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Monika Perl

NEARLY PERFECT SETS IN THE n-FOLD PRODUCTS OF GRAPHS

Abstract. The study of nearly perfect sets in graphs was initiated in [2]. Let $S \subseteq V(G)$. We say that S is a *nearly perfect* set (or is *nearly perfect*) in G if every vertex in $V(G) - S$ is adjacent to at most one vertex in S. A nearly perfect set S in G is called *1-maximal* if for every vertex $u \in V(G) - S$, $S \cup \{u\}$ is not nearly perfect in G. We denote the minimum cardinality of a 1-maximal nearly perfect set in G by $n_p(G)$. We will call the 1-maximal nearly perfect set of the cardinality $n_p(G)$ an $n_p(G) - set$. In this paper, we evaluate the parameter $n_p(G)$ for some n-fold products of graphs. To this effect, we determine 1-maximal nearly perfect sets in the n-fold Cartesian product of graphs and in the n-fold strong product of graphs.

Keywords: dominating sets, product of graphs.

Mathematics Subject Classification: 05C69, 05C70.

1. INTRODUCTION

Let G be a simple graph and $u \in V(G)$. By $N_G(u)$ we denote the *open neighbourhood* of u; i.e., $N_G(u) = \{v \colon uv \in E(G)\}$. We say that the subset $A \subseteq V(G)$ is called *independent* in G if $N_G(x) \cap A = \emptyset$ for every vertex $x \in A$. We denote the cardinality of the maximum independent set in G by $\alpha(G)$. The subset $A \subseteq V(G)$ is called a *perfect dominating* set in G if every vertex outside A has exactly one neighbour in A. We say that the subset $S \subseteq V(G)$ is a *nearly perfect* set (or is *nearly perfect*) in G if every vertex in $V(G) - S$ is adjacent to at most one vertex in S. Note that the set $S = \emptyset$ is a nearly perfect set in an arbitrary graph. A nearly perfect set S in G is called *1-maximal* if for every vertex $u \in V(G) - S$, $S \cup \{u\}$ is not nearly perfect in G. Observe that $V(G)$ is a 1-maximal nearly perfect set in G. We denote the minimum cardinality of a 1-maximal nearly perfect set in G by $n_p(G)$. We will call the 1-maximal nearly perfect set of the cardinality $n_p(G)$ an $n_p(G) - set$. Let $n \geq 2$. The *n-fold Cartesian product of graphs* $G_1, \ldots, G_n$ is the graph $\mathbf{X}_{i=1}^{n} G_i$ with $V(\mathbf{X}_{i=1}^{n} G_i) = V(G_1) \times \cdots \times V(G_n) = X_{i=1}^{n} V(G_i)$, where $(x_1, \ldots, x_n)(y_1, \ldots, y_n) \in E(\mathbf{X}_{i=1}^{n} G_i)$ if

there exists $j \in \{1, 2, \ldots, n\}$ such that $x_j y_j \in E(G_j)$ and $x_i = y_i$ for $i \neq j$. This is equivalent to the inductive definition $\mathbf{X}_{i=1}^{n} G_i = (\mathbf{X}_{i=1}^{n-1} G_i) \times G_n$. The *n-fold strong product of graphs* $G_1, \ldots, G_n$ $(n \geq 2)$ *is the graph* $\bigotimes_{i=1}^{n} G_i$ with $V(\bigotimes X_{i=1}^{n} G_i) = V(G_1) \times \cdots \times V(G_n) = X_{i=1}^{n} V(G_i)$, where $(x_1, \ldots, x_n)(y_1, \ldots, y_n) \in E(\bigotimes_{i=1}^{n} G_i)$ if $(x_1, \ldots, x_n)(y_1, \ldots, y_n) \in E(\mathbf{X}_{i=1}^{n} G_i)$ or if $x_i y_i \in E(G_i)$ for $i = 1, 2, \ldots, n$. This is equivalent to the inductive definition $\bigotimes_{i=1}^{n} G_i = (\bigotimes_{i=1}^{n-1} G_i) \otimes G_n$.
For concepts not defined here, see [1].

