

Mourice Ouma Otieno

ON QUASI-SIMILARITY AND w -HYPONORMAL OPERATORS

Abstract. In this paper, it is shown that a Putnam-Fuglede type commutativity theorem holds for w -hyponormal operators, the normal parts of quasi-similar w -hyponormal operators are unitarily equivalent and a w -hyponormal spectral operator is normal.

Keywords: w -hyponormal operators, quasi-similarity.

Mathematics Subject Classification: 47B20.

1. INTRODUCTION

Let H be an infinite dimensional complex Hilbert space and let $B(H)$ denote the algebra of operators from H to itself (= bounded linear transformations).

Given $A, B \in B(H)$, define $\delta_{A,B}: B(H) \rightarrow B(H)$ by

$$\delta_{A,B}(X) = AX - XB,$$

for some operator X . Hence, $X \in \ker \delta_{A,B}$ ($X \in \ker \delta_{A^*,B^*}$) will denote $AX - XB = 0$ ($A^*X - XB^* = 0$).

The classical Putnam-Fuglede Theorem [16, p. 104] says that if A and B^* are normal operators such that $AX = XB$ for some operator X , then also $A^*X = XB^*$ (if A and B^* are normal operators, then $\ker \delta_{A,B} = \ker \delta_{A^*,B^*}$).

Over the years, a number of authors have considered the problem of the extension of the classical Putnam-Fuglede Theorem to a class (or classes) of operators more general than the class of normal operators. Here the particular classes which have drawn a lot of attention are those consisting of either hyponormal or M -hyponormal or dominant or k -quasi-hyponormal operators. (See [11, 13, 15, 19, 23–26, 28] and some of the references there.)

Recently, Jeon, Tanahashi and Uchiyama ([19, 26]) have shown that Putnam-Fuglede Theorem holds true in the case of p -hyponormal and log-hyponormal operators.

In the first part of this note, we extend these results ([19, 26]) to the class of w -hyponormal operators.

For $p > 0$, recall that ([1, 2, 10, 12, 19]) an operator A is said to be p -hyponormal if $(A^*A)^p \geq (AA^*)^p$. A p -hyponormal operator is called hyponormal if $p = 1$, semi-hyponormal if $p = \frac{1}{2}$. An invertible operator A is called log-hyponormal if $\log(A^*A) \geq \log(AA^*)$.

Let $A = U|A|$ be the polar decomposition of A ; then following ([1, 2]), we define the first **Aluthge transform** of A by $\tilde{A} = |A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}$ and define the second **Aluthge transform** of A by $\tilde{\tilde{A}} = |\tilde{A}|^{\frac{1}{2}}\tilde{U}|\tilde{A}|^{\frac{1}{2}}$, where $\tilde{A} = \tilde{U}|\tilde{A}|$ is the polar decomposition of \tilde{A} . An operator A is said to be w -hyponormal if $|\tilde{A}| \geq |A| \geq |\tilde{\tilde{A}}|$.

It is well known that the class of w -hyponormal operators contains both the p - and log-hyponormal operators. But neither the class of p -hyponormal operators nor the class of log-hyponormal operators contains the other. Also, if an operator A is w -hyponormal, then \tilde{A} is semi-hyponormal and $\tilde{\tilde{A}}$ is hyponormal.

If an operator A is p -hyponormal, then $\ker A \subset \ker A^*$, and if A is log-hyponormal, then $\ker A = \ker A^*$. However, if A is w -hyponormal, then it is not known whether the kernel condition $\ker A \subset \ker A^*$ holds. It is well known that there exists an example of a w -hyponormal operator A with the property that $\ker A$ is not a subset of $\ker A^*$. Nevertheless, there are several properties that p -hyponormal operators share with w -hyponormal operators A or w -hyponormal operators A with $\ker A \subset \ker A^*$ ([3, 5]).

