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k-PERFECT GEODOMINATING SETS IN GRAPHS

Abstract. A perfect geodominating set in a graph G is a geodominating set S such that any vertex $v \in V(G)\backslash S$ is geodominated by exactly one pair of vertices of S. A k-perfect geodominating set is a geodominating set S such that any vertex $v \in V(G)\backslash S$ is geodominated by exactly one pair x, y of vertices of S with $d(x, y) = k$. We study perfect and k-perfect geodomination numbers of a graph G.

Keywords: geodominating set, perfect geodomination number, pendant vertex, pendant edge.

Mathematics Subject Classification: 05C69.

1. INTRODUCTION

For vertices x and y in a connected graph G, the distance $d(x, y)$ is the length of a shortest $x - y$ path in G. An $x - y$ path of length $d(x, y)$ is called an $x - y$ geodesic. A vertex v is said to *lie* in an $x - y$ geodesic P if v is an *internal* vertex of P. The *closed interval* $I[x, y]$ consists of x, y and all vertices lying in some $x - y$ geodesic of G, while for $S \subseteq V(G)$,

$$I[S] = \cup_{x,y \in S} I[x, y].$$

A set S of vertices is a *geodetic set* if $I[S] = V(G)$, and the minimum cardinality of a geodetic set is the *geodetic number* $g(G)$. A geodetic set of cardinality $g(G)$ is called a g-set (cf. [1–6])

Geodetic concepts were studied from the point of view of domination (cf. [2]). Geodetic sets and the geodetic number were referred to as *geodominating* sets and the *geodomination* number (cf. [2]). These expressions we adopt in this paper.

A pair x, y of vertices in a nontrivial connected graph G is said to *geodominate* a vertex v of G if either $v \in \{x, y\}$ or v lies in an $x - y$ geodesic of G. A set S of vertices of G is a geodominating set if every vertex of G is geodominated by some pair of vertices of S. For a graph G and an integer $k \geq 1$, a vertex v of G is *k-geodominated* by a pair x, y of distinct vertices in G if v is geodominated by x, y and $d(x, y) = k$.

A set S of vertices of G is a k-geodominating set of G if each vertex v in $V(G)\backslash S$ is k-geodominated by some pair of distinct vertices of S. The minimum cardinality of a k-geodominating set of G is its *k-geodomination number* $g_k(G)$. A k-geodomination set of cardinality $g_k(G)$ is called a g_k-set of G.

Uniform and *essential* geodominating sets are introduced in [1]. A set S of vertices in a connected graph G is *uniform* if the distance between every two vertices of S is the same fixed number. A geodominating set S is *essential* if for every two vertices u, v in S, there exists a vertex $w \in V(G)\backslash\{u, v\}$ which lies in a $u - v$ geodesic but in no $x - y$ geodesic for $x, y \in S$ and $\{x, y\} \neq \{u, v\}$.

The *cartesian product* of two graphs G, H, denoted by $G \times H$, is the graph with vertex set $V(G) \times V(H)$ specified by putting (u, v) adjacent to $(u^{'}, v^{'})$ if and only if (1) $u = u^{'}$ and $vv^{'} \in E(H)$, or (2) $v = v^{'}$ and $uu^{'} \in E(G)$. This graph has $|V(G)|$ copies of H as rows and $|V(H)|$ copies of G as columns.

All graphs in this paper are connected and for an edge $e = uv$ of a graph G with $\deg(u) = 1$ and $\deg(v) > 1$, we call e a *pendant edge* and u a *pendant vertex*.

2. DEFINITION

A *perfect geodominating set* in a graph G is a geodominating set S such that any vertex $v \in V(G)\backslash S$ is geodominated by exactly one pair of vertices of S and the cardinality of a minimum perfect geodominating set in G is its perfect geodomination number $g_p(G)$.

Let $k \geq 1$ be an integer. A *k-perfect geodominating set* is a geodominating set S such that any vertex $v \in V(G)\backslash S$ is geodominated by exactly one pair x, y of vertices of S with $d(x, y) = k$. The cardinality of a minimum k-perfect geodominating set in G is its k-perfect geodomination number $g_{kp}(G)$.

By definition, any k-perfect geodominating set is both a k-geodominating set and a perfect geodominating set. We refer a $g_{kp}(G)$-set to a k-perfect geodominating set of size $g_{kp}(G)$ and a $g_p(G)$-set to a perfect geodominating set of size $g_p(G)$. Thus for any graph G there is $g_{1p}(G) = |V(G)|$ and also:

1) $g_p(G) \geq g(G)$;
2) $|V(G)| \geq g_{kp}(G) \geq g_k(G) \geq 2$;
3) if $g(G) = 2$, then $g_p(G) = g(G) = 2$;
4) $g_{kp}(G) \geq g_p(G), \quad k \geq 2$.

3. EXAMPLES

In this section we determine the perfect geodomination number and the k-perfect geodomination number of some special classes of graphs. The following are easily verified:

1) $g_p(K_n) = n, g_p(C_{2n+1}) = 3, \ n \geq 1$;
2) $g_p(P_n) = g_p(C_{2n}) = g_p(P_m \times P_n) = 2$;

3) $g_{kp}(K_n) = |V(K_n)|,\ k \geq 2$;
4) $g_p(K_n \times K_n) = n,\ n \geq 2$;
5) $g_p(K_{m,n}) = 4,\ \min\{m, n\} \geq 2$;
6) $g_{2p}(K_{m,n}) = 4,\ \ \min\{m, n\} \geq 2$;
7) $g_{2p}(K_{1,n}) = n + 1$;
8) $g_{2p}(K_m \times K_n) = \max\{m, n\}$;
9) for $k \geq 3$ we have $g_{kp}(K_m \times K_n) = |V(K_m \times K_n)|$, $g_{kp}(K_{m,n}) = |V(K_{m,n})|$ and $g_{kp}(W_n) = |V(W_n)|$.

