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**THE STUDY OF DELSARTE-LIONS TYPE  
BINARY TRANSFORMATIONS,  
THEIR DIFFERENTIAL-GEOMETRIC  
AND OPERATOR STRUCTURE WITH APPLICATIONS.  
PART 2**

**Abstract.** The Gelfand-Levitan integral equations for Delsarte-Lions type transformations in multidimension are studied. The corresponding spectral and analytical properties of Delsarte-Lions transformed operators are analyzed by means of the differential-geometric and topological tools. An approach for constructing Delsarte-Lions type transmutation operators for multidimensional differential expressions is devised.

**Keywords:** Delsarte transmutation operators, generalized de Rham-Hodge differential complex, Delsarte-Lions type transformations, Gelfand-Levitan-Marchenko type integral equations, multikidimensional differential operator pencils.

**Mathematics Subject Classification:** Primary 34A30, 34B05, Secondary 34B15.

1. INTRODUCTION: GENERALIZED DE RHAM-HODGE COMPLEXES  
AND THEIR PROPERTIES

**1.1.** We begin with recalling some differential-geometric properties of Delsarte-Lions type transformations that were discussed in Part 1 for differential operator expressions acting in a multidimensional functional space  $\mathcal{H} = L_1(T;H)$ , where  $T = \mathbb{R}^2$  and  $H := L_2(\mathbb{R}^2; \mathbb{C}^2)$ . They appear to have a deep relationship with classical generalized de Rham-Hodge theory [3–6, 27] devised in the middle of the past century for a set of commuting differential operators defined, in general, on a smooth compact  $m$ -dimensional metric space  $M$ . Concerning our problem of describing the differential-geometric and spectral structure of Delsarte-Lions type transmutations acting in  $\mathcal{H}$ , following [30] we preliminarily consider some backgrounds of a generalized de Rham-Hodge differential complex theory devised for studying these transformations of differential operators. Consider a smooth metric space  $M$  being a

suitably compactified form of the space  $\mathbb{R}^m$ ,  $m \in \mathbb{Z}_+$ . Then on  $M_T := T \times M$  one can define the standard Grassmann algebra  $\Lambda(M_T; \mathcal{H})$  of differential forms on  $T \times M$  and consider an I.V. Skrypnik's [3-6] generalized external anti-differential operator  $d_{\mathcal{L}} : \Lambda(M_T; \mathcal{H}) \rightarrow \Lambda(M_T; \mathcal{H})$  acting as follows: for any  $\beta^{(k)} \in \Lambda^k(M_T; \mathcal{H})$ ,  $k = \overline{0, m}$ ,

$$d_{\mathcal{L}}\beta^{(k)} := \sum_{j=1}^2 dt_j \wedge L_j(t; x|\partial)\beta^{(k)} + \sum_{i=1}^m dx_i \wedge A_i(t; x; \partial)\beta^{(k)} \in \Lambda^{k+1}(M_T; \mathcal{H}), \quad (1.1)$$

where  $A_i \in C^2(T; \mathcal{L}(H))$ ,  $i = \overline{1, m}$ , are some differential operator mappings and

$$L_j(t; x|\partial) := \partial/\partial t_j - L_j(t; x|\partial), \quad (1.2)$$

$j = \overline{1, 2}$ , are suitably defined linear differential operators in  $\mathcal{H}$ , commuting with each other, that is

$$[L_1, L_2] = 0, \quad [A_k, A_i] = 0 \quad \text{and} \quad [L_j, A_i] = 0 \quad (1.3)$$

for all  $j = \overline{1, 2}$  and  $i, k = \overline{1, m}$ . We will put, in general, that differential expressions

$$L_j(t; x|\partial) := \sum_{|\alpha|=0}^{n_j(L)} a_{\alpha}^{(j)}(t; x) \frac{\partial^{|\alpha|}}{\partial x^{\alpha}}, \quad (1.4)$$

with coefficients  $a_{\alpha}^{(j)} \in C^1(T; C^{\infty}(M; \text{End} \mathbb{C}^N))$ ,  $|\alpha| = \overline{0, n_j(L)}$ ,  $n_j^{\alpha} \in \mathbb{Z}_+$ ,  $j = \overline{0, 1}$ , define some closed normal operators in the Hilbert space  $H$  for ant  $t \in T$ . It is easy to observe that the generalization of  $d_{\mathcal{L}}$  defined by (1.1) is a generalization of the usual external anti-differentiation

$$d = \sum_{j=1}^m dx_j \wedge \frac{\partial}{\partial x_j} + \sum_{s=1}^2 dt_s \wedge \frac{\partial}{\partial t_s} \quad (1.5)$$

for which, evidently, commutation conditions

$$\left[ \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k} \right] = 0, \quad \left[ \frac{\partial}{\partial t_s}, \frac{\partial}{\partial t_l} \right] = 0, \quad \left[ \frac{\partial}{\partial x_j}, \frac{\partial}{\partial t_s} \right] = 0 \quad (1.6)$$

hold for all  $j, k = \overline{1, m}$  and  $s, l = \overline{1, 2}$ . If now in (1.5) we substitute  $\partial/\partial x_j \rightarrow A_j$ ,  $\partial/\partial t_s \rightarrow L_s$ ,  $j = \overline{1, m}$ ,  $s = \overline{1, 2}$ , we get the anti-differentiation

$$d_{\mathcal{A}} := \sum_{j=1}^m dx_j \wedge A_j(t; x|\partial) + \sum_{j=1}^2 dt_s \wedge L_s(t; x|\partial), \quad (1.7)$$

where the differential expressions  $A_j, L_s : \mathcal{H} \rightarrow \mathcal{H}$  for all  $j, k = \overline{1, m}$  and  $s, l = \overline{1, 2}$ , satisfy the commutation conditions  $[A_j, A_k] = 0$ ,  $[L_s, L_s] = 0$ ,  $[A_j, L_s] = 0$ , then operation (1.7) defines an external generalized anti-differential operator on  $\Lambda(M_T; \mathcal{H})$ , with respect to which the co-chain sequence

$$\mathcal{H} \rightarrow \Lambda^0(M_T; \mathcal{H}) \xrightarrow{d_{\mathcal{A}}} \Lambda^1(M_T; \mathcal{H}) \xrightarrow{d_{\mathcal{A}}} \dots \xrightarrow{d_{\mathcal{A}}} \Lambda^{m+2}(M_T; \mathcal{H}) \xrightarrow{d_{\mathcal{A}}} 0 \quad (1.8)$$

is evidently a cohomological complex, that is  $d_{\mathcal{A}}d_{\mathcal{A}} \equiv 0$ . As anti-differential (1.1) is a particular case of (1.7), we conclude that the corresponding co-chain sequence (1.8) is a cohomological complex too.

**1.2.** Below we will follow the ideas developed in [3, 4, 35]. A differential form  $\beta \in \Lambda(M_{\mathbb{T}}; \mathcal{H})$  will be called  $d_{\mathcal{A}}$ -closed if  $d_{\mathcal{A}}\beta = 0$  and a form  $\gamma \in \Lambda(M_{\mathbb{T}}; \mathcal{H})$  will be called exact or  $d_{\mathcal{A}}$ -homological to zero if there exists on  $M_{\mathbb{T}}$  such a form  $\omega \in \Lambda(M_{\mathbb{T}}; \mathcal{H})$  that  $\gamma = d_{\mathcal{A}}\omega$ .

Consider now the standard [8, 34, 35, 48] algebraic Hodge star-operation

$$* : \Lambda^k(M_{\mathbb{T}}; \mathcal{H}) \longrightarrow \Lambda^{m+2-k}(M_{\mathbb{T}}; \mathcal{H}) \quad (1.9)$$

$k = \overline{0, m+2}$ , defined as follows: if  $\beta \in \Lambda^k(M_{\mathbb{T}}; \mathcal{H})$ , then the form  $*\beta \in \Lambda^{m+2-k}(M_{\mathbb{T}}; \mathcal{H})$  is such that:

—  $\langle \alpha, *\beta \rangle_{(m+2-k)} := \langle \langle \alpha, *\beta \rangle_{\mathbb{C}^N} \rangle_{m+2-k} = \langle \langle \alpha, \wedge \beta \rangle_{\mathbb{C}^N}, d\mu_g \rangle_{m+2}$  for any  $\alpha \in \Lambda^{m+2-k}(M_{\mathbb{T}}; \mathcal{H})$ , where  $d\mu_g$  is an invariant measure on the metric space  $M_{\mathbb{T}}$  with positive definite Riemannian metrics  $g : T(M_{\mathbb{T}}) \times T(M_{\mathbb{T}}) \rightarrow \mathbb{C}$ , the scalar product

$$\langle \sigma_1^{(1)} \wedge \sigma_2^{(1)} \wedge \dots \wedge \sigma_k^{(1)}, \gamma_1^{(1)} \wedge \gamma_2^{(1)} \wedge \dots \wedge \gamma_k^{(1)} \rangle_k := \det\{\langle \sigma_i^{(1)}, \gamma_j^{(1)} \rangle_1 : i, j = \overline{1, k}\};$$