Recall that an operator $A \in B(H)$ is said to be dominant if for each $\lambda \in \mathbf{C}$, there exists a positive number M_λ such that

$$(A - \lambda)(A - \lambda)^* \leq M_\lambda(A - \lambda)^*(A - \lambda).$$

If the constants M_λ are bounded by a positive operator M , then A is said to be M -hyponormal.

Clearly the following inclusions hold and are known to be proper.

$$\text{Hyponormal} \subseteq p\text{-Hyponormal}(0 < p < 1) \subseteq w\text{-Hyponormal} \subseteq \text{Paranormal}$$

and

$$\text{Log-hyponormal} \subseteq w\text{-Hyponormal} \subseteq \text{Paranormal}.$$

An operator $X \in B(H)$ is called a quasiaffinity if X is both injective and has a dense range. Two operators A and B are said to be quasi-similar if there exist quasiaffinities X and Y such that $X \in \ker \delta_{A,B}$ and $Y \in \ker \delta_{B,A}$.

The operator A is said to be pure if there exists no non-trivial reducing subspace N of H such that the restriction of A to N ($A|_N$) is normal and is said to have a normal direct summand if it is not pure.

Recall that every $A \in B(H)$ has a direct sum decomposition $A = A_1 \oplus A_2$, where A_1 and A_2 are normal and pure parts, respectively. Of course, in the sum decomposition, either A_1 or A_2 may be absent. Given $X \in B(H)$, let $\overline{\text{ran} X}$ and $(\ker X)^\perp$ denote the closure of the range and the orthogonal complement of the kernel of X .

Jeon and Duggal [12] have shown that the normal parts of quasi-similar p -hyponormal operators are unitarily equivalent, a p -hyponormal operator compactly

quasi-similar to an isometry is unitary and a p -hyponormal operator which is also a spectral operator turns out to be normal.

Jeon, Tanahashi and Uchiyama [19] proved that results similar to those in [12] hold true for the class of log-hyponormal operators.

In the second part of this paper, we carry on with these results to the more general case of w -hyponormal operators and show that similar results still hold true. The major tools used to show these results is the second Aluthge transform and the kernel condition $\ker A \subset \ker A^*$.

2. A PUTNAM-FUGLEDE TYPE THEOREM FOR W -HYPONORMAL OPERATORS

In this section, we start by proving a Putnam-Fuglede type commutativity theorem for w -hyponormal operators.

Theorem 1. *If A and B^* are w -hyponormal operators with $\ker A \subset \ker A^*$ and $\ker B^* \subset \ker B$ such that $X \in \ker \delta_{A,B}$ for some operator X , then $X \in \ker \delta_{A^*,B^*}$, $\overline{\text{ran} X}$ reduces A , $(\ker X)^\perp$ reduces B and $A|_{\overline{\text{ran} X}}$ and $B|_{(\ker X)^\perp}$ are unitarily equivalent normal operators.*

To prove Theorem 1, we need the following lemma.

Lemma 2. ([3, Theorem 2.4]). *If A is w -hyponormal, then \widetilde{A} is semi-hyponormal and \widetilde{A} is hyponormal.*

Proof of Theorem 1. Decompose A and B^* into normal and pure parts as $A = A_1 \oplus A_2$ and $B^* = B_1^* \oplus B_2^*$. Let $X = [X_{ij}]_{i,j=1}^2$ and

$$\widetilde{X}_{22} = |\widetilde{A}_2|^{\frac{1}{2}} |A_2|^{\frac{1}{2}} X_{22} |B_2^*|^{\frac{1}{2}} |\widetilde{B}_2^*|^{\frac{1}{2}}.$$