2. MAIN RESULTS

In [3], the following proposition is proved.

Proposition 1. *If S_i is a nearly perfect set in G_i for $i = 1, 2$, then $S_1 \times S_2$ is a nearly perfect set in $G_1 \times G_2$.*

In our further investigations, a nearly perfect set S_i in G_i is not empty and $S_i \neq V(G_i)$ for $i = 1, 2, \ldots, n$.

Theorem 1. *Let $S_i \subseteq V(G_i)$ for $i = 1, 2, \ldots, n$. The subset $S = X_{i=1}^{n} S_i \subseteq V(\boldsymbol{X}_{i=1}^{n} G_i)$ is a nearly perfect set in $\boldsymbol{X}_{i=1}^{n} G_i$ if and only if a subset S_i is a nearly perfect set in G_i for each $i = 1, 2, \ldots, n$.*

Proof. Let S_i be an arbitrary nonempty subset of $V(G_i)$ for $i = 1, 2, \ldots, n$. Suppose that there exists $j \in \{1, 2, \ldots, n\}$ such that S_j is not nearly perfect in G_j. We will show that $S = X_{i=1}^{n} S_i$ is not nearly perfect in $\mathbf{X}_{i=1}^{n} G_i$. If S_j is not nearly perfect in G_j, then there exists a vertex $w_j \in V(G_j) - S_j$ such that $|N_{Gj}(w_j) \cap S_j| \geq 2$. Hence, say $u_j, v_j \in N_{Gj}(w_j) \cap S_j$. By $S_i \neq \emptyset$, we can take a vertex $w_i \in S_i$ for each $i \neq j$. Therefore, the vertex $\mathbf{w} = (w_1, \ldots, w_n) \in V(\mathbf{X}_{i=1}^{n} G_i) - S$ is adjacent to at least two different vertices $(w_1, \ldots, w_{j-1}, u_j, w_{j+1}, \ldots, w_n)$ and $(w_1, \ldots, w_{j-1}, v_j, w_{j+1}, \ldots, w_n)$ in S. Thus, the set $S = X_{i=1}^{n} S_i$ is not nearly perfect in $\mathbf{X}_{i=1}^{n} G_i$.

The "if" part is proved by induction on n. Proposition 1 guarantees that the result holds for $n = 2$. Suppose $n > 3$ and assume that if S_i is nearly perfect in G_i for $i = 1, 2, \ldots, n-1$, then $X_{i=1}^{n-1} S_i$ is a nearly perfect set in $\mathbf{X}_{i=1}^{n-1} G_i$. The rest of the proof runs as in the case $n = 2$, with $G_1 = \mathbf{X}_{i=1}^{n-1} G_i$ and $G_2 = G_n$. Thus, $(X_{i=1}^{n-1} S_i) \times S_n = X_{i=1}^{n} S_i$ is a nearly perfect set in $(\mathbf{X}_{i=1}^{n-1} G_i) \times G_n = \mathbf{X}_{i=1}^{n} G_i$. We conclude from the inductive definition of the n-fold Cartesian product that $S = X_{i=1}^{n} S_i$ is a nearly perfect set in $\mathbf{X}_{i=1}^{n} G_i$, which completes the proof. □

Proposition 2. *Let $S_i \subseteq V(G_i)$ and let $S_i \neq \emptyset$ for $i = 1, 2, \ldots, n$. The subset $S = X_{i=1}^{n} S_i \subseteq V(\boldsymbol{X}_{i=1}^{n} G_i)$ is independent in $\boldsymbol{X}_{i=1}^{n} G_i$ if and only if the subset S_i is independent in G_i for each $i = 1, 2, \ldots, n$.*