$\langle \sigma_i^{(1)}, \gamma_j^{(1)} \rangle_1 := \langle \hat{g}^{-1}\sigma_i^{(1)}, \hat{g}^{-1}\gamma_j^{(1)} \rangle_g$ , 1-forms  $\sigma_i^{(1)}, \gamma_j^{(1)} \in \Lambda^1(M_{\mathbb{T}})$ ,  $i, j = \overline{1, k}$ , and  $\hat{g} : T(M_{\mathbb{T}}) \rightarrow T^*(M_{\mathbb{T}})$  is the canonical isomorphism, generated by the metrics  $\langle \cdot, \cdot \rangle_g$  on  $T(M_{\mathbb{T}})$ ;

- $(m - k + 2)$ -dimensional volume  $|*\beta|$  of form  $*\beta$  equals  $k$ -dimensional volume  $|\beta|$  of the form  $\beta$ ;
- the  $(m+2)$ -dimensional measure  $\langle \beta, \wedge *\beta \rangle \geq 0$  under the fixed orientation on  $M_{\mathbb{T}}$ .

Further, on the space  $\Lambda(M_{\mathbb{T}}; \mathcal{H})$  define the following natural scalar product: for any  $\beta, \gamma \in \Lambda^k(M_{\mathbb{T}}; \mathcal{H})$ ,  $k = \overline{0, m}$ :

$$(\beta, \gamma) := \int_{M_{\mathbb{T}}} \langle \beta, *\gamma \rangle_{\mathbb{C}^N}. \quad (1.10)$$

Subject to scalar product (1.10), one can naturally construct the corresponding Hilbert space

$$\mathcal{H}_{\Lambda}(M_{\mathbb{T}}) := \bigoplus_{k=0}^{m+2} \mathcal{H}_{\Lambda}^k(M_{\mathbb{T}}) \quad (1.11)$$

suitable for our further consideration. Notice that the Hodge star-operation satisfies the following easily verified property: for any  $\beta, \gamma \in \mathcal{H}_{\Lambda}^k(M_{\mathbb{T}})$ ,  $k = \overline{0, m}$ ,

$$(\beta, \gamma) = (*\beta, *\gamma), \quad (1.12)$$

that is the Hodge operation  $* : \mathcal{H}_{\Lambda}(M_{\mathbb{T}}) \rightarrow \mathcal{H}_{\Lambda}(M_{\mathbb{T}})$  is unitary and its standard adjoint with respect to scalar product (1.10) operation satisfies the condition  $(*)' = (*\text{'})^{-1}$ .

Denote by  $d'_{\mathcal{L}}$  the formally adjoint expression to weak differential operation (1.1). By means of the operations  $d'_{\mathcal{L}}$  and  $d_{\mathcal{L}}$  in the  $\mathcal{H}_{\Lambda}(M_{\mathbb{T}})$  one can naturally define [3, 8, 34, 35, 37] a generalized Laplace-Hodge operator  $\Delta_{\mathcal{L}} : \mathcal{H}_1(M_{\mathbb{T}}) \longrightarrow \mathcal{H}_1(M_{\mathbb{T}})$  as

$$\Delta_{\mathcal{L}} = d'_{\mathcal{L}}d_{\mathcal{L}} + d'_{\mathcal{L}}d_{\mathcal{L}}. \quad (1.13)$$

Take a form  $\beta \in \mathcal{H}_{\Lambda}(M_{\mathbb{T}})$  satisfying the equality

$$\Delta_{\mathcal{L}}\beta = 0. \quad (1.14)$$

Such a form is called [35] harmonic. One can also verify that a harmonic form  $\beta \in \mathcal{H}_{\Lambda}(M_{\mathbb{T}})$  satisfies simultaneously the following two adjoint conditions:

$$d'_{\mathcal{L}}\beta = 0, \quad d_{\mathcal{L}}\beta = 0 \quad (1.15)$$

easily stemming from (1.13) and (1.14).

**Lemma 1.1.** *The following differential operators in  $\mathcal{H}_{\Lambda}(M_{\mathbb{T}})$*

$$d_{\mathcal{L}}^* := *d'_{\mathcal{L}}(*)^{-1} \quad (1.16)$$

defines also a new external anti-differential operations in  $\mathcal{H}_1(M_{\mathbb{T}})$ , subject to which the respectively dual to (1.8) co-chain sequence

$$\mathcal{H} \longrightarrow \Lambda^0(M_{\mathbb{T}}; \mathcal{H}) \xrightarrow{d_{\mathcal{L}}^*} \Lambda^1(M_{\mathbb{T}}; \mathcal{H}) \xrightarrow{d_{\mathcal{L}}^*} \dots \xrightarrow{d_{\mathcal{L}}^*} \Lambda^{m+2}(M_{\mathbb{T}}; \mathcal{H}) \xrightarrow{d_{\mathcal{L}}^*} 0 \quad (1.17)$$

is a cohomological complex.

*Proof.* Really, the statement holds true owing to the property  $d_{\mathcal{L}}^*d_{\mathcal{L}}^* = 0$ , following from definition (1.16).  $\square$

**1.3.** Denote further by  $\mathcal{H}_{\Lambda(\mathbb{L})}^k(M_{\mathbb{T}})$ ,  $k = \overline{0, m+2}$ , the cohomology groups of  $d_{\mathcal{L}}$ -closed and by  $\mathcal{H}_{\Lambda(\mathcal{L}^*)}^k(M_{\mathbb{T}})$ ,  $k = \overline{0, m+2}$ , the cohomology groups of  $d_{\mathcal{L}^*}$ -closed differential forms, respectively, and by  $\mathcal{H}_{\Lambda(\mathcal{L}^*\mathcal{L})}^k(M_{\mathbb{T}})$ ,  $k = \overline{0, m+2}$ , the abelian groups of harmonic differential forms from the Hilbert sub-spaces  $\mathcal{H}_{\Lambda}^k(M_{\mathbb{T}})$ ,  $k = \overline{0, m+2}$ . Before formulating next results, define the standard Hilbert-Schmidt rigged chain [12, 13] of positive and negative Hilbert spaces of differential forms

$$\mathcal{H}_{\Lambda,+}^k(M_{\mathbb{T}}) \subset \mathcal{H}_{\Lambda}^k(M_{\mathbb{T}}) \subset \mathcal{H}_{\Lambda,-}^k(M_{\mathbb{T}}), \quad (1.18)$$

the corresponding hereditary rigged chains of harmonic forms:

$$\mathcal{H}_{\Lambda(\mathcal{L}^*\mathcal{L}),+}^k(M_{\mathbb{T}}) \subset \mathcal{H}_{\Lambda(\mathcal{L}^*\mathcal{L})}^k(M_{\mathbb{T}}) \subset \mathcal{H}_{\Lambda(\mathcal{L}^*\mathcal{L}),-}^k(M_{\mathbb{T}}) \quad (1.19)$$

and cohomology groups:

$$\begin{aligned} \mathcal{H}_{\Lambda(\mathcal{L}),+}^k(M_{\mathbb{T}}) &\subset \mathcal{H}_{\Lambda(\mathcal{L})}^k(M_{\mathbb{T}}) \subset \mathcal{H}_{\Lambda(\mathcal{L}),-}^k(M_{\mathbb{T}}), \\ \mathcal{H}_{\Lambda(\mathcal{L}^*),+}^k(M_{\mathbb{T}}) &\subset \mathcal{H}_{\Lambda(\mathcal{L}^*)}^k(M_{\mathbb{T}}) \subset \mathcal{H}_{\Lambda(\mathcal{L}^*),-}^k(M_{\mathbb{T}}) \end{aligned} \quad (1.20)$$

for any  $k = \overline{0, m+2}$ . Assume also that generalized Laplace-Hodge operator (1.13) is reduced upon the space  $\mathcal{H}_\Lambda^0(M)$ . Now by reasoning similar to those in [8, 35] one can formulate a slightly generalized [3–5, 35] de Rham-Hodge theorem.