Now since $X_{22} \in \ker \delta_{A_2, B_2^*}$, it follows that $\widetilde{X}_{22} \in \ker \delta_{\widetilde{A}_2, (\widetilde{B}_2^*)^*}$, where \widetilde{A}_2 and \widetilde{B}_2^* ($= \widetilde{(B_2)^*}$) are hyponormal operators by Lemma 2. Applying Putnam-Fuglede Theorem for hyponormal operators, $\widetilde{X}_{22} \in \ker \delta_{\widetilde{A}_2, \widetilde{B}_2^*}$, $\overline{\text{ran} \widetilde{X}_{22}}$ reduces \widetilde{A}_2 , $(\ker \widetilde{X}_{22})^\perp$ reduces \widetilde{B}_2^* , and $\widetilde{A}_2|_{\overline{\text{ran} \widetilde{X}_{22}}}$ and $\widetilde{B}_2^*|_{(\ker \widetilde{X}_{22})^\perp}$ are unitarily equivalent normal operators. In particular, \widetilde{A}_2 and \widetilde{B}_2^* have normal direct summands. Now by an argument similar to that used in the proof of the converse of Lemma 1 of [10], it is seen that A_2 and B_2^* have normal direct summands. Since A_2 and B_2^* are pure, $\widetilde{X}_{22} = 0$ and hence $X_{22} = 0$. Using a similar argument to the operator equations $X_{21} \in \ker \delta_{A_2, B_1^*}$ and $X_{12} \in \ker \delta_{A_1, B_2}$, we get that $X_{21} = 0 = X_{12}$. Applying Putnam-Fuglede Theorem to the operator equation $X_{11} \in \ker \delta_{A_1, B_1^*}$, $X_{11} \in \ker \delta_{A_1^*, B_1^*}$ and consequently $X \in \ker \delta_{A^*, B^*}$ and the result follows. \square

Corollary 3. *Let $X \in \ker \delta_{A,B}$ for some operator X . If either A is a pure w -hyponormal operator and B^* is w -hyponormal or A is w -hyponormal and B^* is a pure w -hyponormal operator with $\ker A \subset \ker A^*$ and $\ker B^* \subset \ker B$, then $X = 0$.*

Theorem 1 holds in particular for w -hyponormal $A(B^*)$ and hyponormal $B^*(A)$. A generalisation to a dominant operator A is given in the following Corollary.

Corollary 4. *If $X \in \ker \delta_{A,B}$ for some dominant operator A , a w -hyponormal operator B^* with $\ker B^* \subset \ker B$ and $X \in B(H)$, then $X \in \ker \delta_{A^*,B^*}$, $\text{ran } X$ reduces A , $(\ker X)^\perp$ reduces B and $A|_{\overline{\text{ran } X}}$ and $B|_{(\ker X)^\perp}$ are unitarily equivalent normal operators.*

Proof. Decompose A and B^* into normal and pure parts and Let $X = [X_{ij}]_{i,j=1}^2$. Apply Theorem 1 of [13] to $X_{21} \in \ker \delta_{A_2,B_1}$ and $A_2 X_{22} |B_2^*|^{\frac{1}{2}} |\widetilde{B_2^*}|^{\frac{1}{2}} = X_{22} |B_2^*|^{\frac{1}{2}} |\widetilde{B_2^*}|^{\frac{1}{2}} (\widetilde{B_2^*})^*$. Then apply Theorem 1 to $X_{12} \in \ker \delta_{A_1,B_2}$ and the proof follows. \square

3. THE NORMAL PARTS OF QUASI-SIMILAR W -HYPONORMAL OPERATORS

Douglas ([9]) proved that quasi-similar normal operators are unitarily equivalent. This result was extended by Conway ([8]), who proved that the normal parts of quasi-similar subnormal operators are unitarily equivalent. In the same paper, Conway gave an example which shows that the pure parts of quasi-similar subnormal operators are not necessarily quasi-similar. While working on the class of hyponormal operators, Clary [7] proved that quasi-similar hyponormal operators are unitarily equivalent. This result was further extended by Williams [29] to a more general class of dominant operators. Recently Jeon and Duggal ([12]) and Jeon and others ([19]) extended the result of Conway ([8]) to the class of p -hyponormal and log-hyponormal operators, respectively. Let us recall that the classes of p or log-hyponormal operators and dominant operators are independent of each other.