Proof. Let S_i be an arbitrary nonempty subset of $V(G_i)$ for $i = 1, 2, \ldots, n$ and let $S = X_{i=1}^{n} S_i \subseteq V(\mathbf{X}_{i=1}^{n} G_i)$ be independent in $\mathbf{X}_{i=1}^{n} G_i$. Suppose that there exists $j \in \{1, 2, \ldots, n\}$ such that S_j is not independent in G_j $(S_j \neq \emptyset)$. This means that there exist vertices $u_j, v_j \in S_j$ such that $u_j v_j \in E(G_j)$. Therefore, the vertices $(w_1, \ldots, w_{j-1}, u_j, w_{j+1}, \ldots, w_n) = \mathbf{w}_u$ and $(w_1, \ldots, w_{j-1}, v_j, w_{j+1}, \ldots, w_n) = \mathbf{w}_v$, where $w_i \in S_i$ for $i = 1, 2, \ldots, n$, are neighbouring in $\mathbf{X}_{i=1}^n G_i$. Moreover, vertices $\mathbf{w}_u, \mathbf{w}_v \in S$. Thus, S is not independent in $\mathbf{X}_{i=1}^n G_i$.

Now, let S_i be an arbitrary independent subset of $V(G_i)$ for $i = 1, 2, \ldots, n$. By the definition of n-fold Cartesian product, it follows that the set $S = X_{i=1}^n S_i$ is independent in $\mathbf{X}_{i=1}^n G_i$, as required. □

Theorem 2. *Let $S_i \subseteq V(G_i)$ for $i = 1, 2, \ldots, n$. The subset $S = X_{i=1}^n S_i \subseteq V(\boldsymbol{X}_{i=1}^n G_i)$ is a 1-maximal nearly perfect independent set in $\boldsymbol{X}_{i=1}^n G_i$ if and only if the subset S_i is a 1-maximal nearly perfect independent set and a perfect dominating set in G_i for each $i = 1, 2, \ldots, n$.*

Proof. Let $S_i \subseteq V(G_i)$ be an arbitrary nonempty set for $i = 1, 2, \ldots, n$. Suppose that $S = X_{i=1}^n S_i$ is a 1-maximal nearly perfect independent set in $\mathbf{X}_{i=1}^n G_i$. By Theorem 1 and Proposition 2, S_i is independent and nearly perfect in G_i for each $i = 1, 2, \ldots, n$. We just have to show that the nearly perfect set S_i in G_i is 1-maximal and perfect dominating in G_i for each $i = 1, 2, \ldots, n$.

First we prove that the nearly perfect set S_i in G_i is 1-maximal in G_i for $i = 1, 2, \ldots, n$. On the contrary, suppose that there exists $j \in \{1, 2, \ldots, n\}$ such that S_j is not 1-maximal in G_j. From this it follows that there exists a vertex $w_j \in V(G_j) - S_j$ such that $S_j \cup \{w_j\}$ is a nearly perfect set in G_j. Therefore, by Theorem 1, the set $S_{w_j} = S_1 \times \cdots \times S_{j-1} \times (S_j \cup \{w_j\}) \times S_{j+1} \times \cdots \times S_n$ is nearly perfect in $\mathbf{X}_{i=1}^n G_i$. So, every vertex outside S_{w_j} has at most one neighbour in S_{w_j}. Thus, it has at most one neighbour in $S \cup \{\mathbf{x}_{w_j}\} \subseteq S_{w_j}$, where $\mathbf{x}_{w_j} = (x_1, \ldots, x_{j-1}, w_j, x_{j+1}, \ldots, x_n)$. Moreover, every vertex from $S_{w_j} - S$ has at most one neighbour in S, since S is nearly perfect in $\mathbf{X}_{i=1}^n G_i$. By the assumption that S_i is independent in G_i for $i = 1, 2, \ldots, n$, we see that for every vertex $x_i \in S_i$, $N_{G_i}(x_i) \cap S_i = \emptyset$ for $i = 1, 2, \ldots, n$. For this reason the vertex $\mathbf{x}_{w_j} = (x_1, \ldots, x_{j-1}, w_j, x_{j+1}, \ldots, x_n) \in S_{w_j} - S$ does not have neighbours in $S_{w_j} - S$. From the above, it follows that $S \cup \{\mathbf{x}_{w_j}\}$ is a nearly perfect set in $\mathbf{X}_{i=1}^n G_i$. This contradicts the 1-maximality of S.