The groups of harmonic forms  $\mathcal{H}_{\Lambda, -}^k(M_T)$ ,  $k = \overline{0, m+2}$ , are, respectively, isomorphic to the homology groups  $(H_k(M_T; \mathbb{C}))^{|\Sigma|}$ ,  $k = \overline{0, m+2}$ , where  $H_k(M_T; \mathbb{C})$  is the  $k$ -th cohomology group of the manifold  $M_T$  with complex coefficients, the set  $\Sigma \subset \mathbb{C}^p$ ,  $p \in \mathbb{Z}_+$ , is a set of suitable “spectral” parameters marking the linear space of independent  $d_{\mathcal{L}}^*$ -closed 0-form from  $\mathcal{H}_{\Lambda(\mathcal{L}), -}^0(M_T)$  and, moreover, the following direct sum decompositions

$$\begin{aligned} \mathcal{H}_{\Lambda, -}^k(M_T) &= \mathcal{H}_{\Lambda(\mathcal{L}^* \mathcal{L}), -}^k(M_T) \oplus \Delta_{\mathcal{L}} \mathcal{H}_{\Lambda, -}^k(M_T) = \\ &= \mathcal{H}_{\Lambda(\mathcal{L}^* \mathcal{L}), -}^k(M_T) \oplus d_{\mathcal{L}} \mathcal{H}_{\Lambda, -}^{k-1}(M_T) \oplus d'_{\mathcal{L}} \mathcal{H}_{\Lambda, -}^{k+1}(M_T) \end{aligned}$$

hold for any  $k = \overline{0, m+2}$ .

Another result of this type was earlier formulated in [3–6] and reads as the following generalized de Rham-Hodge theorem.

**Theorem 1.2.** *The generalized cohomology groups  $\mathcal{H}_{\Lambda(\mathcal{L}), +}^k(M_T)$ ,  $k = \overline{0, m+2}$ , are, respectively, isomorphic to the cohomology groups  $(\mathbb{H}^k(M_T; \mathbb{C}))^{|\Sigma|}$ ,  $k = \overline{0, m+2}$ .*

*Proof.* A proof of this theorem is based on some special sequence [3–6] of differential Lagrange type identities. □

Define the following closed subspace

$$\mathcal{H}_0^* := \{\varphi^{(0)}(\eta) \in \mathcal{H}_{\Lambda(\mathcal{L}^*), -}^0(M_T) : d_{\mathcal{L}}^* \varphi^{(0)}(\eta) = 0, \varphi^{(0)}(\eta)|_{\Gamma}, \eta \in \Sigma\} \quad (1.21)$$

for some smooth  $(m+1)$ -dimensional hypersurface  $\Gamma \subset M_T$  and  $\Sigma \subset (\sigma(L) \cap \bar{\sigma}(L)) \times \Sigma_{\sigma} \subset \mathbb{C}^p$ , where  $\mathcal{H}_{\Lambda(\mathcal{L}^*), -}^0(M_T)$  is, as above, a suitable Hilbert-Schmidt rigged [12, 13] zero-order cohomology group Hilbert space from the co-chain given by (1.20),  $\sigma(L)$  and  $\sigma(L^*)$  are, respectively, mutual generalized spectra of the sets of differential operators  $L$  and  $L^*$  in  $H$  at  $t = 0 \in T$ . Thereby, the dimension  $\dim \mathcal{H}_0^* = \text{card } \Sigma := |\Sigma|$  is assumed to be known. The next lemma earlier stated by I.V. Skrypnik [3–6] is utmost importance meaning for a proof of Theorem (1.2).

**Lemma 1.3.** *There exists a set of differential  $(k+1)$ -forms  $Z^{(k+1)}[\varphi^{(0)}(\eta), d_{\mathcal{L}} \psi^{(k)}] \in \Lambda^{k+1}(M_T; \mathbb{C})$ ,  $k = \overline{0, m+2}$ , and a set of  $k$ -forms  $Z^{(k)}[\varphi^{(0)}(\eta), \psi^{(k)}] \in \Lambda^k(M_T; \mathbb{C})$ ,  $k = \overline{0, m+2}$ , parametrized by the set  $\Sigma \ni \eta$ , being semilinear in  $(\varphi^{(0)}(\eta), \psi^{(k)}) \in \mathcal{H}_0^* \times \mathcal{H}_{\Lambda, -}^k(M_T)$ , such that*

$$Z^{(k+1)}[\varphi^{(0)}(\eta), d_{\mathcal{L}} \psi^{(k)}] = dZ^k[\varphi^{(0)}(\eta), \psi^{(k)}] \quad (1.22)$$

for all  $k = \overline{0, m+2}$  and  $\eta \in \Sigma$ .

*Proof.* A proof is based on the following Lagrange type identity generalizing identity (3.3) of Part 1 and holding for any pair  $(\varphi^0(\eta), \psi(k)) \in \mathcal{H}_0^* \times \mathcal{H}_{\Lambda, -}^k(M_T)$ :

$$\begin{aligned} 0 &= \langle d_{\mathcal{L}}^* \phi^{(0)}(\eta), \wedge * (\psi^{(k)} \wedge \bar{\gamma}) \rangle = \langle *d'_{\mathcal{L}}(*)^{-1} \varphi^{(0)}(\eta), *(\psi^{(k)} \wedge \bar{\gamma}) \rangle = \\ &= \langle *d'_{\mathcal{L}}(*)^{-1} \phi^{(0)}(x), \wedge \psi^{(k)} \wedge \bar{\gamma} \rangle = \\ &= \langle (* )^{-1} \varphi^{(0)}(\eta), \wedge d_{\mathcal{L}} \psi^{(k)} \wedge \bar{\gamma} \rangle + Z^{(k+1)}[\psi^{(0)}(\eta), \wedge d_{\mathcal{L}} \psi^{(k)}] \wedge \bar{\gamma} = \\ &= \langle (* )_{-1} \varphi^{(0)}(\eta), \wedge d_{\mathcal{L}} \psi^{(k)} \wedge \bar{\gamma} \rangle + dZ^{(k)}[\varphi^{(0)}(\eta), \psi^{(k)}] \wedge \bar{\gamma}, \end{aligned}$$

where  $Z^{(k+1)}[\varphi^{(0)}(\eta), d_{\mathcal{L}} \psi^{(k)}] \in \Lambda^{k+1}(M_T; \mathbb{C})$ ,  $k = \overline{0, m+2}$ , and  $Z^{(k)}[\varphi^{(0)}(\eta), \psi^{(k)}] \in \Lambda^k(M_T; \mathbb{C})$ ,  $k = \overline{0, m+2}$ , are some semilinear differential forms on  $M_T$  parametrized by a parameter  $\lambda \in \Sigma$ , and  $\bar{\gamma} \in \Lambda^{m+1-k}(M_T; \mathbb{C})$  is an arbitrary constant  $(m+1-k)$ -form. Thereby, the semilinear differential  $(k+1)$ -forms  $Z^{(k+1)}[\varphi^{(0)}(\eta), d_{\mathcal{L}} \psi^{(k)}] \in \Lambda^{k+1}(M_T; \mathbb{C})$  and  $k$ -forms  $Z^{(k)}[\varphi^{(0)}(\eta), \psi^{(k)}] \in \Lambda^k(M_T; \mathbb{C})$ ,  $k = \overline{0, m+2}$ ,  $\lambda \in \Sigma$ , constructed above are exactly those searched for in the Lemma.  $\square$

**1.4.** Now based on Lemma (1.3), one can construct the cohomology group isomorphism claimed in Theorem 1.2 formulated above. Namely, following [3–5], let us take some singular simplicial [34, 35] complex  $\mathcal{K}(M_T)$  of the metric space  $M_T$  and introduce a set of linear mappings  $B_{\lambda}^{(k)} : \mathcal{H}_{\Lambda, -}^k M_T \longrightarrow C_k(M_T; \mathbb{C})$ ,  $k = \overline{0, m+2}$ ,  $\lambda \in \Sigma$ , where  $C_k(M_T; \mathbb{C})$ ,  $k = \overline{0, m+2}$ , are free abelian groups over the field  $\mathbb{C}$  generated by all  $k$ -chains of singular simplexes  $S^{(k)} \subset M_T$ ,  $k = \overline{0, m+2}$ , respectively, from the simplicial complex  $\mathcal{K}(M_T)$ , as follows:

$$B_{\lambda}^{(k)}(\psi^{(k)}) := \sum_{S^{(k)} \in C_k(M_T; \mathbb{C})} S^{(k)} \int_{S^{(k)}} Z^{(k)}[\varphi^{(0)}(\lambda), \psi^{(k)}] \quad (1.23)$$

with  $\psi^{(k)} \in \mathcal{H}_{\Lambda, -}^k(M_T)$ ,  $k = \overline{0, m+2}$ . The following theorem proved fearlier in [3–5] with use of mappings (1.23) holds.