In this section, we extend the results of ([12]) and ([19]) to a more general class of w -hyponormal operators.

Theorem 5. *Let A and B be quasi-similar w -hyponormal operators with $\ker A \subset \ker A^*$ and $\ker B \subset \ker B^*$, respectively. Let $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$, where A_1, B_1 and A_2, B_2 are the normal and pure parts, respectively. Then A_1 and B_1 are unitarily equivalent. Furthermore, there exist bounded operators X^* and Y^* with dense ranges such that $X^* \in \ker \delta_{A_2,B_2}$ and $Y^* \in \ker \delta_{B_2,A_2}$.*

To prove the theorem, we need the following results which we will state as Lemmas.

Lemma 6. ([3, Theorem 2.6]). *Let A be w -hyponormal with $\ker A \subset \ker A^*$. If \widetilde{A} is normal, then $\widetilde{A} = A$.*

The following lemma is well known and a proof is included for completeness.

Lemma 7. *Let A be w -hyponormal with $\ker A \subset \ker A^*$ and B be a normal operator. If there exists an operator $X \in B(H)$ with a dense range such that $X \in \ker \delta_{A,B}$, then A is normal.*

Proof. Decompose $A = A_1 \oplus A_2$ into its normal and pure parts, respectively. Let $A_2 = U_2|A_2|$, $\widetilde{A}_2 = |A_2|^{\frac{1}{2}} U |A_2|^{\frac{1}{2}}$ and $\widetilde{\widetilde{A}}_2 = |\widetilde{A}_2|^{\frac{1}{2}} \widetilde{U} |\widetilde{A}_2|^{\frac{1}{2}}$.

A_2 being pure, it is injective and $|A_2|^{\frac{1}{2}}$ is a quasiaffinity. Also since A_1 is normal, $\widetilde{\widetilde{A}} = \widetilde{\widetilde{A}}_1 \oplus \widetilde{\widetilde{A}}_2 = A_1 \oplus \widetilde{\widetilde{A}}_2$.

Now if we let $T = |\widetilde{A}_2|^{\frac{1}{2}} |A_2|^{\frac{1}{2}}$, then by a simple computation, $\widetilde{\widetilde{A}}_2 T = T A_2$ and T is a quasiaffinity.

Also if we let $Z = I_H \oplus T$, then clearly Z is also a quasiaffinity such that $\widetilde{\widetilde{A}} Z = Z A$, where $\widetilde{\widetilde{A}}$ is a hyponormal operator.

Thus $\widetilde{\widetilde{A}} Z X = Z A X = Z X B$ and by ([24]), $\widetilde{\widetilde{A}}$ is normal. Hence by Lemma 6, we get the result. \square

With these results, we are now ready to prove the Theorem.

Proof of Theorem 5. By the given hypotheses, there exist quasiaffinities X and Y say such that $X \in \ker \delta_{A,B}$ and $Y \in \ker \delta_{B,A}$. Let $X = [X_{ij}]_{i,j=1}^2$ and $Y = [Y_{ij}]_{i,j=1}^2$ with respect to decompositions of A and B respectively. Then by a simple matrix calculation, we obtain $X_{21} \in \ker \delta_{A_2, B_1}$ and $Y_{21} \in \ker \delta_{B_2, A_1}$. Next we show that $X_{21} = 0 = Y_{21}$. Let $M = \overline{\text{ran}(X_{21})}$, then the subspace M is invariant under A_2 and by Theorem 1, $X_{21} \in \ker \delta_{A_2^*, B_1^*}$ and M reduces A_2 . Let $A_2^1 = A_2|_M$, then by Lemma 5 of [27], A_2^1 is w -hyponormal. Now define $X_{21}^1 : H \rightarrow M$ by $X_{21}^1 x = X_{21} x$ for each $x \in H$. Then X_{21}^1 has a dense range and satisfies the equation $X_{21}^1 \in \ker \delta_{A_2^1, B_1}$. By Lemma 7, A_2^1 is normal, which contradicts the fact that A_2 is pure and hence $X_{21} = 0$. Similarly $Y_{21} = 0$. Thus X_{11} and Y_{11} are injective and since $X_{11} \in \ker \delta_{A_1, B_1}$ and $Y_{11} \in \ker \delta_{B_1, A_1}$, by Lemma 1.1 of [29], A_1 and B_1 are unitarily equivalent. Also notice that X_{22} and Y_{22} have dense ranges and that $X_{22} \in \ker \delta_{A_2, B_2}$ and $Y_{22} \in \ker \delta_{B_2, A_2}$. Hence the proof is complete. \square