Now, we prove that S_i is a perfect dominating set in G_i for $i = 1, 2, \ldots, n$. On the contrary, suppose that there exists $j \in \{1, 2, \ldots, n\}$ such that S_j is not perfect dominating in G_j, i.e., there is a vertex $u_j \in V(G_j) - S_j$ such that $|N_{G_j}(u_j) \cap S_j| \neq 1$. Since S_j is a nearly perfect set in G_j, we see that there must be $|N_{G_j}(u_j) \cap S_j| = 0$. Let $u_i \in V(G_i) - S_i$ for every $i \neq j$. In $\mathbf{X}_{i=1}^n G_i$, $\mathbf{u} = (u_1, u_2, \ldots, u_n) \in X_{i=1}^n (V(G_i) - S_i)$ does not have neighbours in S. From the definition of the n-fold Cartesian product, it follows that the vertex $\mathbf{u}$ can have neighbours only in a set $A_1 \cup A_2$, where $A_1 = V(G_1) \times \cdots \times V(G_{j-1}) \times \{u_j\} \times V(G_{j+1}) \times \cdots \times V(G_n)$ and $A_2 = \{u_1\} \times \cdots \times \{u_{j-1}\} \times (V(G_j) - S_j) \times \{u_{j+1}\} \times \cdots \times \{u_n\}$. But vertices of A_1 do not have neighbours in S, because $|N_{G_j}(u_j) \cap S_j| = 0$, and vertices of A_2 do not have neighbours in S either, because A_2 is a subset of $X_{i=1}^n (V(G_i) - S_i)$. Thus, $S \cup \{\mathbf{u}\}$ is a nearly perfect set in $\mathbf{X}_{i=1}^n G_i$, which contradicts the 1-maximality of S.

Now, let S_i be a 1-maximal nearly perfect independent set in G_i. From Theorem 1 and Proposition 2, it follows that S is a nearly perfect independent set in $\mathbf{X}_{i=1}^n G_i$. It only remains to show that the nearly perfect set S in $\mathbf{X}_{i=1}^n G_i$ is 1-maximal in $\mathbf{X}_{i=1}^n G_i$. The proof is by induction on n. Let $n = 2$ and let (y_1, y_2) be an arbitrary vertex outside $S_1 \times S_2$. Without loos of generality, we have two cases to consider.

1. Let $y_2 \in S_2$. From the 1-maximality of S_1 (as a subset of G_1), it follows that $S_1 \cup \{y_1\}$ is not nearly perfect in G_1. Hence, we can find a vertex $v \in V(G_1) - (S_1 \cup \{y_1\})$ having at least two neighbours in $\{(y_1, y_2)\} \cup \{(s_1, y_2) | s_1 \in S_1\}$ which is obviously a subset of $S_1 \times S_2 \cup \{(y_1, y_2)\}$. Consequently, $S_1 \times S_2 \cup \{(y_1, y_2)\}$ is not nearly perfect in $G_1 \times G_2$.
2. Let $y_1 \in V(G_1) - S_1$ and $y_2 \in V(G_2) - S_2$. By assumption, there are vertices $s_1 \in N_{G_1}(y_1) \cap S_1$ and $s_2 \in N_{G_2}(y_2) \cap S_2$. In $G_1 \times G_2$, (y_1, s_2) has at least two neighbours in $S_1 \times S_2 \cup \{(y_1, y_2)\}$, i.e., (y_1, y_2) and (s_1, s_2). Thus, $S_1 \times S_2 \cup \{(y_1, y_2)\}$ is not nearly perfect in $G_1 \times G_2$.

Thus, $S_1 \times S_2$ is a 1-maximal nearly perfect set in $G_1 \times G_2$.

