All required conditions are fulfilled in a $[1, 2, 1]$-colouring of a complete graph K_4 if one assigns the vertices of K_4 to the training days of the teams and the edges to the matches between them.

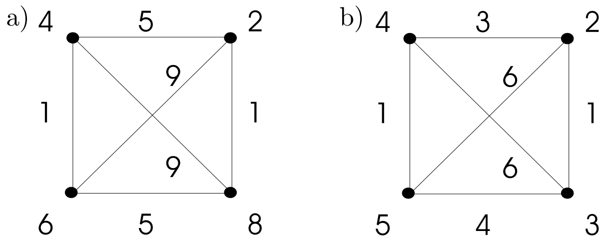

Fig. 1. A $[2, 4, 1]$-colouring (a) and a $[1, 2, 1]$-colouring of K_4 (b)

Figure 1 shows examples of a $[2, 4, 1]$-colouring with 9 colours and a $[1, 2, 1]$-colouring with 6 colours of the complete graph K_4. The right picture shows then that one can arrange a schedule for the considered soccer tournament round fulfilling all the desired conditions in six days.

Kemnitz and Marangio [3] have proved the following properties for $[r, s, t]$-colourings.

Lemma 1.1. *If $H \subseteq G$, then*

$$\chi_{r,s,t}(H) \leq \chi_{r,s,t}(G).$$

Lemma 1.2. *If $r' \leq r$, $s' \leq s$, $t' \leq t$, then*

$$\chi_{r',s',t'}(G) \leq \chi_{r,s,t}(G).$$

Lemma 1.3. *If G is non-trivial, then*

1. $\chi_{r,0,0}(G) = r(\chi(G) - 1) + 1$,
2. $\chi_{0,s,0}(G) = s(\chi'(G) - 1) + 1$.

Lemma 1.4. *For the $[r, s, t]$-chromatic number of a graph G, there holds*

$$\max\{r(\chi(G)-1)+1, s(\chi'(G)-1)+1, t+\Delta(G)\} \leq \chi_{r,s,t}(G) \leq r(\chi(G)-1)+s(\chi'(G)-1)+t+1,$$

if $|V(G)| \geq 2$, $G \neq \overline{K_n}$ and $s, t \geq 1$ (where $\Delta(G)$ is the maximum degree of G and $\overline{K_n}$ is the empty graph).

Remark 1.5. *Note that the lower bound obtained by Kemnitz and Marangio* [3],

$$\max\{r(\chi(G)-1)+1, s(\chi'(G)-1)+1, t+1\} \leq \chi_{r,s,t}(G) \leq r(\chi(G)-1)+s(\chi'(G)-1)+t+1$$

is slightly improved.
For paths P_n, $n \geq 3$, we obtain

$$\max\{r+1, s+1, t+2\} \leq \chi_{r,s,t}(P_n) \leq r+s+t+1.$$

(for $n = 2$, $\chi'(G) = 1$, so we would have $\max\{r+1, t+2\} \leq \chi_{r,s,t}(P_2) \leq r+t+1$*).*

Proof. By Lemma 1.3 and 1.2,
$r(\chi(G)-1)+1 = \chi_{r,0,0}(G) \leq \chi_{r,s,t}(G)$
$s(\chi'(G)-1)+1 = \chi_{0,s,0}(G) \leq \chi_{r,s,t}(G)$

On the other hand, the star $K_{1,\Delta}$ induced by a vertex of maximum degree $\Delta(G) = \Delta$ and its adjacent vertices needs at least $t+\Delta$ colours: to colour the Δ edges we have to use at least $s(\Delta-1)+1$ colours, which is greater or equal to Δ, if s is at least 1. Let us look for the colour of the "central vertex" in the different cases.

If this colour is smaller than the smallest one for the edges or greater than the greatest one, we are using at least $t+\Delta$ colours. If the colour for the vertex "fits" between the colours of two edges, the difference between these two colours must be at least $2t$, hence the total number of colours used to colour the edges is at least $2t+\Delta-1$, which is at least $t+\Delta$, if $t \geq 1$.

Then, by Lemma 1.1, $t+\Delta \leq \chi_{r,s,t}(K_{1,\Delta}) \leq \chi_{r,s,t}(G)$. Hence, $\max\{r(\chi(G)-1)+1, s(\chi'(G)-1)+1, t+\Delta\} \leq \chi_{r,s,t}(G)$. Observe that if $s = 0$, $t+\Delta$ would not be a lower bound but $t+1$, but for the other bounds nothing changes.

For the upper bound, it is enough to find a possible $[r, s, t]$-colouring with the desired number of colours. If we (see [3]) colour the vertices of G with colours $0, r, \ldots, r(\chi(G)-1)$ and the edges with colours $r(\chi(G)-1)+t, r(\chi(G)-1)+t+s, \ldots, r(\chi(G)-1)+t+s(\chi'(G)-1)$, an $[r, s, t]$-colouring of G is obtained. □

1.1. SHARPNESS OF THE LOWER BOUNDS

In this subsection the sharpness of some of the lower bounds introduced by Kemnitz and Marangio [3] will be proved.

Lemma 1.1.1. *For any graph G,*
if $r \geq \lceil \frac{\Delta(G)}{\chi(G)-1} \rceil s + 2t$, or $r \geq \lceil \frac{\Delta(G)+2-\chi(G)}{\chi(G)-1} \rceil s + 2t$ and $s < 2t$,
then $\chi_{r,s,t}(G) = (\chi(G)-1)r+1$,
and if $s \geq r+2t$ and $r < 2t$ or $2s \geq 3r+2t$,
then $\chi_{r,s,t}(G) = (\chi'(G)-1)s+1$.

Proof. (1.1) If $r \geq \lceil \frac{\Delta(G)}{\chi(G)-1} \rceil s + 2t$, the elements of the graph could be coloured as follows:

The vertices are coloured with the $\chi(G)$ colours $1, r+1, 2r+1, \ldots, (\chi(G)-1)r+1$. The edges use the following colours: $a_1 = t+1, a_2 = s+t+1, \ldots, a_{p+1} = ps+t+1, a_{p+2} = \max\{a_{p+1}+s, r+t+1\}$, $a_{np+2} = \max\{a_{np+1}+s, nr+t+1\}$, for all n, and

$a_{j+1} = a_j + s$, for all $j \geq p+2$ and $j \neq np+1$ for some n, where $p := \lceil \frac{\Delta(G)}{\chi(G)-1} \rceil$.
In this way we placed the colours of the edges in the intervals between the colours of the vertices. Then, using the colouring defined above, in an attempt to use no more than $(\chi(G)-1)r+1$ colours, the edges can receive $p+1+p(\chi(G)-2) \geq \Delta(G)+1 \geq \chi'(G)$ different colours, where the inequality $\chi'(G) \leq \Delta(G)+1$ is known as Vizing's Theorem [4].

(1.2) If $s < 2t$ and $r \geq \lceil \frac{\Delta(G)+2-\chi(G)}{\chi(G)-1} \rceil s + 2t$, the vertices can receive colours from the list $1, r+1, \ldots, (\chi(G)-1)r+1$ and the edges from the following a_i, defined as $a_1 = t+1$, $a_{i+1} = a_i + s$ for $i \neq n(p+1)-1$ for some n, and $a_{n(p+1)} = nr+t+1$ for all n, where $p := \lceil \frac{\Delta(G)+2-\chi(G)}{\chi(G)-1} \rceil$.
Then, similarly as above, among the $(\chi(G)-1)(r-1)$ remaining colours fewer than $(\chi(G)-1)r+1$, there are $(p+1)(\chi(G)-1) \geq \Delta(G)+1$ possible colours for the edges, which is a sufficient number.