From the Theorem, we immediately obtain the following corollaries.

Corollary 8. *If A and B are quasi-similar w -hyponormal operators with $\ker A \subset \ker A^*$, then $\sigma_e(A) = \sigma_e(B)$.*

Proof. Decompose A and B into its normal and pure parts, respectively. Now by the Theorem and [31, Corollary 2.9], the proof follows. \square

Remark. As an application, we are able to answer Clary [7] and Conway's [8] famous question affirmatively.

Corollary 9. *If a w -hyponormal operator A with $\ker A \subset \ker A^*$ is quasi-similar to a normal operator B , then A and B are unitarily equivalent normal operators.*

Corollary 10. *If A and B are quasi-similar w -hyponormal operators with $\ker A \subset \ker A^*$, then B is pure whenever A is pure.*

In Theorem 1, if X is a quasiaffinity, then $\overline{\text{ran}X} = H$ and $(\ker X)^\perp = H$. Accordingly, the following corollary is true.

Corollary 11. *If A and B^* are w -hyponormal operators with $\ker A \subset \ker A^*$ and $\ker B^* \subset \ker B$ such that $X \in \ker \delta_{A,B}$ for some quasiaffinity operator X , then A and B are unitarily equivalent normal operators.*

Remark. First, observe that from Corollary 11, we immediately recapture Corollary 9.

Also in Theorem 1, if X is a quasiaffinity, then $X \in \ker \delta_{A,B}$ implies $X \in \ker \delta_{A^*,B^*}$, where A is w -hyponormal operator with $\ker A \subset \ker A^*$ and B is a normal operator.

However, in the sequel, we wish to give an alternative proof (as a generalisation of Corollary 9) of this result.

Theorem 12. *Let A be w -hyponormal with $\ker A \subset \ker A^*$ and B be a normal operator. If there exists a quasiaffinity $X \in B(H)$ such that $X \in \ker \delta_{A,B}$, then $X \in \ker \delta_{A^*,B^*}$.*

We will need the following results.

Lemma 13. ([20]). *If B is a normal operator on H , then*

$$\bigcap (B - \lambda)H = \{0\} \quad \text{for } \lambda \in \mathbf{C}.$$

Lemma 14. ([21]). *Let $A, B \in B(H)$ be such that $0 \leq B \leq T(A - \lambda)(A - \lambda)^*$ for each $\lambda \in \mathbf{C}$, where T is a positive real number. Then for every $x \in B^{\frac{1}{2}}H$, there exists a bounded function $f : \mathbf{C} \rightarrow H$ such that $(A - \lambda)f(\lambda) \equiv x$.*

Proof of Theorem 12. Decompose $A = A_1 \oplus A_2$ into normal and pure parts, respectively. By Lemma 2, \tilde{A} is hyponormal and using Lemma 7, there exists a quasiaffinity W such that $\tilde{A}W = WA$. Hence

$$\tilde{A}WX = WAX = WXB$$

and $\tilde{A}Z = ZB$, where $Z = WX$ and Z is a quasiaffinity. Thus $Z^*\tilde{A}^* = B^*Z^*$, where Z^* is also a quasiaffinity.