**Theorem 1.4.** *The set of operators (1.23) parametrized by  $\lambda \in \Sigma$  realizes the cohomology group isomorphism referred to in Theorem 1.2.*

*Proof.* One can get a proof of this theorem passing over, in (1.23), to the corresponding cohomology  $\mathcal{H}_{\Lambda(\mathcal{L}), -}^k(M_T)$  and homology  $H_k(M_T; \mathbb{C})$  groups of  $M_T$  for every  $k = \overline{0, m+2}$ . Taking an element  $\psi^{(k)} := \psi^{(k)}(\mu) \in \mathcal{H}_{\Lambda(\mathcal{L}), -}^k(M_T)$ ,  $k = \overline{0, m+2}$ , solving the equation  $d_{\mathcal{L}} \psi^{(k)}(\mu) = 0$  with  $\mu \in \Sigma_k$  being some set of the related “spectral” parameters marking elements of the subspace  $\mathcal{H}_{\Lambda(\mathcal{L}), -}^k(M_T)$ , from (1.23) and (1.22), one can easily conclude that  $dZ^{(k)}[\varphi^{(0)}(\lambda), \psi^{(k)}(\mu)] = 0$  for all  $(\lambda, \mu) \in \Sigma \times \Sigma_k$ ,  $k = \overline{0, m+2}$ . Owing to the Poincare lemma [33–35], this, in particular, means that there exist differential  $(k-1)$ -forms  $\Omega^{(k-1)}[\varphi^{(0)}(\lambda), \psi^{(k)}(\mu)] \in \Lambda^{k-1}(M; \mathbb{C})$ ,  $k = \overline{0, m+2}$ , such that

$$Z^{(k)}[\varphi^{(0)}(\lambda), \psi^{(k)}(\mu)] = d\Omega^{(k-1)}[\varphi^{(0)}(\lambda), \psi^{(k)}(\mu)] \quad (1.24)$$

for all pairs  $(\varphi^{(0)}(\lambda), \psi^{(k)}(\mu)) \in \mathcal{H}_0^* \times \mathcal{H}_{\Lambda(\mathcal{L}),-}^k(M_T)$  parametrized by  $(\lambda, \mu) \in \Sigma \times \Sigma_k$ ,  $k = \overline{0, m+2}$ . As a result of passing to the homology groups  $H_k(M_T; \mathbb{C})$ ,  $k = \overline{0, m+2}$ , on the right-hand side of (1.23), owing to the standard Stokes theorem [33–35], it can be shown that the mappings

$$B_\lambda^{(k)} : \mathcal{H}_{\Lambda(\mathcal{L}),-}^k(M_T) \longrightarrow H_k(M_T; \mathbb{C}) \tag{1.25}$$

are isomorphisms for every  $k = \overline{0, m+2}$  and  $\lambda \in \Sigma$ . Making further use of the Poincare duality [8, 34, 35] between the homology groups  $H_k(M_T; \mathbb{C})$ ,  $k = \overline{0, m+2}$ , and the cohomology groups  $H^k(M; \mathbb{C})$ ,  $k = \overline{0, m+2}$ , respectively, one finally obtains the statement claimed of Theorem 1.4.  $\square$

## 2. THE STRUCTURE OF DELSARTE-LIONS TYPE OPERATORS IN MULTIDIMENSION

**2.1.** Now take into account that our differential operators  $L_j : \mathcal{H} \longrightarrow \mathcal{H}$ ,  $j = \overline{1, 2}$ , are of special form (1.2). Also assume that differential expressions (1.4) are normal closed operators defined on dense subspace  $D(\mathcal{L}) \subset L_2(M; \mathbb{C}^N)$ .

Then due to Theorem 1.4 one can find such a pair  $(\varphi^{(0)}(\lambda), \psi^{(k)}(\mu)) \in \mathcal{H}_0^* \times \mathcal{H}_{\Lambda(\mathcal{L}),-}^k(M_T)$  parametrized by elements  $(\lambda, \mu) \in \Sigma \times \Sigma_k$ , for which the equivalent equality

$$B_\lambda^{(m)}(\psi^{(0)}(\mu)dx) = S_{(t;x)}^{(m)} \int_{\partial S_{(t;x)}^{(m)}} \Omega^{(m-1)}[\varphi^{(0)}(\lambda), \psi^{(0)}(\mu)dx] \tag{2.1}$$

holds, where  $S_{(t;x)}^{(m)} \in C_m(M_T; \mathbb{C})$  is some arbitrary but fixed element parametrized by a chosen point  $(t; x) \in \partial S_{(t;x)}^{(m)}$ . Consider the following integral expressions

$$\begin{aligned} \Omega_{(t;x)}(\lambda, \mu) &:= \int_{\sigma_{(t;x)}^{(m-1)}} \Omega^{(m-1)}[\varphi^{(0)}(\lambda), \psi^{(0)}(\mu)dx], \\ \Omega_{(t_0;x_0)}(\lambda, \mu) &:= \int_{\sigma_{(t_0;x_0)}^{(m-1)}} \Omega^{(m-1)}[\phi^{(0)}(\lambda), psi_{(0)}(\mu)dx], \end{aligned} \tag{2.2}$$

with a point  $(t_0; x_0) \in M_T \cap \partial S_{(t_0;x_0)}^{(m)}$  being fixed, the boundaries  $\sigma_{(t;x)}^{(m-1)} := \partial S_{(t;x)}^{(m)}$ ,  $\sigma_{(t_0;x_0)}^{(m-1)} := \partial S_{(t_0;x_0)}^{(m)}$  being assumed to be homological to each other as  $(t; x_0) \longrightarrow (t; x) \in M_T$ ,  $(\lambda, \mu) \in \Sigma \times \Sigma_k$ , and interpret them as the kernels [12, 13] of the corresponding invertible integral operators of Hilbert-Schmidt type  $\Omega_{(t;x)}, \Omega_{(t_0;x_0)} : L_2^\rho(\Sigma; \mathbb{C}) \longrightarrow L_2^\rho(\Sigma; \mathbb{C})$ , where  $\rho$  is some finite Borel measure on the parameter set  $\Sigma$ . Now define the invertible operators expressions

$$\Omega_\pm : \psi^{(0)}(\mu) \longrightarrow \tilde{\psi}^{(0)}(\mu) \tag{2.3}$$

for  $\psi^{(0)}(\mu)dx \in \mathcal{H}_{\Lambda(\mathcal{L}),-}^m(M_{\mathbb{T}})$  and some  $\tilde{\psi}^{(0)}(\mu)dx \in \mathcal{H}_{\Lambda(\mathcal{L}),-}^m(M_{\mathbb{T}})$ ,  $\mu \in \Sigma$ , where, by definition, for any  $\eta \in \Sigma$

$$\begin{aligned} \tilde{\psi}^{(0)}(\eta) &:= \psi^{(0)}(\eta) \cdot \Omega_{(t;x)}^{-1} \cdot \Omega_{(t_0;x_0)} = \\ &= \int_{\Sigma} d\rho(\mu) \int_{\Sigma} d\rho(\xi) \psi^{(0)}(\mu) \Omega_{(t;x)}^{-1}(\mu, \xi) \Omega_{(t_0;x_0)}(\xi, \eta), \end{aligned} \quad (2.4)$$

being motivated by expression (2.1). Namely, consider the following diagram

$$\begin{array}{ccc} \mathcal{H}_{\Lambda(\mathcal{L}),-}^m(M_{\mathbb{T}}) & \xrightarrow{\Omega_{\pm}} & \mathcal{H}_{\Lambda(\tilde{\mathcal{L}}),-}^m(M_{\mathbb{T}}), \\ B_{\lambda}^{(m)} \downarrow & \swarrow \tilde{B}_{\lambda}^{(m)} & \\ H_m(M_{\mathbb{T}}; \mathbb{C}) & & \end{array} \quad (2.5)$$

which is assumed to be commutative for some another co-chain complex

$$\mathcal{H} \longrightarrow \Lambda^0(M_{\mathbb{T}}; \mathcal{H}) \xrightarrow{d_{\tilde{\mathcal{L}}}} \Lambda^1(M_{\mathbb{T}}; \mathcal{H}) \xrightarrow{d_{\tilde{\mathcal{L}}}} \dots \xrightarrow{d_{\tilde{\mathcal{L}}}} \Lambda^{m+2}(M_{\mathbb{T}}; \mathcal{H}) \xrightarrow{d_{\tilde{\mathcal{L}}}} 0. \quad (2.6)$$