(2) If $s \geq r+2t$, the elements of the graph can be coloured using the following colours: for the edges, the colours $1, s+1, \ldots, (\chi'(G)-1)s+1$; so the remaining colours fewer than $(\chi'(G)-1)s+1$ are divided into $\chi'(G)-1$ intervals, each containing $s-1$ colours. And for the vertices $t+1, r+t+1$, which fit in the first interval of colours for the edges; $\max\{s, 2r\}+t+1, \max\{s+r, 3r\}+t+1$ that are in the second interval (because if $r < 2t$, then $2s+1 \geq s+r+2t+1 > 3r+2t+1$ and in the other case if $2s \geq 3r+2t$, then $2s+1 \geq \max\{s+r, 3r\}+2t+1$); and finally $\max\{2s, 4r\}+t+1, \max\{3s, 5r\}+t+1, \ldots, \max\{(\chi'(G)-2)s, \chi'(G)r\}+t+1$, where each one lays in one of the intervals defined above.

In this way, there are $\chi'(G)+1$ possible colours for the vertices; this number is greater or equal to $\Delta(G)+1$ and hence greater or equal to $\chi(G)$, where the last inequality is a consequence of Brooks Theorem [1]. □

Lemma 1.1.2. *For the star $K_{1,n}$ with n leaves, $\chi_{1,1,t}(K_{1,n}) = t + \Delta(K_{1,n})$, if $t < n$.*

Proof. Kemnitz and Marangio [3] proved that for any bipartite graph G, $t+\Delta(G) \leq \chi_{1,1,t}(G) \leq t+\Delta(G)+1$. Then, if v_0 is the root of the star, the n edges are noted as $e_1, e_2, \ldots, e_n$ and v_i is the leaf adjacent to e_i, we could colour $K_{1,n}$ as follows: $c(v_0) = 1$, $c(v_1) = 2t+1$, $c(v_i) = 2$ for all $i = 2 \ldots n$ and $c(e_j) = t+j$ for all $j = 1 \ldots n$, which is a $[1,1,t]$-colouring with $t+n$ colours, hence $\chi_{1,1,t}(K_{1,n}) = t + \Delta(K_{1,n})$. □

Using the cited property for bipartite graphs given by Kemnitz and Marangio, the lower bound for bipartite graphs could be improved.

Theorem 1.1.3. *For any bipartite graph G, if $t \geq \Delta(G)$, then $\chi_{r,s,t}(G) \geq t+\Delta(G)+1$.*

Proof. Considering the star $K_{1,\Delta}$ induced by a vertex of the maximum degree $\Delta(G) = \Delta$ in G and its adjacent vertices, let us prove that if $t \geq \Delta$, there is no $[1,1,t]$ colouring with $t+\Delta$ colours.
Suppose $\chi_{1,1,t}(K_{1,\Delta}) \leq t+\Delta$. If $c(v_0) < c(e_i) < c(v_i)$ (or in the symmetric situation) for some i, where v_0 is the root of the star and e_i is incident to v_0 and v_i, then $c(v_i)$(or $c(v_0)$) $\geq 2t+1 > t+\Delta$, which is a contradiction.
Then, $c(v_0), c(v_i) < c(e_i)$ (or the symmetric situation holds) for all i. Since $s = 1$,

all edges should receive different colours. Let $c(e_1) < c(e_2) < \cdots < c(e_\Delta)$. Then $c(e_1) \geq t+2$ and $c(e_{\Delta+1}) \geq t+\Delta+1$, a contradiction.
Hence, $\chi_{1,1,t}(K_{1,\Delta}) = t+\Delta+1$. Then, by Lemma 1.1 and Lemma 1.2, $\chi_{r,s,t}(G) \geq \chi_{r,s,t}(K_{1,\Delta}) \geq \chi_{1,1,t}(K_{1,n})$. □

The previous Lemma 1.1 and Remark 1.5 will be used to determine bounds for the $[r, s, t]$-chromatic number.

The following Lemma will be useful in reducing the number of cases to be considered.

Lemma 1.6. *If c is an $[r, s, t]$-colouring with colours from $\{1, \ldots, k\}$, then c' with $c'(x) = k + 1 - c(x)$ is also an $[r, s, t]$-colouring with colours from $\{1, \ldots, k\}$.*

Proof. If $1 \leq c(x) \leq k$, then $1 \leq k+1-c(x) \leq k$. And if $|c(x) - c(y)| \geq d$, then $|(k+1-c(x)) - (k+1-c(y))| = |c(y) - c(x)| \geq d$. □

In [3], general bounds and exact values of the $[r, s, t]$-chromatic number have been determined for complete graphs and for graphs with $\min\{r, s, t\} = 0$.
In this paper, we determine the $[r, s, t]$-chromatic number for paths.

2. SOME LOWER BOUNDS FOR $\chi_{R,S,T}(P_N)$

In the following theorems, lemmas and observations, we will use the following notation for the vertices and edges of a path

$$(\ldots, c(e_0), \mathbf{c(v_0)}, c(e_1), \mathbf{c(v_1)}, c(e_2), \mathbf{c(v_2)}, c(e_3), \ldots),$$

where $\ldots, v_0, v_1, \ldots$ are vertices and $\ldots, e_0, e_1, \ldots$ edges of the considered path, such that $e_i = v_{i-1}v_1$.

2.1. LOWER BOUNDS FOR $\chi_{R,S,T}(P_2)$ AND $\chi_{R,S,T}(P_3)$

Observation 2.1.1. *We consider a path P_2 given by $v_0e_1v_1$. We may assume that $c(v_0) \leq c(v_1)$. Then, by Lemma 1.6, all possible constellations of colours of its elements can be reduced to the following two:*
If $c(v_0) \leq c(v_1) \leq c(e_1)$, then $k \geq r+t+1$.
If $c(v_0) \leq c(e_1) \leq c(v_1)$, then $k \geq \max\{2t+1, r+1\}$.
Hence

$$k \geq \min\{r+t+1, \max\{2t+1, r+1\}\} = \begin{cases} r+1 & \text{if } r \geq 2t; \\ 2t+1 & \text{if } t \leq r < 2t; \\ r+t+1 & \text{if } r < t. \end{cases}$$

Then for any path of order $n \geq 2$, by Lemma 1.1, the previous lower bounds hold.

Observation 2.1.2. *We consider a path P_3 given by $v_0e_1v_1e_2v_2$. By symmetry, it can be assumed that $c(e_1) \leq c(e_2)$. We now distinguish three main cases, that due to Lemma 1.6 can be reduced to two, and its several subcases:*
(Observe that, by Remark 1.5, we may assume $k \leq r+s+t+1$. Hence we can omit the cases for which $k \geq k_0 > r+s+t+1$)

1. *If $c(e_1) \leq c(e_2) \leq c(v_1)$, then $k \geq s+t+1$.*

1.1. *If $c(v_0) \leq c(e_1) \leq c(e_2) \leq c(v_1)$, then $k \geq \max\{r+1, s+2t+1\}$.*
For this constellation, let us analyze the 5 possible situations, corresponding to the 5 possible relations between the colour of v_2 and the colours of the other elements of P_3.
If $c(v_2) \leq c(v_0) \leq c(e_1) \leq c(e_2) \leq c(v_1)$, then $k \geq \max\{r+1, s+2t+1\}$.
If $c(v_0) \leq c(v_2) \leq c(e_1) \leq c(e_2) \leq c(v_1)$, then $k \geq \max\{r+1, s+2t+1\}$.
If $c(v_0) \leq c(e_1) \leq c(v_2) \leq c(e_2) \leq c(v_1)$, then $k \geq \max\{r+t+1, s+2t+1, 3t+1\}$.
If $c(v_0) \leq c(e_1) \leq c(e_2) \leq c(v_2) \leq c(v_1)$, then $k \geq r+s+2t+1 \geq r+s+t+1$.
If $c(v_0) \leq c(e_1) \leq c(e_2) \leq c(v_1) \leq c(v_2)$, then $k \geq r+s+2t+1 \geq r+s+t+1$.
In the same way, the following cases are treated using shortened tables.