Now let

$$x \in (\tilde{A}^*\tilde{A} - \tilde{A}\tilde{A}^*)^{\frac{1}{2}}H,$$

then by Lemma 14, there exists a bounded function $f : \mathbf{C} \rightarrow H$ such that $(\tilde{A}^* - \lambda)f(\lambda) \equiv x$, for each $\lambda \in \mathbf{C}$.

Hence

$$Z^*x = Z^*(\tilde{A}^* - \lambda)f(\lambda) = (B^* - \lambda)f(\lambda)Z^* \in \text{ran}(B^* - \lambda),$$

for each $\lambda \in \mathbf{C}$. By Lemma 13, $Z^*x = 0$ and $x = 0$. Hence by Lemma 6 and Putnam-Fuglede Theorem, the result follows. \square

4. W -HYPONORMAL OPERATORS QUASI-SIMILAR TO SPECTRAL OPERATORS

A spectral operator (in the sense of Dunford) is an operator with a countable additive resolution of the identity defined on the Borel sets of the complex plane. If A is spectral, then it has the canonical decomposition $A = R + S$, where R and S are its scalar and radical parts, respectively. For details of spectral operators, see [14, 15, 17, 18, 22].

In [22] (or [15]), it is shown that an M -hyponormal spectral operator is normal. However, Example 2 of [24] shows that a dominant operator may be quasinilpotent and hence spectral, without necessarily being normal.

Recently, Jean and Duggal ([12]) have shown that a p -hyponormal spectral operator is normal.

In this section, we try to extend this result to a more general case of w -hyponormal operators.

The following result is Lemma 2 of [15] (see also [22]).

Lemma 15. *Let A be an M -hyponormal operator and suppose there exists an operator X with a dense range and a spectral operator B such that $X \in \ker \delta_{A,B}$. Then there exist a positive operator P , a normal operator N and a quasinilpotent operator Q such that $(A - N)P = PQ$ and $AN = NA$.*

From the Lemma, the following Corollary ([15, Corollary 4]) is immediate.

Corollary Q. *If $X \in \ker \delta_{A,B}$, where B is spectral, A is hyponormal and X has a dense range, then A is normal, B is a scalar operator and is similar to A .*

The following result is an extension of Corollary Q above.

Theorem 16. *Let A be w -hyponormal with $\ker A \subset \ker A^*$ and B be a spectral operator. If there exists an operator $X \in B(H)$ with a dense range such that $X \in \ker \delta_{A,B}$, then A is normal, B is a scalar operator and is similar to A .*

Proof. Decompose $A = A_1 \oplus A_2$ into normal and pure parts, respectively. Then, as in the proof of Lemma 7, there exists a quasiaffinity Z such that $\tilde{A}Z = ZA$, where \tilde{A} is a hyponormal operator.

Thus $\tilde{A}ZX = ZAX = ZXB$ and, by Corollary Q \tilde{A} is normal. Hence by Lemma 6, A is normal, B is a scalar and is similar to A . \square

Corollary 17. *Let A be w -hyponormal with $\ker A \subset \ker A^*$ and B be a spectral operator. If there exists a quasiaffinity $X \in B(H)$ such that $X \in \ker \delta_{A,B}$, then A is normal, B is a scalar operator and is similar to A .*

Corollary 18. ([12, Theorem 11], [19, Theorem 12]). *Let A be log or p -hyponormal and B be a spectral operator. If there exists a quasiaffinity $X \in B(H)$ such that $X \in \ker \delta_{A,B}$, then A is normal, B is a scalar operator and is similar to A .*

Acknowledgments

I wish to express my deep gratitude to the International Science Programme (ISP) of Uppsala University, Sweden, for funding my visit to Lund University which enabled me to start preparing this note. My gratitude also goes to the Department of Mathematics, LTH, Lund University, Sweden, for their hospitality and making their facilities available to me during the preparation of this article.

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Mourice O. Otieno

University of Nairobi
School of Mathematics
P.O. Box 30197
Nairobi, Kenya

Received: March 28, 2006.