Suppose $n > 2$ and assume the result holds for all sequences of graphs with less than n elements. Hence, $X_{i=1}^{n-1} S_i$ is a 1-maximal nearly perfect set in $\mathbf{X}_{i=1}^{n-1} G_i$. We can apply the reasoning used in the case $n = 2$ with $G_1' = \mathbf{X}_{i=1}^{n-1} G_i$ and $G_2' = G_n$. Since S_n is a 1-maximal nearly perfect set in G_n, $(X_{i=1}^{n-1} S_i) \times S_n$ is a 1-maximal nearly perfect set in $(\mathbf{X}_{i=1}^{n-1} G_i) \times G_n$. By the inductive definition of the n-fold Cartesian product, $S = X_{i=1}^{n} S_i$ is a 1-maximal nearly perfect set in $\mathbf{X}_{i=1}^{n} G_i$, which proves the theorem. □

Corollary 1. *If S_i is an $\alpha(G_i)$-set for each $i = 1, 2, \ldots, n$, then $n_p(\boldsymbol{X}_{i=1}^{n} G_i) \leq |X_{i=1}^{n} S_i| = \prod_{i=1}^{n} |S_i| = \prod_{i=1}^{n} \alpha(G_i)$.*

Corollary 2. *If S_i is an independent $n_p(G_i) - set$ and a perfect dominating set in G_i for each $i = 1, 2, \ldots, n$, then $n_p(\boldsymbol{X}_{i=1}^{n} G_i) \leq |X_{i=1}^{n} S_i| = \prod_{i=1}^{n} |S_i| = \prod_{i=1}^{n} n_p(G_i)$.*

It has been proved in [2] that the $n_p(P_{3k+1}) - sets$ and $n_p(C_{3k}) - sets$, for $k \geq 1$, are independent and perfect dominating in P_{3k+1} and C_{3k}, respectively.

Theorem 2 leads immediately to the following conclusion.

Corollary 3. *If $n_p(\boldsymbol{X}_{i=1}^{n} G_i) = 1$, then $n_p(G_i) = 1$ for each $i = 1, 2, \ldots, n$.*

It is easily seen that $n_p(G_i) = 1$ for each $i = 1, 2, \ldots, n$ does not always imply $n_p(\mathbf{X}_{i=1}^{n} G_i) = 1$. For example, let G_1 be a graph such that $V(G_1) = \{x_0, x_1, x_2, x_3\}$ and $E(G_1) = \{x_0x_1, x_1x_2, x_2x_3, x_3x_1\}$ and let G_2 be a cycle on three vertices. Then $n_p(G_i) = 1$ for each $i = 1, 2$ but $n_p(G_1 \times G_2) = 2$.

Theorem 3. *Let $S_i \subseteq V(G_i)$, for $i = 1, 2, \ldots, n$. If $S = X_{i=1}^{n} S_i \subseteq V(\bigotimes_{i=1}^{n} G_i)$ is a nearly perfect set in $\bigotimes_{i=1}^{n} G_i$, then S_i is a nearly perfect set in G_i for each $i = 1, \ldots, n$.*

Proof. Our proof starts with the observation that $\bigotimes_{i=1}^{n} G_i$ is an overgraph of $\mathbf{X}_{i=1}^{n} G_i$ and $V(\bigotimes_{i=1}^{n} G_i) = V(\mathbf{X}_{i=1}^{n} G_i)$. Consequently, if $S \subseteq V(\bigotimes_{i=1}^{n} G_i) = V(\mathbf{X}_{i=1}^{n} G_i)$ is a nearly perfect set in $\bigotimes_{i=1}^{n} G_i$, then S is also a nearly perfect set in $\mathbf{X}_{i=1}^{n} G_i$. By Theorem 1, we obtain the desired conclusion. □