1.2. *If $c(e_1) \leq c(v_0) \leq c(e_2) \leq c(v_1)$, then $k \geq \max\{r+t+1, s+t+1, 2t+1\}$.*

Table 1

$c(v_2) \leq c(e_1) \leq c(v_0) \leq c(e_2) \leq c(v_1)$	$k \geq \max\{r+t+1, s+t+1, 2t+1\}$
$c(e_1) \leq c(v_2) \leq c(v_0) \leq c(e_2) \leq c(v_1)$	
$c(e_1) \leq c(v_0) \leq c(v_2) \leq c(e_2) \leq c(v_1)$	$k \geq \max\{r+t+1, s+t+1, 3t+1\}$
$c(e_1) \leq c(v_0) \leq c(e_2) \leq c(v_2) \leq c(v_1)$	$k \geq r+s+t+1$
$c(e_1) \leq c(v_0) \leq c(e_2) \leq c(v_1) \leq c(v_2)$	

1.3. *If $c(e_1) \leq c(e_2) \leq c(v_0) \leq c(v_1)$, then $k \geq \max\{r+t+1, s+t+1, r+s+1\}$.*

Table 2

$c(v_2) \leq c(e_1) \leq c(e_2) \leq c(v_0) \leq c(v_1)$	$k \geq \max\{r+t+1, s+t+1, r+s+1, 2t+1\}$
$c(e_1) \leq c(v_2) \leq c(e_2) \leq c(v_0) \leq c(v_1)$	
$c(e_1) \leq c(e_2) \leq c(v_2) \leq c(v_0) \leq c(v_1)$	$k \geq r+s+t+1$
$c(e_1) \leq c(e_2) \leq c(v_0) \leq c(v_2) \leq c(v_1)$	
$c(e_1) \leq c(e_2) \leq c(v_0) \leq c(v_1) \leq c(v_2)$	

1.4. *If $c(e_1) \leq c(e_2) \leq c(v_1) \leq c(v_0)$, then $k \geq r+s+t+1$.*

2. *If* $c(e_1) \le c(v_1) \le c(e_2)$, *then* $k \ge 2t + 1$.

2.1. *If* $c(v_0) \le c(e_1) \le c(v_1) \le c(e_2)$, *then* $k \ge \max\{r + t + 1, s + t + 1, 3t + 1\}$.

Table 3

$c(v_2) \le c(v_0) \le c(e_1) \le c(v_1) \le c(e_2)$	$k \ge \max\{r + t + 1, s + t + 1, 3t + 1\}$
$c(v_0) \le c(v_2) \le c(e_1) \le c(v_1) \le c(e_2)$	
$c(v_0) \le c(e_1) \le c(v_2) \le c(v_1) \le c(e_2)$	
$c(v_0) \le c(e_1) \le c(v_1) \le c(v_2) \le c(e_2)$	$k \ge \max\{r + 3t + 1, 2r + t + 1, s + t + 1\}$
$c(v_0) \le c(e_1) \le c(v_1) \le c(e_2) \le c(v_2)$	$k \ge \max\{2r + 1, s + 2t + 1, 4t + 1\}$

2.2. *If* $c(e_1) \le c(v_0) \le c(v_1) \le c(e_2)$, *then* $k \ge \max\{r + 2t + 1, s + 1\}$.

Table 4

$c(v_2) \le c(e_1) \le c(v_0) \le c(v_1) \le c(e_2)$	$k \ge \max\{r + 2t + 1, s + 1\}$
$c(e_1) \le c(v_2) \le c(v_0) \le c(v_1) \le c(e_2)$	
$c(e_1) \le c(v_0) \le c(v_2) \le c(v_1) \le c(e_2)$	
$c(e_1) \le c(v_0) \le c(v_1) \le c(v_2) \le c(e_2)$	$k \ge \max\{2r + 2t + 1, s + 1\}$
$c(e_1) \le c(v_0) \le c(v_1) \le c(e_2) \le c(v_2)$	$k \ge \max\{r + 3t + 1, 2r + t + 1, s + t + 1\}$

2.3. *If* $c(e_1) \le c(v_1) \le c(v_0) \le c(e_2)$, *then* $k \ge \max\{r + t + 1, s + 1, 2t + 1\}$.

Table 5

$c(v_2) \le c(e_1) \le c(v_1) \le c(v_0) \le c(e_2)$	$k \ge \max\{r + t + 1, 2r + 1, s + 1, 2t + 1\}$
$c(e_1) \le c(v_2) \le c(v_1) \le c(v_0) \le c(e_2)$	
$c(e_1) \le c(v_1) \le c(v_2) \le c(v_0) \le c(e_2)$	$k \ge \max\{r + 2t + 1, s + 1\}$
$c(e_1) \le c(v_1) \le c(v_0) \le c(v_2) \le c(e_2)$	
$c(e_1) \le c(v_1) \le c(v_0) \le c(e_2) \le c(v_2)$	$k \ge \max\{r + 2t + 1, s + t + 1, 3t + 1\}$

2.4. *If* $c(e_1) \le c(v_1) \le c(e_2) \le c(v_0)$, *then* $k \ge \max\{r + t + 1, s + 1, 2t + 1\}$.

Table 6

$c(v_2) \le c(e_1) \le c(v_1) \le c(e_2) \le c(v_0)$	$k \ge \max\{r + t + 1, 2r + 1, s + 1, 2t + 1\}$
$c(e_1) \le c(v_2) \le c(v_1) \le c(e_2) \le c(v_0)$	
$c(e_1) \le c(v_1) \le c(v_2) \le c(e_2) \le c(v_0)$	$k \ge \max\{r + 2t + 1, s + 1\}$
$c(e_1) \le c(v_1) \le c(e_2) \le c(v_2) \le c(v_0)$	$k \ge \max\{r + t + 1, s + t + 1, 3t + 1\}$
$c(e_1) \le c(v_1) \le c(e_2) \le c(v_0) \le c(v_2)$	

Hence, for any path of order $n \ge 3$, *by Lemma 1.1, the* $[r, s, t]$*-chromatic number is bounded from below by the minimum of all these values (where there is no need to consider values larger then any other).*

$$\begin{aligned} k \geq \min\{ & \max\{r+2t+1, s+1\}, \\ & \max\{r+1, s+2t+1\}, \\ & \max\{r+t+1, s+t+1, 2t+1\}, \\ & \max\{r+t+1, 2r+1, s+1, 2t+1\}, \\ & r+s+t+1\}. \end{aligned}$$

Observe that, if $r \leq s+2t$, $s \leq r+2t$ *and* $t \leq r+s$,
$k \geq \min\{r+2t+1, s+2t+1, \max\{r+t+1, s+t+1, 2t+1\}, \max\{r+t+1, 2r+1, s+1, 2t+1\}\}$.

Observation 2.1.3. *If we consider paths that include, as a substructure, a edge-vertex-edge-vertex-edge chain (paths of order greater of equal to* 4*), the same lower bound is also given, exchanging* r *and* s *in all cases (considering the same constellations with the following changes:* $v_i \to e_{i+1}$ *and* $e_i \to v_i$*).*

$$\begin{aligned} k \geq \min\{ & \max\{s+2t+1, r+1\}, \\ & \max\{s+1, r+2t+1\}, \\ & \max\{s+t+1, r+t+1, 2t+1\}, \\ & \max\{s+t+1, 2s+1, r+1, 2t+1\}, \\ & r+s+t+1\}. \end{aligned}$$

And if $r \leq s+2t$, $s \leq r+2t$ *and* $t \leq r+s$,
$k \geq \min\{s+2t+1, r+2t+1, \max\{s+t+1, r+t+1, 2t+1\}, \max\{s+t+1, 2s+1, r+1, 2t+1\}\}$.