Here, by definition, the generalized ‘‘anti-differentiation’’ is

$$d_{\tilde{\mathcal{L}}} = \sum_{j=1}^2 dt_j \wedge \tilde{L}_j(t; x | \partial) \quad (2.7)$$

and

$$\begin{aligned} \tilde{L}_j &= \partial / \partial t_j - \tilde{L}_j(t; x | \partial), \\ \tilde{L}_j(t; x | \partial) &:= \sum_{|\alpha|=0}^{n_j(\tilde{L})} \tilde{a}_{\alpha}^{(j)}(t; x) \frac{\partial^{|\alpha|}}{\partial x^{\alpha}}, \end{aligned} \quad (2.8)$$

where coefficients  $\tilde{a}_{\alpha}^{(j)} \in C^1(\mathbb{T}; C^{\infty}(M; \text{End } \mathbb{C}^N)$ ,  $|\alpha| = \overline{0}, n_j(\tilde{L}), n_j(\tilde{L}) := n_j(L) \in \mathbb{Z}_+, j = \overline{1, 2}$ . The corresponding isomorphisms  $\tilde{B}_{\lambda}^{(m)} : \mathcal{H}_{\Lambda(\tilde{\mathcal{L}}),+}^m(M_{\mathbb{T}}) \longrightarrow H_m(M_{\mathbb{T}}; \mathbb{C})$ ,  $\lambda \in \Sigma$ , act, by definition, as follows:

$$\tilde{B}_{\lambda}^{(m)}(\tilde{\psi}^{(0)}(\mu)dx) = S_{(t;x)}^{(m)} \int_{\partial S_{(t;x)}^{(m)}} \tilde{\Omega}^{(m-1)}[\tilde{\varphi}^{(0)}(\lambda), \tilde{\psi}^{(0)}(\mu)dx] \quad (2.9)$$

where  $\tilde{\varphi}^{(0)}(\lambda) \in \tilde{\mathcal{H}}_0^* \subset \mathcal{H}_{\Lambda(\tilde{\mathcal{L}}^*),-}^0(M_{\mathbb{T}})$ ,  $\lambda \in (\sigma(\tilde{L}) \cap \bar{\sigma}(\tilde{L}^*)) \times \Sigma_{\sigma}$ ,

$$\tilde{\mathcal{H}}_0^* := \{\tilde{\varphi}^{(0)}(\lambda) \in \mathcal{H}_{\Lambda(\tilde{\mathcal{L}}^*),-}^m(M_{\mathbb{T}}) : d_{\tilde{\mathcal{L}}}^* \tilde{\varphi}^{(0)}(x) = 0, \tilde{\varphi}^{(0)}(\lambda)|_{\tilde{\Gamma}} = 0, \lambda \in \Sigma\} \quad (2.10)$$

for a hypersurface  $\tilde{\Gamma} \subset M_{\mathbb{T}}$ . Respectively, one defines the following closed subspace

$$\tilde{\mathcal{H}}_0 := \{\tilde{\psi}^{(0)}(\mu) \in \mathcal{H}_{\Lambda(\tilde{\mathcal{L}}^*),-}^0(M_{\mathbb{T}}) : d_{\tilde{\mathcal{L}}}^* \tilde{\psi}^{(0)}(\lambda) = 0, \tilde{\psi}^{(0)}(\mu)|_{\tilde{\Gamma}} = 0, \mu \in \Sigma\} \quad (2.11)$$

for the hyperspace  $\tilde{\Gamma} \subset M_{\mathbb{T}}$ , introduced above.



Suppose now that elements (2.4) belong to closed subspace (2.11), that is

$$d_{\tilde{\mathcal{L}}}\tilde{\psi}^{(0)}(\mu) = 0. \quad (2.12)$$

Similarly to (2.11), define a closed subspace  $\mathcal{H}_0^* \subset \mathcal{H}_{\Lambda(\mathcal{L}^*),-}^m(M_T)$  as follows:

$$\mathcal{H}_0 := \{\psi^{(0)}(\lambda) \in \mathcal{H}_{\Lambda(\mathcal{L}^*),-}^0(M_T) : d_{\mathcal{L}}\psi^{(0)}(\lambda) = 0, \psi^{(0)}(\lambda)|_{\Gamma} = 0, \lambda \in \Sigma\} \quad (2.13)$$

for all  $\mu \in \Sigma$ . Then owing to the commutativity of diagram (2.5), there exist the corresponding two invertible mappings

$$\Omega_{\pm} : \mathcal{H}_0 \rightarrow \tilde{\mathcal{H}}_0, \quad (2.14)$$

depending on the ways they were extended onto the entire Hilbert space  $\mathcal{H}_{\Lambda,-}^m(M_T)$ . Now extend operators (2.14) onto the entire Hilbert space  $\mathcal{H}_{\Lambda,-}^m(M_T)$  by means of the standard method [26, 43] of variation of constants, taking into account that for kernels  $\Omega_{(t;x)}(\lambda, \mu), \Omega_{(t_0;x_0)}(\lambda, \mu) \in L_2^{(p)}(\Sigma; \mathbb{C}) \otimes L_2^{(p)}(\Sigma; \mathbb{C})$ ,  $\lambda, \mu \in \Sigma$ , one can write down the following relationships:

$$\begin{aligned} \Omega_{(t;x)}(\lambda, \mu) - \Omega_{(t_0;x_0)}(\lambda, \mu) &= \int_{\partial S_{(t;x)}^{(m)}} \Omega^{(m-1)}[\varphi^{(0)}(x), \psi^{(0)}(\mu)dx] - \\ &- \int_{\partial S_{(t_0;x_0)}^{(m)}} \Omega^{(m-1)}[\varphi^{(0)}(\lambda), \psi^{(0)}(\mu)dx] = \\ &= \int_{S_{\pm}^{(m)}(\sigma_{(t;x)}^{(m-1)}, \sigma_{(t_0;x_0)}^{(m-1)})} d\Omega^{(m-1)}[\varphi^{(0)}(\lambda), \psi^{(0)}(\mu)dx] = \\ &= \int_{S_{\pm}^{(m)}(\sigma_{(t;x)}^{(m-1)}, \sigma_{(t_0;x_0)}^{(m-1)})} Z^{(m)}[\varphi^{(0)}(\lambda), \psi^{(0)}(\mu)dx], \end{aligned} \quad (2.15)$$

where, by definition,  $m$ -dimensional open surfaces  $S_{\pm}^{(m)}(\sigma_{(t;x)}^{(m-1)}, \sigma_{(t_0;x_0)}^{(m-1)}) \subset M_T$  are spanned smoothly without self-intersection between two homological cycles  $\sigma_{(t;x)}^{(m-1)} = \partial S_{(t;x)}^{(m)}$  and  $\sigma_{(t_0;x_0)}^{(m-1)} = \partial S_{(t_0;x_0)}^{(m)} \in C_{m-1}(M_T; \mathbb{C})$  in such a way that the boundary  $\partial(S_{+}^{(m)}(\sigma_{(t_0;x_0)}^{(m-1)}, \sigma_{(t_0;x_0)}^{(m-1)}) \cup S_{-}^{(m)}(\sigma_{(t;x)}^{(m-1)}, \sigma_{(t_0;x_0)}^{(m-1)}) = \emptyset$ . Making use of relationship (2.15), one can thereby easily find the following integral operator expressions in  $\mathcal{H}_{-}$ :

$$\Omega_{\pm} = \mathbf{1} - \int_{\Sigma} d\rho(\eta)\tilde{\psi}^{(0)}(\xi)\Omega_{(t_0;x_0)}^{-1}(\xi, \eta) \int_{S_{\pm}^{(m)}(\sigma_{(t;x)}^{(m-1)}, \sigma_{(t_0;x_0)}^{(m-1)})} Z_{(m)}[\varphi^{(0)}(\eta), (\cdot)dx] \quad (2.16)$$

defined for fixed pairs  $(\varphi^{(0)}(\xi), \psi^0(\eta)) \in \mathcal{H}_0^* \times \mathcal{H}_0$  and  $(\tilde{\varphi}^{(0)}(\xi), \tilde{\psi}^0(\mu)) \in \tilde{\mathcal{H}}_0^* \times \tilde{\mathcal{H}}_0$ ,  $\lambda, \mu \in \Sigma$ , being bounded invertible operators of Volterra type [23] on the whole Hilbert space  $\mathcal{H}$ . Thereby, we have proved the following theorem.

**Theorem 2.1.** *Let mappings (2.14) be given by Volterrian operator expressions (2.16), where the semi-linear forms  $Z_{(m)}[\cdot, \cdot]$  are defined by means of relationships (1). Then diagram (2.5) commutes.*

Moreover, for the differential operators  $\tilde{L}_j : \mathcal{H} \rightarrow \mathcal{H}$ ,  $j = \overline{1, 2}$ , one can easily get the following expressions

$$\tilde{L}_j = \Omega_{\pm} L_j \Omega_{\pm}^{-1} \quad (2.17)$$

for  $j = \overline{1, 2}$ , where the left-hand sides of (2.17) do not depend on the signs “ $\pm$ ” of the right-hand sides. Thereby, Volterrian integral operators (2.16) are the Delsarte-Lions transmutation operators, mapping a given set  $\mathcal{L}$  of differential operators into a new set  $\tilde{\mathcal{L}}$  of differential operators transformed via Delsarte expressions (2.17).