Theorem 4. *If $S_i \subseteq V(G_i)$ is an independent nearly perfect set in G_i for each $i = 1, 2, \ldots, n$, then $S = X_{i=1}^{n} S_i \subseteq V(\bigotimes_{i=1}^{n} G_i)$ is a nearly perfect set in $\bigotimes_{i=1}^{n} G_i$.*

Proof. The proof is by induction on n. The proof for the case $n = 2$ was given in [3]. Assume that $X_{i=1}^{n-1} S_i$ is nearly perfect in $\bigotimes_{i=1}^{n-1} G_i$. We can apply the reasoning used in the case $n = 2$ with $G'_1 = \bigotimes_{i=1}^{n-1} G_i$ and $G'_2 = G_n$. Since S_n is a nearly perfect set in G_n, $(X_{i=1}^{n-1} S_i) \times S_n$ is a nearly perfect set in $(\bigotimes_{i=1}^{n-1} G_i) \otimes G_n$. We conclude from the inductive definition of the n-fold strong product that $S = X_{i=1}^{n} S_i$ is a nearly perfect set in $\bigotimes_{i=1}^{n} G_i$, as required. □

Theorem 5. *If $S_i \subseteq V(G_i)$ is a 1-maximal nearly perfect independent set in G_i for each $i = 1, \ldots, n$, then $S = X_{i=1}^{n} S_i \subseteq V(\bigotimes_{i=1}^{n} G_i)$ is a 1-maximal nearly perfect set in $\bigotimes_{i=1}^{n} G_i$.*

Proof. By Theorem 4, S is a nearly perfect set in $\bigotimes_{i=1}^{n} G_i$. The proof is completed by showing that the nearly perfect set S in $\bigotimes_{i=1}^{n} G_i$ is 1-maximal in $\bigotimes_{i=1}^{n} G_i$. The proof is by induction on n. The result for the case $n = 2$ was given in [3]. Assume that $X_{i=1}^{n-1} S_i$ is a 1-maximal nearly perfect set in $\bigotimes_{i=1}^{n-1} G_i$. We can apply the reasoning used in the case $n = 2$ with $G'_1 = \bigotimes_{i=1}^{n-1} G_i$ and $G'_2 = G_n$. Since S_n is a nearly perfect set in G_n, $(X_{i=1}^{n-1} S_i) \times S_n$ is a 1-maximal nearly perfect set in $(\bigotimes_{i=1}^{n-1} G_i) \otimes G_n$. We conclude from the inductive definition of the n-fold strong product that $S = X_{i=1}^{n} S_i$ is a 1-maximal nearly perfect set in $\bigotimes_{i=1}^{n} G_i$, and the proof is complete. □

Corollary 4. *If S_i is an $\alpha(G_i)$-set for each $i = 1, 2, \ldots, n$, then $n_p(\bigotimes_{i=1}^{n} G_i) \leq |X_{i=1}^{n} S_i| = \prod_{i=1}^{n} |S_i| = \prod_{i=1}^{n} \alpha(G_i)$.*

Corollary 5. *If S_i is an independent $n_p(G_i) -$ set for each $i = 1, 2, \ldots, n$, then $n_p(\bigotimes_{i=1}^{n} G_i) \leq |X_{i=1}^{n} S_i| = \prod_{i=1}^{n} |S_i| = \prod_{i=1}^{n} n_p(G_i)$.*

The results from [2] and Corollary 5 give $n_p(P_{3k+1} \otimes C_{3l}) \leq n_p(P_{3k+1}) \cdot n_p(C_{3l}) = (k + 1) \cdot l$.

Theorem 5 leads evidently to the following conclusion.

Corollary 6. *If $n_p(G_i) = 1$ for $i = 1, 2, \ldots, n$, then $n_p(\bigotimes_{i=1}^{n} G_i) = 1$.*

It is easy to check that $n_p(\bigotimes_{i=1}^{n} G_i) = 1$ does not always imply $n_p(G_i) = 1$, for $i = 1, 2, \ldots, n$. For example, $n_p(K_2 \otimes K_2) = 1$ but $n_p(K_2) = 2$.