2.2. GENERAL LOWER BOUNDS FOR $\chi_{R,S,T}(P_N)$

Lemma 2.2.1. *If* $t < r \leq s < r+t$ *and* $2r \geq s+t$*, then*

$$\chi_{r,s,t}(P_n) \geq s+t+1 \text{ for all } n \geq 3.$$

Proof. By Observation 2.1.2, $k \geq \min\{r+2t+1, s+2t+1, s+t+1, 2r+1\} = s+t+1$. □

Lemma 2.2.2. *If* $t \leq r \leq 2t$, $2r > s$ *and* $2r \leq s+t$*, then*

$$\chi_{r,s,t}(P_n) \geq 2r+1 \text{ for all } n \geq 3.$$

Proof. By Observation 2.1.2, $k \geq \min\{r+2t+1, s+2t+1, s+t+1, 2r+1\} = 2r+1$. □

Lemma 2.2.3. *If* $s \geq r$, $s \geq t$ *and* $2r < 3t$*, then*

$$\chi_{r,s,t}(P_n) \geq 2r+1 \text{ for all } n \geq 4.$$

Lemma 2.2.4. *If* $r \geq s$, $r \geq t$ *and* $2s < 3t$*, then*

$$\chi_{r,s,t}(P_n) \geq 2s+1 \text{ for all } n \geq 4.$$

To prove the previous lemmas we will show a detailed procedure to be often used in this paper to solve similar situations. Let us call it "symmetric replacement".

Observe that, in these lemmas, in the assumptions and the statements r and s have been exchanged in all cases. Introducing a new notation: x_i, y_i, for all i, are the elements of the graph (vertices or edges) in the following order $\ldots x_1 y_1 x_2 y_2 \ldots$ and $N(x) = r$ if x_i is a vertex for all i and $N(x) = s$ if x_i is an edge for all i (we use the same convention for y_i's) – both can be reformulated together as follows.

Lemma 2.2.3/2.2.4.

If $N(x) \geq N(y)$, $N(x) \geq t$ and $2N(y) < 3t$, then

$$\chi_{r,s,t}(P_n) \geq 2N(y) + 1 \quad \text{for all } n \geq 4.$$

Observe that in this case the minimum order of the path for which the condition holds is the same. But it will not be like this in general, so we will give the proof and then, depending of the role of x_i and y_i, this value will be fixed.

Proof. Suppose that $k \leq 2N(y)$. By Lemma 1.6, we may assume $c(y_1), c(y_3) < c(y_2)$. Then $c(y_1), c(y_3) \leq N(y)$ and $N(y) + 1 \leq c(y_2) \leq 2N(y)$. By symmetry, let us assume $c(x_2) < c(x_3)$. Then $c(y_2) < c(x_2) < c(x_3)$ is not possible, because $N(x) + N(y) + t + 1 > 2N(y) + 1$.

Case 1. $c(x_2) < c(y_2) < c(x_3)$. Then $c(x_2) \leq 2N(y) - 2t$, $t + 1 \leq c(y_2) \leq 2N(y) - t$ and $c(y_1), c(y_3) \leq N(y) - t$. Now $2N(y) - 3t < 0$ implies $c(y_1) > c(x_2)$ and $c(y_1) \geq t + 1$, a contradiction.

Case 2. $c(x_2) < c(x_3) < c(y_2)$. Then $c(x_2) \leq 2N(y) - t - N(x)$ and $N(x) + 1 \leq c(x_3) \leq 2N(y) - t$. Now $N(x) + t + 1 > N(y)$ implies $c(y_3) < c(x_3)$ and $c(y_3) \leq 2N(y) - 2t$.

If there exists an x_4, then $2N(y) - 3t < 0$ implies $c(x_4) > c(y_3)$ and $c(x_4) \geq t + 1$. Now $2N(y) - 2t < t + 1$ implies $c(x_4) > c(x_3)$ and $c(x_4) \geq 2N(x) + 1$, a contradiction. If there exists no x_4, but there exist an x_1 and y_0, then $2N(y) - 2t - N(x) \leq 2N(y) - 3t$ implies $c(y_1) > c(x_2)$ and $c(y_1) \geq t + 1$. Now $2N(y) - t - 2N(x) \leq 2N(y) - 3t$ implies $c(x_1) > c(x_2)$ and $c(x_1) \geq N(x) + 1$. Next $N(y) - t < N(x) + 1$ implies $c(x_1) > c(y_1)$ and $c(x_1) \geq 2t + 1$. Then $c(y_0) < c(x_1)$ which implies $c(y_0) \leq 2N(y) - t$ and on the other hand, $c(y_0) > c(y_1)$ implies $c(y_0) \geq N(y) + t + 1$, a contradiction with $N(y) \leq 2t$. □

Now let us analyze both situations:

If x_i is a vertex for all i (Lemma 2.2.4), we will be always in the situation where v_3 (x_4) exists (because the existence of e_3 (y_3) has already been assumed). Hence, the elements used are e_1, v_1, e_2, v_2, e_3 and v_3, in other words, a path of order 4. So $\chi_{r,s,t}(P_n) \geq 2s + 1$ for all $n \geq 4$.

On the other hand, if x_i is an edge for all i (Lemma 2.2.3), v_1, e_2, v_2, e_3, v_3 and e_4 were used in one situation and $v_0, e_1, v_1, e_2, v_2, e_3$ and v_3 in the other. Hence, $\chi_{r,s,t}(P_n) \geq 2r + 1$ for all $n \geq 4$.

Lemma 2.2.5. *If* $2r < 3t$ *and* $2r \leq 2t + s$, *then*

$$\chi_{r,s,t}(P_n) \geq 2r + 1 \text{ for all } n \geq 5.$$

Proof. Observe that $2r \leq 2t + s$ implies $r \leq t + s/2 \leq t + s$.
Suppose $k \leq 2r$. By Lemma 1.6, we may assume $c(v_0), c(v_2) < c(v_1)$. Hence, $c(v_0), c(v_2) \leq r$ and $r + 1 \leq c(v_1) \leq 2r$. By symmetry, we can suppose $c(e_1) < c(e_2)$. Then $c(v_1) < c(e_1) < c(e_2)$ is not possible because $r + s + t + 1 > 2r$.
Case 1. $c(e_1) < c(v_1) < c(e_2)$. Then $c(e_1) \leq 2r - 2t$, $r + 1 \leq c(v_1) \leq 2r - t$ and thus $c(v_0), c(v_2) \leq r - t$. Now $2r - 3t < 0$ implies $c(v_0) > c(e_1)$ and $c(v_0) \geq t + 1$, a contradiction.
Case 2. $c(e_1) < c(e_2) < c(v_1)$. Then $c(e_1) \leq 2r - t - s$ and $s + 1 \leq c(e_2) \leq 2r - t$. Now $2r - 2t - s \leq 0$ implies $c(v_0) > c(e_1)$ and $c(v_0) \geq t + 1$. Hence $c(e_1) \leq r - t$.
If there exist an e_0 and v_{-1}, $r-t-s \leq 0$ implies $c(e_0) > c(e_1)$ and $c(e_0) \geq s+1$. Then $r-t < s/2 < s+1$ implies $c(e_0) > c(v_0)$ and $c(e_0) \geq 2t+1$. Thus $c(v_{-1}) < c(e_0)$, which implies $c(v_{-1}) \leq 2r-t$. Furthermore $c(v_{-1}) > c(v_0)$, which implies $c(v_{-1}) \geq r+t+1$, a contradiction.
If there exists neither e_0 nor v_{-1}, but there exist an e_3, v_3, e_4 and v_4, then $s+t+1 > r$ implies $c(v_2) < c(e_2)$ and $c(v_2) \leq 2r - 2t$. Hence $c(e_2) \geq t + 1$. Then $2r - 3t < 0$ implies $c(e_3) > c(v_2)$ and $c(e_3) \geq t + 1$. Now $2r - t - s \leq t$ implies $c(e_3) > c(e_2)$ and $c(e_3) \geq \max\{2s + 1, s + t + 1\}$. Thus, $s + 2t + 1 > 2r$ implies $c(v_3) < c(e_3)$ and $c(v_3) \leq 2r-t$. Furthermore $c(v_3) > c(v_2)$ implies $c(v_3) \geq r+1$. Then $c(e_3) \geq r+t+1$. Now $r + s + t + 1 > 2r$ implies $c(e_4) < c(e_3)$ and $c(e_4) \leq 2r - s$. Next $r + t + 1 > r + (r - s) + 1 > 2r - s$ implies $c(e_4) < c(v_3)$ and $c(e_4) \leq 2r - 2t$. Then $c(v_4) > c(e_4)$ implies $c(v_4) \geq t + 1$ and $c(v_4) < c(v_3)$ implies $c(v_4) \leq 2r - 2t$, a contradiction.