**2.2.** Suppose now that none of the differential operators  $L_j(t; x|\partial)$ ,  $j = \overline{1, 2}$ , considered above depends on the variable  $t \in \mathbb{T} \subset \mathbb{R}_+^2$ . Then, evidently, one can take

$$\begin{aligned} \mathcal{H}_0 &:= \{\psi_{\mu}^{(0)}(\xi) \in L_{2,-}(M; \mathbb{C}^N) : L_j \psi_{\mu}^{(0)}(\xi) = \mu_j \psi_{\mu}^{(0)}(\xi), \\ &\quad j = \overline{1, 2}, \psi_{\mu}^{(0)}(\xi)|_{\bar{\Gamma}} = 0, \mu = (\mu_1, \mu_2) \in \sigma(\tilde{L}) \cap \bar{\sigma}(L^*), \xi \in \Sigma_{\sigma}\}, \\ \tilde{\mathcal{H}}_0 &:= \{\tilde{\psi}_{\mu}^{(0)}(\xi) \in L_{2,-}(M; \mathbb{C}^N) : \tilde{L}_j \tilde{\psi}_{\mu}^{(0)}(\xi) = \mu_j \tilde{\psi}_{\mu}^{(0)}(\xi), \\ &\quad j = \overline{1, 2}, \tilde{\psi}_{\mu}^{(0)}(\xi)|_{\bar{\Gamma}} = 0, \mu = (\mu_1, \mu_2) \in \sigma(\tilde{L}) \cap \bar{\sigma}(L^*), \xi \in \Sigma_{\sigma}\}, \\ \mathcal{H}_0^* &:= \{\varphi_{\lambda}^{(0)}(\eta) \in L_{2,-}(M; \mathbb{C}^N) : L_j^* \varphi_{\lambda}^{(0)}(\eta) = \bar{\lambda}_j \varphi_{\lambda}^{(0)}(\eta), j = \overline{1, 2}, \\ &\quad \varphi_{\lambda}^{(0)}(\eta)|_{\bar{\Gamma}} = 0, \lambda = (\lambda_1, \lambda_2) \in \sigma(\tilde{L}) \cap \bar{\sigma}(L^*), \eta \in \Sigma_{\sigma}\}, \\ \tilde{\mathcal{H}}_0^* &:= \{\tilde{\varphi}_{\lambda}^{(0)}(\eta) \in L_{2,-}(M; \mathbb{C}^N) : \tilde{L}_j^* \tilde{\varphi}_{\lambda}^{(0)}(\eta) = \bar{\lambda}_j \tilde{\varphi}_{\lambda}^{(0)}(\eta), \\ &\quad j = \overline{1, 2}, \tilde{\varphi}_{\lambda}^{(0)}(\eta)|_{\bar{\Gamma}} = 0, \lambda = (\lambda_1, \lambda_2) \in \sigma(\tilde{L}) \cap \bar{\sigma}(L^*), \eta \in \Sigma_{\sigma}\}, \end{aligned} \quad (2.18)$$

and construct the corresponding Delsarte-Lions transmutation operators

$$\begin{aligned} \Omega_{\pm} &= 1 - \int_{\sigma(\tilde{L}) \cap \bar{\sigma}(L^*)} d\rho_{\sigma}(\lambda) \int_{\Sigma_{\sigma} \times \Sigma_{\sigma}} d\rho_{\Sigma_{\sigma}}(\xi) d\rho_{\Sigma_{\sigma}}(\eta) \times \\ &\quad \times \int_{S_{\pm}^{(m)} \sigma_{(t_0; x_0)}^{(m-1)}, \sigma_{(t_0; x_0)}^{(m-1)}} dx \tilde{\psi}_{\lambda}^{(0)}(\xi) \Omega_{x_0}^{-1}(\lambda; \xi; \eta) \tilde{\varphi}_{\lambda}^{(0), \top}(\eta)(\cdot) \end{aligned} \quad (2.19)$$

acting already in the Hilbert space  $L_{2,+}(M; \mathbb{C}^N)$ , where for any  $(\lambda; \xi; \eta) \in (\sigma(\tilde{L}) \cap \bar{\sigma}(L^*) \times \Sigma_{\sigma}^2)$ , kernels

$$\Omega_{(x_0)}(\lambda; \xi; \eta) := \int_{\sigma_{x_0}^{(m-1)}} \Omega^{(m-1)}[\varphi_{\lambda}^{(0)}(\xi), \psi_{\lambda}^{(0)}(\eta) dx] \quad (2.20)$$

for  $(\xi, \eta) \in \Sigma_{\sigma}^2$  and every  $\lambda \in \sigma(\tilde{L}) \cap \bar{\sigma}(L^*)$ , belong to  $L_2^{(\rho)}(\Sigma_{\sigma}; \mathbb{C}) \otimes L_2^{(\rho)}(\Sigma_{\sigma}; \mathbb{C})$ . Moreover, as  $\partial \Omega_{\pm} / \partial t_j = 0$ ,  $j = \overline{1, 2}$ , one gets easily the set of differential expressions

$$\tilde{L}_j(x|\partial) := \Omega_{\pm} L_j(x|\partial) \Omega_{\pm}^{-1} \quad (2.21)$$

$j = \overline{1, 2}$ , also commuting with each other.

Volterrian operators (2.19) possess some additional properties. Namely, define the following Fredholm type integral operator in  $H$

$$\Omega := \Omega_{+}^{-1} \Omega_{-}, \quad (2.22)$$

which can be written in the form

$$\mathbf{\Omega} = \mathbf{1} + \Phi(\mathbf{\Omega}), \tag{2.23}$$

where the operator  $\Phi(\mathbf{\Omega}) \in \mathcal{B}_\infty(H)$ . Moreover, owing to relationships (2.20) one easily concludes that the following commutator conditions

$$[\mathbf{\Omega}, L_j] = 0 \tag{2.24}$$

hold for  $j = \overline{1, 2}$ .

Denote now by  $\hat{\Phi}(\mathbf{\Omega}) \in H_- \otimes H_-$  and  $\hat{K}_+(\mathbf{\Omega}), \hat{K}_-(\mathbf{\Omega}) \in H_- \otimes H_-$  the kernels corresponding [12, 13] to operators  $\Phi \in \mathcal{B}_\infty(H)$  and  $\mathbf{\Omega}_\pm \in \mathcal{B}_\infty(H)$ . Then, owing to the fact that  $\sup K_+(\mathbf{\Omega}) \cap \sup K_-(\mathbf{\Omega}) = \emptyset$ , from (2.22) and (2.23), one gets the well known Gelfand-Levitan-Marchenko linear integral equation

$$\hat{K}_+(\mathbf{\Omega}) + \Phi(\hat{\mathbf{\Omega}}) + \hat{K}_+(\mathbf{\Omega})_+ \cdot \hat{\Phi}(\mathbf{\Omega}) = \hat{K}_-(\mathbf{\Omega}), \tag{2.25}$$

allowing to find the kernel  $\hat{K}_+(\mathbf{\Omega}) \in H_- \otimes H_-$  for  $(t; x) \in \sup K_+(\mathbf{\Omega}_+)$  factorizing Fredholmian operator (2.22). Conditions (2.24) can be rewritten suitably as follows:

$$(L_{j,ext} \otimes \mathbf{1})\hat{\Phi}(\mathbf{\Omega}) = (1 \otimes L_{j,ext}^*)\hat{\Phi}(\mathbf{\Omega}), \tag{2.26}$$

where  $L_{j,ext} \in \mathcal{L}(H_-)$ ,  $j = \overline{1, 2}$ , and their adjoint  $L_{j,ext}^* \in \mathcal{L}(H_-)$ ,  $j = \overline{1, 2}$ , are the respective extensions [12, 26] of the differential operators  $L_j$  and  $L_j^* \in \mathcal{L}(H)$ ,  $j = \overline{1, 2}$ .

Concerning relationships (2.21), one can write down [12, 26] kernel conditions similar to (2.26):

$$(\tilde{L}_{j,ext} \otimes \mathbf{1})\hat{K}_\pm(\mathbf{\Omega}) = (1 \otimes L_{j,ext}^*)\hat{K}_\pm(\mathbf{\Omega}), \tag{2.27}$$

where, as above,  $\tilde{L}_{j,ext} \in \mathcal{L}(H_-)$ ,  $j = \overline{1, 2}$ , are the respective rigging extensions of the differential operators  $\tilde{L}_j \in \mathcal{L}(H)$ ,  $j = \overline{1, 2}$ .

**2.3.** Proceed now to analyzing the question of the general differential expression structure of transformed operator expression (2.17). It is evident that conditions (2.25) and (2.26) on the kernels  $\hat{K}_\pm(\mathbf{\Omega}) \in \mathcal{H}_- \otimes \mathcal{H}_-$  of Delsarte-Lions transmutation operators are necessary for operator expression (2.17) to exist and be differential. Consider the question whether these conditions are sufficient.