Theorem 6. *Let $S_i \subseteq V(G_i)$ for $i = 1, 2, \ldots, n$. If $S = X_{i=1}^{n} S_i \subseteq V(\bigotimes_{i=1}^{n} G_i)$ is a 1-maximal nearly perfect set in $\bigotimes_{i=1}^{n} G_i$ and S contains an isolated vertex of $\bigotimes_{i=1}^{n} G_i$, then S_i is a 1-maximal nearly perfect set in G_i for each $i = 1, 2, \ldots, n$.*

Proof. Our proof starts with the observation that if S is a 1-maximal nearly perfect set in $\bigotimes_{i=1}^{n} G_i$, then $S \neq \emptyset$ and, in consequence, $S_i \neq \emptyset$ for every $i = 1, 2, \ldots n$. Therefore, by Theorem 3, S_i is a nearly perfect set in G_i for each $i = 1, 2, \ldots, n$. It suffices to prove that the nearly perfect set S_i in G_i is 1-maximal for each $i = 1, 2, \ldots, n$. Suppose, contrary to our claim, that there exists $j \in \{1, 2, \ldots, n\}$ such that S_j is not 1-maximal in G_j. Thus, there exists a vertex $w_j \in V(G_j) - S_j$ such that $S_j \cup \{w_j\}$ is a nearly perfect set in G_j. Since S contains an isolated vertex of $\bigotimes_{i=1}^{n} G_i$, say $\mathbf{x} = (x_1, \ldots, x_n) \in S$, we conclude that x_i (as a vertex of G_i) does not have neighbours in S_i, for $i = 1, 2, \ldots, n$. Therefore, the vertex $\mathbf{x}_{w_j} = (x_1, \ldots, x_{j-1}, w_j, x_{j+1}, \ldots, x_n) \in S_1 \times \cdots \times \{w_j\} \times S_{j+1} \times \cdots \times S_n = S_{w_j}$ does not have neighbours in S_{w_j}. By the definition of the n-fold strong product, it follows that the vertex $\mathbf{x}_{w_j}$ has neighbours in a set $\{x_1\} \times \cdots \times \{x_{j-1}\} \times N_{G_j}(w_j) \times \{x_{j+1}\} \times \cdots \times \{x_n\}$

only. But $\{x_1\}\times\cdots\times\{x_{j-1}\}\times(S_j\cup\{w_j\})\times\{x_{j+1}\}\times\cdots\times\{x_n\}$ is a nearly perfect set in the induced subgraph $\langle\{x_1\}\times\cdots\times\{x_{j-1}\}\times V(G_j\times\{x_{j+1}\}\times\cdots\times\{x_n\}\rangle_{\bigotimes_{i=1}^n G_i}$ since $S_j\cup\{w_j\}$ is a nearly perfect in G_j. From the above it follows that for every vertex $\mathbf{y}=(y_1,\ldots,y_n)$ outside $S\cup\{\mathbf{x}_{w_j}\}$ the inequality $|N_{\bigotimes_{i=1}^n G_i}(\mathbf{y})\cap(S\cup\{\mathbf{x}_{w_j}\})|\leq 1$ holds. Hence $S\cup\{\mathbf{x}_{w_j}\}$ is a nearly perfect set in $\bigotimes_{i=1}^n G_i$, which contradicts the 1-maximality of S. □

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REFERENCES

[1] R. Diestel, *Graph Theory,* Springer-Verlag, New York, 1997.

[2] J.E. Dunbar, F.C. Harris, S.M. Hedetniemi, S.T. Hedetniemi, A.A. McRae, R.C. Laskar, *Nearly perfect sets in graphs*, Discrete Mathematics **138** (1995), 229–246.

[3] M. Kwaśnik, M. Perl, *Nearly perfect sets in products of graphs*, Opuscula Mathematica **24/2** (2004), 177–180.

Monika Perl
mperl@ps.pl

Szczecin University of Technology
Institute of Mathematics
al. Piastów 48/49, Szczecin, Poland

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