Hence, $\chi_{r,s,t}(P_n) \geq 2r + 1$ for all $n \geq 5$. □

Lemma 2.2.6. *If* $2s < 3t$ *and* $2s \leq 2t + r$, *then*

$$\chi_{r,s,t}(P_n) \geq 2s + 1 \text{ for all } n \geq 6.$$

Proof. Directly by "the symmetric replacement" in the proof of Lemma 2.2.5. □

Lemma 2.2.7. *If* $t \leq r \leq 2t$ *and* $s \leq r \leq s + t$, *then*

$$\chi_{r,s,t}(P_n) \geq r + t + 1 \text{ for all } n \geq 3.$$

Proof. By Observation 2.1.2, $k \geq \min\{r+2t+1, s+2t+1, r+t+1, 2r+1\} = r+t+1$. □

Lemma 2.2.8. *If* $t \leq s \leq 2t$ *and* $r \leq s \leq r + t$, *then*

$$\chi_{r,s,t}(P_n) \geq s + t + 1 \text{ for all } n \geq 4.$$

Proof. We can use the lower bound given in Observation 2.1.3 and obtain $k \geq \min\{s+2t + 1, r + 2t + 1, s + t + 1, 2s + 1\} = s + t + 1$. □

3. $\chi_{R,S,T}(P_N)$

Theorem 3.1. *If P_2 is a path of order 2, then*

$$\chi_{r,s,t}(P_2) = \begin{cases} r+1 & \text{if } r \geq 2t; \\ 2t+1 & \text{if } t \leq r < 2t; \\ r+t+1 & \text{if } r < t. \end{cases}$$

Proof. Since just one edge is considered in this case, $\chi_{r,s,t} = \chi_{r,s',t}$ for all s and s'. Then the proof is a direct consequence of Lemma 4 in [3], which shows that if $\chi(G) = 2$ (which is the case):

$$\chi_{r,0,t}(G) = \begin{cases} r+1 & \text{if } r \geq 2t; \\ 2t+1 & \text{if } t \leq r < 2t; \\ r+t+1 & \text{if } r < t. \end{cases}$$

□

Theorem 3.2. *If $r \geq s + 2t$, then*

$$\chi_{r,s,t}(P_n) = r + 1 \text{ for all } n \geq 3.$$

Proof. The following colouring

$$(\ldots, t+1, \mathbf{r{+}1}, s+t+1, \mathbf{1}, t+1, \mathbf{r{+}1}, s+t+1, \ldots)$$

shows that $\chi_{r,s,t}(P_n) \leq r + 1$ for all n. So, by Remark 1.5, we conclude that $\chi_{r,s,t}(P_n) = r+1$ for all $n \geq 2$. Observe that this includes $n = 2$ and $r \geq s+2t \geq 2t$, what has already been proved. □

Theorem 3.3. *If $s \geq r + 2t$, then*

$$\chi_{r,s,t}(P_n) = s + 1 \text{ for all } n \geq 3.$$

Proof. Similarly, the following colouring

$$(\ldots, s+1, \mathbf{t{+}1}, 1, \mathbf{r{+}t{+}1}, s+1, \mathbf{t{+}1}, 1, \ldots)$$

and Remark 1.5 show that $\chi_{r,s,t}(P_n) = s + 1$ for all $n \geq 3$. □

Theorem 3.4. *If $s \leq r < s + t$ and $r \geq 2t$, then*

$$\chi_{r,s,t}(P_n) = r + t + 1 \text{ for all } n \geq 3.$$

Proof. The following colouring

$$(\ldots, t+1, \mathbf{1}, r+t+1, \mathbf{r{+}1}, t+1, \mathbf{1}, r+t+1, \ldots)$$

shows that $\chi_{r,s,t}(P_n) \leq r+t+1$ for all n. Observation 2.1.2 shows that $k \geq \min\{r+2t+1, s+2t+1, r+t+1, 2r+1\} = r+t+1$ for $n \geq 3$. Hence, $\chi_{r,s,t}(P_n) = r+t+1$ for all $n \geq 3$. □

Theorem 3.5. *If $s+t \leq r < s+2t$ and $r \geq 2t$, then*

$$\chi_{r,s,t}(P_n) = s+2t+1 \text{ for all } n \geq 3.$$

Proof. The following colouring

$$(\ldots, t+1, \mathbf{1}, s+t+1, \mathbf{s{+}2t{+}1}, t+1, \mathbf{1}, s+t+1, \ldots)$$

shows that $\chi_{r,s,t}(P_n) \leq s+2t+1$ for all n, and a lower bound for $n \geq 3$ is given by Observation 2.1.2, $k \geq \min\{r+2t+1, s+2t+1, r+t+1, 2r+1\} = s+2t+1$. Hence, $\chi_{r,s,t}(P_n) = s+2t+1$ for all $n \geq 3$. □

Theorem 3.6. *If $r \leq s < r+t$ and $s \geq 2t$, then*

$$\chi_{r,s,t}(P_n) = \begin{cases} 2r+1 & \text{if } 2r < s+t \text{ for } n=3; \\ s+t+1 & \text{else.} \end{cases}$$

Proof. Observation 2.1.3 gives the lower bound $k \geq \min\{s+2t+1, r+2t+1, s+t+1, 2s+1\} = s+t+1$ for $n \geq 4$ and an upper bound is given by the following colouring

$$(\ldots, s+1, \mathbf{t{+}1}, 1, \mathbf{s{+}t{+}1}, s+1, \mathbf{t{+}1}, 1, \ldots).$$

Hence, $\chi_{r,s,t}(P_n) = s+t+1$ for all $n \geq 4$.