For studying this question, let us consider Volterrian operators (2.19) with kernels satisfying the conditions (2.25) and (2.26), assuming that suitable oriented surfaces  $S_\pm^{(m)}(\sigma_{(t;x)^{(m-1)}}, \sigma_{(t_0;x_0)^{(m-1)})} \in C_m(M_T; \mathbb{C})$  can be given as follows:

$$\begin{aligned} S_+^{(m)}(\sigma_{(t;x)^{(m-1)}}, \sigma_{(t_0;x_0)^{(m-1)})} &= \{(t'; x') \in M_T : t' = P(t; x|x') \in T\} \\ S_-^{(m)}(\sigma_{(t;x)^{(m-1)}}, \sigma_{(t_0;x_0)^{(m-1)})} &= \{(t'; x') \in M_T : t' = P(t; x|x') \in T \setminus [t_0, t]\}, \end{aligned} \tag{2.28}$$

where a mapping  $P \in C^\infty(M_T \times M; T)$  is piecewise smooth and such that the boundaries  $\partial S_\pm^{(m)}(\sigma_{(t;x)^{(m-1)}}, \sigma_{(t_0;x_0)^{(m-1)})} = \pm(\sigma_{(t;x)^{(m-1)}} - \sigma_{(t_0;x_0)^{(m-1)})}$  with cycles  $\sigma_{(t;x)^{(m-1)}}^{\pm}$  and  $\sigma_{(t_0;x_0)^{(m-1)}}^{\pm} \in \mathcal{K}(M_T)$  are homological to each other for any choice of points  $(t_0; x_0)$  and

$(t; x) \in M_T$ . Then by means of some simple but cumbersome calculations, based on considerations from [41] and [9], one can see that the resulting expressions on the left-hand side of

$$\tilde{L} = L + [K_{\pm}(\Omega), L] \cdot \Omega_{\pm}^{-1} \quad (2.29)$$

are differential ones exactly equal to each other, if the expression for an operator  $L \in \mathcal{L}(\mathcal{H})$  was differential too.

Concerning the inverse operators  $\Omega_{\pm}^{-1} \in \mathcal{B}(\mathcal{H})$  appearing in (2.29), one can notice here that, owing to the functional symmetry between closed subspaces  $\mathcal{H}_0$  and  $\tilde{\mathcal{H}}_0 \subset \tilde{\mathcal{H}}_-$ , defining relationships (2.14) and (2.4) are reversible, that is, there exist inverse operator mappings  $\Omega_{\pm}^{-1} : \mathcal{H}_0 \rightarrow \mathcal{H}_0$ , such that

$$\Omega_{\pm}^{-1} : \tilde{\psi}^{(0)}(\lambda) \longrightarrow \psi^{(0)}(\lambda) := \tilde{\psi}^{(0)}(\lambda) \cdot \tilde{\Omega}_{(t;x)}^{-1} \tilde{\Omega}_{(t;x)} \quad (2.30)$$

for some suitable kernels  $\tilde{\Omega}_{(t;x)}(\lambda, \mu)$  and  $\tilde{\Omega}_{(t_0;x_0)}(\lambda, \mu) \in L_2^{(\rho)}(\Sigma; \mathbb{C}) \otimes L_2^{(\rho)}(\Sigma; \mathbb{C})$ , related naturally to the transformed differential expression  $\tilde{L} \in \mathcal{L}(\mathcal{H})$ . Thereby, owing to expressions (2.30), one can write down the following integral operator expressions similar to (2.19):

$$\begin{aligned} \Omega_{\pm}^{-1} = \mathbf{1} - \int_{\Sigma} d\rho(\xi) \int_{\Sigma} d\rho(\eta) \psi^{(0)}(\xi) \tilde{\Omega}_{t_0;x_0}^{-1}(\xi, \eta) \times \\ \times \int_{S_{\pm}^{(m)}(\sigma_{(t;x)}^{(m-1)}, \sigma_{(t_0;x_0)}^{(m-1)})} \tilde{Z}^{(m)}[\tilde{\varphi}^{(0)}(\eta), (\cdot) dx], \end{aligned} \quad (2.31)$$

defined for fixed pairs  $(\tilde{\varphi}^{(0)}(\xi), \tilde{\psi}^{(0)}(\eta)) \in \tilde{\mathcal{H}}_0^* \times \tilde{\mathcal{H}}_0$  and  $(\varphi^{(0)}(\xi), \psi^{(0)}(\eta)) \in \mathcal{H}_0^* \times \mathcal{H}_0$ ,  $\xi, \eta \in \Sigma$ , and being bounded invertible operators of Volterra type on the entire Hilbert space  $\mathcal{H}$ . In particular, the compatibility conditions  $\Omega_{\pm} \Omega_{\pm}^{-1} = \mathbf{1} = \Omega_{\pm}^{-1} \Omega_{\pm}$  must be fulfilled identically on  $\mathcal{H}$ , involving some restrictions identifying both measures  $\rho$  and  $\Sigma$  and suitable asymptotic conditions on the coefficient functions of the differential expression  $L \in \mathcal{L}$ . Restrictions of this type were already mentioned before in [42, 44, 45], where in particular the relationships with the local and nonlocal Riemann problems were discussed. identically measures  $\rho$  and  $\Sigma$  and possible asymptotic conditions of coefficient functions of the differential expression  $L \in \mathcal{L}$ . Such kinds of restrictions were already mentioned before in [42, 44, 45, 48], where the relationships with the local and nonlocal Riemann problems were discussed.

**2.4.** Within the framework of the general construction presented above one can give a natural interpretation of so called Backlund transformations for coefficient function of a given differential operator expression  $L \in \mathcal{L}(\mathcal{H})$ . Namely, following the symbolic considerations in [47], we reinterpret the approach devised there for constructing the Backlund transformations making use of the techniques based on the theory of Delsarte transmutation operators. Let us define two Delsarte-Lions transformed differential operator expressions

$$L_1 = \Omega_{1,\pm} L \Omega_{1,\pm}^{-1}, \quad L_2 = \Omega_{2,\pm} L \Omega_{2,\pm}^{-1}, \quad (2.32)$$

where  $\Omega_{1,+}, \Omega_{2,-} \in \mathcal{B}(\mathcal{H})$  are some Delsarte transmutation Volterrian operators in  $\mathcal{H}$  with non-intersecting Borel spectral measures  $\rho_1$  and  $\rho_2$  on  $\Sigma$ , but such that the following conditions

$$\Omega_{1,+}^{-1} \Omega_{1,-} = \Omega = \Omega_{2,+}^{-1} \Omega_{2,-}. \tag{2.33}$$

Making now use of conditions (2.32) and relationships (2.33), one easily finds that the operator  $B := \Omega_{2,-} \Omega_{1,+}^{-1} \in \mathcal{B}(\mathcal{H})$  satisfies the following operator equations:

$$L_2 B = B L_1, \quad \Omega_{2,\pm} B = B \Omega_{1,\pm}, \tag{2.34}$$

which motivate the next definition.

**Definition 2.2.** *An invertible symbolic mapping  $B : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})$  will be called a Delsarte-Backlund transformation of an operator  $L_1 \in \mathcal{L}(\mathcal{H})$  if there holds the condition*

$$[QB, L_1] = 0 \tag{2.35}$$

for some linear differential expression  $Q \in \mathcal{L}(\mathcal{H})$ .

Condition (2.35) can be realized as follows. Take any differential expression  $q \in \mathcal{L}(\mathcal{H})$  satisfying the symbolic equation

$$[qB, L] = 0 \tag{2.36}$$

Then, making use of the transformations like those in (2.32), from (2.33) one finds that

$$[QB, L_1] = 0, \tag{2.37}$$

where, owing to (2.34),

$$QB := \Omega_{1,+} q B \Omega_{1,+}^{-1} = \Omega_{1,+} q \Omega_{2,+}^{-1} B. \tag{2.38}$$

Therefore, the expression  $Q = \Omega_{1,+} q \Omega_{2,+}^{-1}$  appears to be differential too, owing to condition (2.34).

The above consideration related to the symbolic mapping  $B : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})$  gives rise to an effective tool for constructing self-Backlund transformations for coefficients of given differential operator expressions  $L_1, L_2 \in \mathcal{L}(\mathcal{H})$  having many applications in soliton theory.