For $n=3$, if $2r \geq s+t$, Lemma 2.2.1 shows that $\chi_{r,s,t}(P_3) = s+t+1$. And if $2r < s+t$, the following colouring

$$(\mathbf{1}, 2r+1, \mathbf{r{+}1}, 1, \mathbf{2r{+}1})$$

(observe that $r+t > s \geq 2t$ implies $r > t$, therefore $2r > s$) shows that $\chi_{r,s,t}(P_3) \leq 2r+1$. Then $r < 2t$ ($2r < s+t$ implies $r < s-r+t < 2t$, because $s < r+t$, hence $s-r < t$) and $s < 2r$ ($s < r+t < 2r$). Hence, Lemma 2.2.2 can be applied to conclude that $\chi_{r,s,t}(P_3) = 2r+1$. □

Theorem 3.7. *If $r+t \leq s < r+2t$ and $s \geq 2t$, then*

$$\chi_{r,s,t}(P_n) = \begin{cases} s+1 & \text{if } r < 2t \text{ and } (r < t \text{ or } 2r \leq s) \text{ for } n=3; \\ 2r+1 & \text{if } s < 2r < 4t \text{ and } r \geq t \text{ for } n=3; \\ r+2t+1 & \text{else.} \end{cases}$$

Proof. Observation 2.1.3 gives the lower bound $k \geq \min\{s+2t+1, r+2t+1, s+t+1, 2s+1\} = r+2t+1$ and then the following colouring

$$(\ldots, r+2t+1, \mathbf{t{+}1}, 1, \mathbf{r{+}t{+}1}, r+2t+1, \mathbf{t{+}1}, 1, \ldots)$$

shows that $\chi_{r,s,t}(P_n) = r+2t+1$ for all $n \geq 4$.
For $n=3$, if $r \geq 2t$, Observation 2.1.2 shows that $k \geq \min\{r+2t+1, s+2t+1, s+t+1, 2r+1\} = r+2t+1$. Hence $\chi_{r,s,t}(P_3) = r+2t+1$ with the previous colouring. If $r < 2t$, one possible colouring is

$$(\mathbf{1}, s+1, \max\{\mathbf{t+1}, \mathbf{r+1}\}, 1, \max\{\mathbf{2r+1}, \mathbf{r+t+1}\}).$$

Hence, if $t > r$ or ($r \geq t$ and $2r \leq s$), then by Remark 1.5 $\chi_{r,s,t}(P_3) = s + 1$. On the other hand, if $r \geq t$ and $2r > s$, then by the colouring and Lemma 2.2.2 (because $2r < (r + t) + t \leq s + t$), $\chi_{r,s,t}(P_3) = 2r + 1$. □

Theorem 3.8. *For $t < r, s < 2t$, the following conditions hold true:*

1. *If $n = 3$, then*

$$\chi_{r,s,t}(P_3) = \begin{cases} r+t+1 & \text{if } s < r; \\ s+t+1 & \text{if } r \leq s \text{ and } 2r \geq s+t; \\ 2r+1 & \text{if } r \leq s \text{ and } 2r < s+t. \end{cases}$$

2. *If ($3t \leq 2r$, $3t \leq 2s$ for $n \geq 4$) or ($3t \leq 2r$, $3t > 2s$ for $n \geq 5$) or ($3t > 2r$, $3t \leq 2s$ for $n \geq 6$), then*

$$\chi_{r,s,t}(P_n) = 3t + 1.$$

3. *If $3t \leq 2r$ and $3t > 2s$ for $n = 4$, then*

$$\chi_{r,s,t}(P_n) = \begin{cases} 2s+1 & \text{if } 2s > r+t; \\ r+t+1 & \text{if } 2s \leq r+t. \end{cases}$$

4. *If $3t > 2r$ and $3t \leq 2s$ for $n = 4$ or 5, then*

$$\chi_{r,s,t}(P_n) = \begin{cases} 2r+1 & \text{if } 2r > s+t; \\ s+t+1 & \text{if } 2r \leq s+t. \end{cases}$$

5. *If $3t > 2r$ and $3t > 2s$, then*

$$\chi_{r,s,t}(P_n) = \begin{cases} 2r+1 & \text{if } (s<r \text{ for } n\geq 5) \text{ or } (r<s \text{ and } 2r>s+t \text{ for } n=4 \text{ or } 5); \\ 2s+1 & \text{if } (r<s \text{ for } n\geq 6) \text{ or } (s<r \text{ and } 2s>r+t \text{ for } n=4); \\ r+t+1 & \text{if } s<r \text{ and } 2s\leq r+t \text{ for } n=4; \\ s+t+1 & \text{if } r<s \text{ and } 2r\leq s+t \text{ for } n=4 \text{ or } 5. \end{cases}$$

Proof. Observe that in this case, we have a triple $[r, s, t]$, which satisfies the conditions of Lemmas 2.2.7 and 2.2.8, hence we have the lower bounds:
$\chi_{r,s,t}(P_n) \geq r + t + 1$ for all $n \geq 3$, if $s \leq r$, and $\chi_{r,s,t}(P_n) \geq \max\{r + t + 1, s + t + 1\}$ for all $n \geq 4$.

(1) If $n = 3$, then we distinguish the following three cases:

If $s < r$, then the following colouring:

$$(\mathbf{r+1}, r+t+1, \mathbf{1}, t+1, \mathbf{2t+1})$$

and the lower bound show that $\chi_{r,s,t}(P_3) = r + t + 1$.

If $r \leq s$, then the following colouring:

$$(\mathbf{s+1}, s+t+1, \mathbf{1}, t+1, \mathbf{2t+1})$$

shows that $\chi_{r,s,t}(P_3) \leq s+t+1$. If $2r \geq s+t$, then by Lemma 2.2.1, $\chi_{r,s,t}(P_3) = s+t+1$. And if $2r < s+t$, then P_3 could be coloured as follows:

$$(\mathbf{1}, 2r+1, \mathbf{r{+}1}, 1, \mathbf{2r{+}1}),$$

which, together with Lemma 2.2.2, shows that $\chi_{r,s,t}(P_3) = 2r+1$.

(2) If $3t \leq 2r$ or $3t \leq 2s$, then the colouring

$$(\ldots, 2t+1, \mathbf{t{+}1}, 1, \mathbf{3t{+}1}, 2t+1, \mathbf{t{+}1}, 1, \ldots)$$

shows that $\chi_{r,s,t}(P_n) \leq 3t+1$ for all n.

Suppose $k \leq 3t$. Now if $3t \leq 2r$, then by Lemma 1.6, we may suppose $c(v_0), c(v_2) < c(v_1)$. Hence, $c(v_0), c(v_2) \leq 3t-r$ and $r+1 \leq c(v_1) \leq 3t$. By symmetry, let us assume $c(e_1) < c(e_2)$. Then $c(e_2) > c(e_1) > c(v_1)$ is not possible, because $r+s+t+1 > 3t$.
Case 1. $c(e_1) < c(v_1) < c(e_2)$. Then $c(e_1) \leq t$, $r+1 \leq c(v_1) \leq 2t$ and thus $c(v_0), c(v_2) \leq 2t-r$. Then $c(v_0) > c(e_1)$, which implies $c(v_0) \geq t+1$, a contradiction.
Case 2. $c(e_1) < c(e_2) < c(v_1)$. Then $c(e_1) \leq 2t-s$ and $s+1 \leq c(e_2) \leq 2t$. And this implies $c(v_0) > c(e_1)$ and $c(v_0) \geq t+1$.
If there exist an e_0 and v_{-1}, then $c(e_0) > c(e_1)$ and $c(e_0) \geq s+1$, thus $c(e_0) > c(v_0)$ and $c(e_0) \geq 2t+1$. Therefore, $c(v_{-1}) < c(e_0)$ and $c(v_{-1}) \leq 2t$, hence $c(v_{-1}) < c(v_0)$ and $c(v_{-1}) \leq 3t-2r \leq 0$, which is a contradiction.
If there exists neither e_0 nor v_{-1}, but there exist an e_3, v_3, e_4 and v_4, then $c(v_2) < c(e_2)$ and $c(v_2) \leq t$. Therefore, $c(e_3) > c(v_2)$ and $c(e_3) \geq t+1$, which implies $c(e_3) > c(e_2)$ and $c(e_3) \geq 2s+1$. This holds if and only if $3t > 2s$.
In this case, $c(v_3) < c(e_3)$ implies $c(v_3) \leq 2t$ and $c(v_3) > c(v_2)$ implies $c(v_3) \geq r+1$. Hence, $c(e_3) \geq r+t+1$. Then $c(e_4) < c(e_3)$ and $c(e_4) \leq 3t-s$ imply $c(e_4) < c(v_3)$ and $c(e_4) \leq t$. Therefore, $c(v_4) > c(e_4)$ and $c(v_4) \geq t+1$, which implies $c(v_4) > c(v_3)$ and $c(v_4) \geq 2r+1 > 3t$, a contradiction.

Hence, if $3t \leq 2r$ and $3t \leq 2s$, then $\chi_{r,s,t}(P_n) = 3t+1$ for all $n \geq 4$, and if $3t \leq 2r$ and $3t > 2s$, then $\chi_{r,s,t}(P_n) = 3t+1$ for all $n \geq 5$.

If $3t \leq 2s$, the application of "the symmetric replacement" in the proof of the case $3t \leq 2r$ shows that if $3t \leq 2s$ and $3t > 2r$, then $\chi_{r,s,t}(P_n) = 3t+1$ for all $n \geq 6$.

(3) If $3t \leq 2r$ and $3t > 2s$ for $n = 4$, the colouring

$$(\mathbf{t{+}1}, 1, \mathbf{r{+}t{+}1}, s+1, \mathbf{1}, \max\{2s+1, r+t+1\}, \mathbf{r{+}1})$$

shows that $\chi_{r,s,t}(P_4) = 2s+1$, if $2s > r+t$ by Lemma 2.2.4. And if $2s \leq r+t$, then from the lower bound, $\chi_{r,s,t}(P_4) = r+t+1$.

(4) If $3t > 2r$ and $3t \leq 2s$ for $n = 4$ or 5, the colouring

$$(\mathbf{s{+}t{+}1}, t+1, \mathbf{1}, s+t+1, \mathbf{r{+}1}, 1, \max\{\mathbf{2r+1}, \mathbf{s+t+1}\}, s+1, \mathbf{1})$$

shows that $\chi_{r,s,t}(P_n) = 2r+1$ for $n = 4$ or 5, if $2r > s+t$ by Lemma 2.2.3. And if $2r \leq s+t$, then from the lower bound, $\chi_{r,s,t}(P_n) = s+t+1$ for $n = 4$ or 5.

(5) Let $3t > 2r$ and $3t > 2s$.

If $s < r$, then the colouring

$$(\ldots, 2r+1, \mathbf{r{+}1}, 1, \mathbf{2r{+}1}, r+1, \mathbf{1}, 2r+1, \ldots)$$

and Lemma 2.2.5 show that $\chi_{r,s,t}(P_n) = 2r+1$ for all $n \geq 5$. If $n = 4$, then we are basically in the same situation as in case (3). Hence, if $2s > r+t$, then $\chi_{r,s,t}(P_4) = 2s+1$ and, if $2s \leq r+t$, then $\chi_{r,s,t}(P_4) = r+t+1$.

If $r < s$, by Lemma 2.2.6, $\chi_{r,s,t}(P_n) = 2s+1$ for all $n \geq 6$ with the colouring

$$(\ldots, 2s+1, \mathbf{s{+}1}, 1, \mathbf{2s{+}1}, s+1, \mathbf{1}, 2s+1, \ldots)$$

and, if $n = 4$ or $n = 5$, we could use the same argument as in case *d)*. Hence, if $2r > s+t$, then $\chi_{r,s,t}(P_n) = 2r+1$ for n equal to 4 or 5, and if $2r \leq s+t$, then $\chi_{r,s,t}(P_n) = s+t+1$ for n equal to 4 or 5. □

Theorem 3.9. *If $s \leq t \leq r < 2t$ and $s < r$, then*

$$\chi_{r,s,t}(P_n) = \begin{cases} s+2t+1 & \text{if } s \leq 2r-2t \text{ and } ((r \geq s+t \text{ and } n \geq 3) \text{ or } n \geq 5); \\ 2r+1 & \text{if } s > 2r-2t \text{ and } n \geq 5; \\ r+t+1 & \text{otherwise.} \end{cases}$$

Proof. a) If $s \leq 2r-2t$, then the colouring

$$(\ldots, s+t+1, \mathbf{1}, t+1, \mathbf{s{+}2t{+}1}, s+t+1, \mathbf{1}, t+1, \ldots)$$

shows that $\chi_{r,s,t}(P_n) \leq s+2t+1$ for all n.

If $r \geq s+t$, then Observation 2.1.2 gives the lower bound $k \geq \min\{r+2t+1, s+2t+1, r+t+1, 2r+1\} = s+2t+1$, for $n \geq 3$. Hence, $\chi_{r,s,t}(P_n) = s+2t+1$ for all $n \geq 3$.

If $r < s+t$, suppose $k \leq s+2t$, which is at most $2r$. Then, by Lemma 1.6, we may assume $c(v_0), c(v_2) < c(v_1)$. Hence $c(v_0), c(v_2) \leq s+2t-r$ (observe that $s+2t-r > t$) and $r+1 \leq c(v_1) \leq s+2t$. By symmetry, let us assume $c(e_1) < c(e_2)$, then $c(e_1) \leq 2t$ and $s+1 \leq c(e_2) \leq s+2t$.

Case 1. $c(e_2) > c(v_1)$. Then $r+1 \leq c(v_1) \leq s+t$ and thus $c(e_1) < c(v_1)$ and $c(e_1) \leq s$. Then $c(v_0) > c(e_1)$ implies $c(v_0) \geq t+1$ and now $c(v_0) < c(v_1)$ implies $c(v_0) \leq s+t-r$, a contradiction.

Case 2. $c(e_2) < c(v_1)$. Then $s+1 \leq c(e_2) \leq s+t$ and $c(e_1) \leq t$. This implies $c(v_0) > c(e_1)$, $t+1 \leq c(v_0) \leq s+2t-r$ and $e_1 \leq s+t-r$.
If there exist an e_0 and v_{-1}, then $c(e_0) > c(e_1)$ and $c(e_0) \geq s+1$. Thus $c(e_0) > c(v_0)$ and $c(e_0) \geq 2t+1$. And thus $c(v_{-1}) < c(e_0)$, which implies $c(v_{-1}) \leq s+t$ and $c(v_{-1}) > c(v_0)$; hence $c(v_{-1}) \geq r+t+1$, a contradiction.
If there exists neither e_0 nor v_{-1}, but there exist an e_3, v_3, e_4 and v_4, then $c(v_2) < c(e_2)$ implies $c(v_2) \leq s$ and $c(e_2) \geq t+1$. Then $c(e_3) > c(v_2)$ and $c(e_3) \geq t+1$ and thus, $c(e_3) > c(e_2)$ and $c(e_3) \geq s+t+1$. Then $c(v_3) < c(e_3)$, hence $c(v_3) \leq s+t$. And $c(v_3) > c(v_2)$ thus $c(v_3) \geq r+1$, which implies $c(e_3) \geq r+t+1$. Then $c(e_4) < c(e_3)$ and $c(e_4) \leq 2t$. Hence $c(e_4) < c(v_3)$, $c(e_4) \leq s$ and $c(v_4) > c(e_4)$, which implies $c(v_4) \geq t+1$, $c(v_4) > c(v_3)$ and $c(v_4) \geq 2r+1$, a contradiction.