**2.5.** Return now to studying the structure of Delsarte-Lions transformations for a polynomial differential operator pencil

$$L(\lambda; x|\partial) := \sum_{j=0}^{n(L)} L_j(x|\partial) \lambda^j, \tag{2.39}$$

where  $r(L) \in \mathbb{Z}_+$  and  $\lambda \in \mathbb{C}$  is a complex-valued parameter. We need to find the Delsarte-Lions transformations  $\Omega_{\lambda,\pm} \in \mathcal{B}(\mathcal{H})$ ,  $\lambda \in \mathbb{C}$ , corresponding to (2.39) such that, for some polynomial differential operators pencil  $\tilde{L}(\lambda; x|\partial) \in \mathcal{L}(\mathcal{H})$ , the following Delsarte-Lions [2] condition

$$\tilde{\mathbf{L}}\mathbf{\Omega}_{\lambda,\pm} = \mathbf{\Omega}_{\lambda,\pm}\mathbf{L} \quad (2.40)$$

holds for almost all  $\lambda \in \mathbb{C}$ . In order to find such transformations  $\mathbf{\Omega}_{\lambda,\pm} \in \mathcal{B}(\mathcal{H})$ , let us consider a differential operator  $\mathbf{L}_\tau(x|\partial) \in \mathcal{L}(\mathcal{H}_\tau)$ , depending on  $\tau \in \mathbb{R}$ , where

$$\mathbf{L}_\tau(x|\partial) := \sum_{j=0}^{n(L)} L_j(x|\partial)\partial^j / \partial\tau^j, \quad (2.41)$$

acting in the functional space  $\mathcal{H}_\tau = C^{q(L)}(\mathbb{R}; \mathcal{H})$  for some  $q(L) \in \mathbb{Z}_+$ . Then one can easily construct the corresponding Delsarte-Lions transformations  $\mathbf{\Omega}_{\tau,\pm} \in \mathcal{B}(\mathcal{H}_\tau)$  of Volterra type for some differential operator expression

$$\tilde{\mathbf{L}}_\tau(x|\partial) := \sum_{j=0}^{n(L)} \tilde{L}_j(x|\partial)\partial^j / \partial\tau^j, \quad (2.42)$$

for which the following Delsarte-Lions [2] transmutation conditions

$$\tilde{\mathbf{L}}_\tau\mathbf{\Omega}_{\tau,\pm} = \mathbf{\Omega}_{\tau,\pm}\mathbf{L}_\tau \quad (2.43)$$

hold in  $\mathcal{H}_\tau$ . Thus, making use of the results obtained above, one can write down:

$$\begin{aligned} \mathbf{\Omega}_{\tau,\pm} = & \mathbf{1} - \int_{\Sigma} d\rho_{\Sigma}(\xi) \int_{\Sigma} d\rho_{\Sigma}(\eta) \tilde{\psi}_{\tau}^{(0)}(\lambda; \xi) \Omega_{(\tau_0; x_0)}^{-1}(\lambda; \xi, \eta) \times \\ & \times \int_{S_{\pm}^{(m)}(\sigma_{(\tau; x)}^{(m-1)}, \sigma_{(\tau_0; x_0)}^{(m-1)})} Z^{(m)}[\varphi_{\tau}^{(0)}(\lambda; \eta), (\cdot) dx], \end{aligned} \quad (2.44)$$

defined by means of the following closed subspaces  $\mathcal{H}_{\tau,0} \subset \mathcal{H}_{\tau,-}$  and  $\mathcal{H}_{\tau,0}^* \subset \mathcal{H}_{\tau,-}^*$  :

$$\begin{aligned} \mathcal{H}_{\tau,0} & := \{ \psi_{\tau}^{(0)}(\lambda; \xi) \in \mathcal{H}_{\tau,-} : \mathbf{L}_\tau \psi_{\tau}^{(0)}(\lambda; \xi) = 0, \psi_{\tau}^{(0)}(\lambda; \xi)|_{\tau=0} = \\ & = \psi^{(0)}(\lambda; \xi) \in \mathcal{H}, \mathbf{L} \psi^{(0)}(\lambda; \xi) = 0, \psi^{(0)}(\lambda; \xi)|_{\Gamma} = 0, \lambda \in \mathbb{C}, \xi \in \Sigma \}, \\ \mathcal{H}_{\tau,0}^* & := \{ \varphi_{\tau}^{(0)}(\lambda; \eta) \in \mathcal{H}_{\tau,-}^* : \mathbf{L}_\tau \varphi_{\tau}^{(0)}(\lambda; \eta) = 0, \varphi_{\tau}^{(0)}(\lambda; \eta)|_{\tau=0} = \\ & = \varphi^{(0)}(\lambda; \eta) \in \mathcal{H}^*, \mathbf{L} \varphi^{(0)}(\lambda; \eta) = 0, \varphi^{(0)}(\lambda; \eta)|_{\Gamma} = 0, \lambda \in \mathbb{C}, \eta \in \Sigma \}. \end{aligned} \quad (2.45)$$

Based on representations (2.28) and related results obtained in [21, 48] one can formulate the following important lemma.

**Lemma 2.3.** *Let differential expression (2.42) be Delsarte-Lions transformed by means of Volterrian transmutation operators (2.44), satisfying conditions (2.43). Then it will remain to be differential iff the surfaces  $S_{\pm}^{(m)}(\sigma_{(\tau; x)}^{(m-1)}, \sigma_{(\tau_0; x_0)}^{(m-1)})$  are generated by suitably chosen standard simplicial polyhedra  $\sigma_{(\tau; x)}^{(m-1)}$  and  $\sigma_{(\tau_0; x_0)}^{(m-1)} \in \mathbb{R}^m \times \mathbb{R}$ .*

Recalling now that our operators  $L_j \in \mathcal{L}(\mathcal{H})$ ,  $j = \overline{0, r(L)}$ , do not depend on the parameter  $\tau \in \mathbb{R}$ , from (2.44) one can easily derive:

$$\begin{aligned} \Omega_{\pm} &= \mathbf{1} - \int_{\Sigma} d\rho_{\Sigma}(\xi) \int_{\Sigma} d\rho_{\Sigma}(\eta) \tilde{\psi}^{(0)}(\lambda; \xi) \Omega_{(x_0)}^{-1}(\lambda; \xi, \eta) \times \\ &\quad \times \int_{S_{\pm}^{(m)}(\sigma_{(x)}^{(m-1)}, \sigma_{(x_0)}^{(m-1)})} Z_0^{(m)}[\varphi^{(0)}(\lambda; \eta), (\cdot) dx], \end{aligned}$$

where we have put  $\sigma_x^{(m-1)} := \sigma_{(\tau_0; x)}^{(m-1)}$ ,  $\sigma_{x_0}^{(m-1)} := \sigma_{(\tau_0; x_0)}^{(m-1)} \in C_{m-1}(\mathbb{R}^m; \mathbb{C})$  and

$$Z_0^{(m)}[\varphi^{(0)}(\lambda; \eta), \psi^{(0)} dx] := Z^{(m)}[\varphi_{\tau}^{(0)}(\lambda; \eta), \psi_{\tau}^{(0)} dx] |_{d\tau=0}. \tag{2.46}$$

The closed subspaces  $\mathcal{H}_0 \in \mathcal{H}_-$  and  $\mathcal{H}_0^* \in \mathcal{H}_-^*$  corresponding to (2) are given as follows:

$$\begin{aligned} \mathcal{H}_0 &:= \{\psi^{(0)}(\lambda; \xi) \in \mathcal{H}_- : L\psi^{(0)}(\lambda; \xi) = 0, \psi^{(0)}(\lambda; \xi)|_{\Gamma} = 0, \lambda \in \mathbb{C}, \xi \in \Sigma\}, \tag{2.47} \\ \mathcal{H}_{\tau, 0}^* &:= \{\varphi^{(0)}(\lambda; \eta) \in \mathcal{H}_-^* : L\varphi^{(0)}(\lambda; \eta) = 0, \varphi^{(0)}(\lambda; \eta)|_{\Gamma} = 0, \lambda \in \mathbb{C}, \eta \in \Sigma\}. \end{aligned}$$

As a consequence, the following theorem is true.

**Theorem 2.4.** *Let Volterrian operator expressions (2) be properly defined for almost all  $\lambda \in \mathbb{C}$ , meromorphic functions with respect to the parameter  $\lambda \in \mathbb{C}$ . Then Delsarte-Lions transformed differential expressions (2.39) remain differential and polynomial with respect to the parameter  $\lambda \in \mathbb{C}$  too.*

Thereby, making use of expressions (2) one can construct the Delsarte-Lions transformed linear differential pencil  $\tilde{L} \in \mathcal{L}(\mathcal{H})$ , whose coefficients are related to those of the pencil  $L \in \mathcal{L}(\mathcal{H})$  via some Backlund type relationships useful for applications (see [26, 28, 43, 48]) in soliton theory.

**Acknowledgments**

*The authors are grateful to their friends and colleagues for fruitful discussions of some aspects of the generalized de Rham-Hodge theory and Delsarte-Lions transmutation operators aspects, including, in particular, their applications to differential operator pencils, presented in the article.*

*The first author was supported in part by a local AGH grant, for which he cordially thanks.*

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*Received: April 26, 2006.*