Hence, $\chi_{r,s,t}(P_n) = s+2t+1$ for all $n \geq 5$.

b) If $s > 2r-2t$, then the colouring

$$(\ldots, r+1, \mathbf{1}, 2r+1, \mathbf{r+1}, 1, \mathbf{2r+1}, r+1, \ldots)$$

shows that $\chi_{r,s,t}(P_n) \leq 2s+1$ for all n. Now Lemma 2.2.5 may be applied, hence $\chi_{r,s,t}(P_n) = 2r+1$ for all $n \geq 5$.

c) If $s > 2r-2t$ or ($s \leq 2r-2t$ and $r < s+t$) and $n \leq 4$, then the colouring

$$(\mathbf{t+1}, 1, \mathbf{r+t+1}, t+1, \mathbf{1}, r+t+1, \mathbf{r+1})$$

and Lemma 2.2.7 show that $\chi_{r,s,t}(P_n) = r+t+1$ for $n = 3, 4$. □

Theorem 3.10. *If $r \leq t \leq s < 2t$ and $r < s$, then*

$$\chi_{r,s,t}(P_n) = \begin{cases} r+2t+1 & \text{if } r \leq 2s-2t \text{ and } ((s \geq r+t \text{ and } n \geq 4) \text{ or } n \geq 6); \\ 2s+1 & \text{if } r > 2s-2t \text{ and } n \geq 6; \\ 2t+1 & \text{if } n = 3; \\ s+t+1 & \text{otherwise.} \end{cases}$$

Proof. a) If $r \leq 2s-2t$, then the colouring

$$(\ldots, r+2t+1, \mathbf{t+1}, 1, \mathbf{r+t+1}, r+2t+1, \mathbf{t+1}, 1, \ldots)$$

shows that $\chi_{r,s,t}(P_n) \leq r+2t+1$ for all n.

If $s \geq r+t$, then by Observation 2.1.3, $k \geq \min\{s+2t+1, r+2t+1, s+t+1, 2s+1\} = r+2t+1$. Hence, $\chi_{r,s,t}(P_n) = r+2t+1$ for all $n \geq 4$.

If $s < r+t$, then by "the symmetric replacement" in the proof of Theorem 3.9, for $r < s+t$, $\chi_{r,s,t}(P_n) = r+2t+1$ for all $n \geq 6$.

b) If $r > 2s - 2t$, then the colouring

$$(\ldots, s+1, \mathbf{1}, 2s+1, \mathbf{s{+}1}, 1, \mathbf{2s{+}1}, s+1, \ldots)$$

shows that $\chi_{r,s,t}(P_n) \leq 2s+1$ for all n. Hence, by Lemma 2.2.6 $\chi_{r,s,t}(P_n) = 2s+1$ for all $n \geq 6$.

c) If $r > 2s - 2t$ or ($r \leq 2s - 2t$ and $s < r + t$) and $n = 4$ or $n = 5$, then the colouring

$$(\mathbf{2t{+}1}, t+1, \mathbf{1}, s+t+1, \mathbf{s{+}1}, 1, \mathbf{s{+}t{+}1}, s+1, \mathbf{1})$$

and Lemma 2.2.8 show that $\chi_{r,s,t}(P_n) = s+t+1$ for $n = 4$ or 5.

d) If $n = 3$, then the colouring

$$(\mathbf{1}, 2t+1, \mathbf{t{+}1}, 1, \mathbf{2t{+}1})$$

shows that $\chi_{r,s,t}(P_3) \leq 2t+1$. By Observation 2.1.2, we get $k \geq \min\{r+2t+1, s+2t+1, s+t+1, 2t+1\} = 2t+1$. Hence, $\chi_{r,s,t}(P_3) = 2t+1$. □

Theorem 3.11. *If* $r, s \leq t < r+s$, *then*

$$\chi_{r,s,t}(P_n) = 2t+1 \textit{ for all } n \geq 3.$$

Proof. The colouring

$$(\ldots, t+1, \mathbf{1}, 2t+1, \mathbf{t{+}1}, 1, \mathbf{2t{+}1}, t+1, \ldots)$$

shows that $\chi_{r,s,t}(P_n) \leq 2t+1$ for all n. By Observation 2.1.2, we get $k \geq \min\{r+2t+1, s+2t+1, 2t+1, 2t+1\} = 2t+1$ for all $n \geq 3$. Hence, $\chi_{r,s,t}(P_n) = 2t+1$ for all $n \geq 3$. □

Theorem 3.12. *If* $t \geq r+s$, *then*

$$\chi_{r,s,t}(P_n) = r+s+t+1 \textit{ for all } n \geq 3.$$

Proof. By Remark 1.5, the bound $\chi_{r,s,t}(P_n) \leq r+s+t+1$ holds for all n, because the colouring

$$(\ldots, r+s+t+1, \mathbf{1}, r+t+1, \mathbf{r{+}1}, r+s+t+1, \mathbf{1}, r+t+1, \ldots)$$

is always possible. Then, by Observation 2.1.2, $k \geq \min\{r+2t+1, s+2t+1, 2t+1, 2t+1, r+s+t+1\} = r+s+t+1$. Hence, $\chi_{r,s,t}(P_n) = r+s+t+1$ for all $n \geq 3$. □

All results presented in this section are summarized in Table 7.

Table 7

$\chi_{[r,s,t]}(P_n)$	Size	Conditions		
$r+1$	2	$r \geq 2t$		
$2t+1$	2	$t \leq r < 2t$		
$r+t+1$	2	$r < t$		
$r+1$	≥ 3	$r \geq s+2t$		
$s+1$	≥ 3	$s \geq r+2t$		
$r+t+1$	≥ 3	$(s \leq r < s+t) \wedge (r \geq 2t)$		
$s+2t+1$	≥ 3	$(s+t \leq r < s+2t) \wedge (r \geq 2t)$		
$2r+1$	3	$(r \leq s < r+t) \wedge (s \geq 2t)$		$2r < s+t$
$s+t+1$	≥ 3			$2r \geq s+t \vee n \geq 4$
$s+1$	3	$(r+t \leq s < r+2t) \wedge (s \geq 2t)$		$r < 2t \wedge (r < t \vee 2r \leq s)$
$2r+1$	3			$s < 2r < 4t \wedge r \geq t$
$r+2t+1$	≥ 3			$r \geq 2t \vee n \geq 4$
$r+t+1$	3	$t < r, s < 2t$	$s < r$	
$s+t+1$			$r \leq s$	$2r \geq s+t$
$2r+1$				$2r < s+t$
$3t+1$	≥ 4		$(3t \leq 2r \wedge 3t \leq 2s) \vee$ $(3t \leq 2r \wedge 3t > 2s \wedge n \geq 5) \vee$ $(3t > 2r \wedge 3t \leq 2s \wedge n \geq 6)$	
$2s+1$	4		$3t \leq 2r \wedge 3t > 2s$	$2s > r+t$
$r+t+1$				$2s \leq r+t$
$2r+1$	4,5		$3t > 2r \wedge 3t \leq 2s$	$2r > s+t$
$s+t+1$				$2r \leq s+t$
$2r+1$	≥ 4		$3t > 2r \wedge 3t > 2s$	$(r < s \wedge 2r > s+t \wedge n = 4,5)$ $\vee (s < r \wedge n \geq 5)$
$2s+1$	≥ 4			$(s < r \wedge 2s > r+t \wedge n = 4)$ $\vee (r < s \wedge n \geq 6)$
$r+t+1$	4			$s < r \wedge 2s \leq r+t$
$s+t+1$	4,5			$r < s \wedge 2r \leq s+t$
$r+t+1$	3,4	$(s \leq t \leq r < 2t) \wedge (s < r)$	$r < s+t \vee s > 2r-2t$	
$s+2t+1$	≥ 3		$s \leq 2r-2t \wedge (r \geq s+t \vee n \geq 5)$	
$2r+1$	≥ 5		$s > 2r-2t$	
$2t+1$	3	$(r \leq t \leq s < 2t) \wedge (r < s)$		
$s+t+1$	4,5		$s < r+t \vee r > 2s-2t$	
$r+2t+1$	≥ 4		$r \leq 2s-2t \wedge (s \geq r+t \vee n \geq 6)$	
$2s+1$	≥ 6		$r > 2s-2t$	
$2t+1$	≥ 3	$r, s \leq t < r+s$		
$r+s+t+1$	≥ 3	$t \geq r+s$		

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