Now we determine the k-perfect geodomination numbers of P_n, C_n and $K_2 \times P_n$:

Example 1. $$g_{kp}(P_n) = \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor + 1, & k = 2; \\ \left\lceil \frac{n}{k} \right\rceil + 2, & 3 \leq k \leq n-2, n \overset{k}{\equiv} 0; \\ \left\lceil \frac{n}{k} \right\rceil, & 3 \leq k \leq n-2, n \overset{k}{\equiv} 1; \\ \left\lfloor \frac{n}{2} \right\rfloor + 2, & 3 \leq k \leq n-2, n \overset{k}{\equiv} 2; \\ \left\lfloor \frac{n}{2} \right\rfloor + 3, & 3 \leq k \leq n-2, n \overset{k}{\equiv} 3, 4, \ldots, k-1. \end{cases}.$$

Proof. Let P_n be the path of length $n \geq 2$ with the vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $\{v_i v_{i+1} : i = 1, 2, \ldots, n-1\}$. For each k-perfect geodominating set S, there is $|S| \geq \left\lceil \frac{n}{k} \right\rceil$. If $|S| = \left\lceil \frac{n}{2} \right\rceil$, $n = 2t$, $t \geq 1$, then there exists such i that $\{v_i, v_{i+1}\} \cap S = \emptyset$, so S is not a 2-perfect geodominating set. On the other hand, $S = \{v_1, v_3, v_5, \ldots, v_n\}$ for n odd and $S = \{v_1, v_2, v_4, v_6, \ldots, v_n\}$ for n even are 2-perfect geodominating sets, hence $g_{2p}(P_n) = \left\lfloor \frac{n}{2} \right\rfloor + 1$.

Let $n \overset{k}{\equiv} 0$ and S be a subset of vertices with $|S| = \left\lceil \frac{n}{k} \right\rceil$ or $\left\lceil \frac{n}{k} \right\rceil + 1$; since $\{v_1, v_n\} \subseteq S$, then S is not a k-perfect geodominating set and by considering the k-perfect geodominating set $\{v_1, v_{k+1}, v_{2k+1}, \ldots, v_{(\lfloor \frac{n}{k} \rfloor - 1)k}, v_{(\lfloor \frac{n}{k} \rfloor - 1)k+1}, v_n\}$ we obtain $g_{kp}(P_n) = \left\lceil \frac{n}{k} \right\rceil + 2$.

If $n \overset{k}{\equiv} 1$, then $\{v_1, v_{k+1}, v_{2k+1}, \ldots, v_n\}$ is a k-perfect geodominating set, so $g_{kp}(P_n) = \left\lceil \frac{n}{k} \right\rceil$. The other cases are similarly verified. $\square$

Example 2. $$g_{kp}(C_n) = \begin{cases} \left\lceil \frac{n}{k} \right\rceil, & n \overset{k}{\equiv} 0, 1 \\ \left\lceil \frac{n}{k} \right\rceil + 1, & \text{otherwise} \end{cases}, \quad k \geq 2.$$

Proof. Let C_n be the n-cycle, $n \geq 3$, with the vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $\{v_i v_{i+1} : i = 1, 2, \ldots, n-1\} \cup \{v_n v_1\}$. It is clear that for each integers $n \geq 3$ and $k \geq 2$, $g_{kp}(C_n) \geq \left\lceil \frac{n}{k} \right\rceil$. On the other hand considering the following k-perfect geodominating sets:

$S = \{v_1, v_{k+1}, v_{2k+1}, \ldots, v_{(\lfloor \frac{n}{k} \rfloor - 1)k+1}\}$ for $n \overset{k}{\equiv} 0$, and

$S = \{v_1, v_{k+1}, v_{2k+1}, \ldots, v_{\lfloor \frac{n}{k} \rfloor k+1}\}$ for $n \overset{k}{\equiv} 1$ imply that $g_{kp}(C_n) = \left\lceil \frac{n}{k} \right\rceil$ when $n \overset{k}{\equiv} 0, 1$.

From now on, let $n \overset{k}{\neq} 0, 1$. We show that no subset of G with size $\lceil \frac{n}{k} \rceil$ is a k-perfect geodominating set. Suppose that S is a k-perfect geodominating set for C_n with size $\lceil \frac{n}{k} \rceil$. It is easily seen that there is a vertex $v \in V(G) \backslash S$ which is not k-geodominated by two vertices of S, which is a contradiction; hence, $g_{kp}(C_n) \geq \lceil \frac{n}{k} \rceil + 1$. On the other hand, let $n = \lfloor \frac{n}{k} \rfloor k + l$, $2 \leq l < k$ and let $T = \{v_1, v_{k+1}, v_{2k+1}, \ldots, v_{\lfloor \frac{n}{k} \rfloor k+1}, v_{k-l+1}\}$. Then T is a k-perfect geodominating set. Hence, $g_{kp}(C_n) = \lceil \frac{n}{k} \rceil + 1$. □

A similar proof shows that $g_k(C_n) = \begin{cases} \lceil \frac{n}{k} \rceil, & n \overset{k}{\equiv} 0, 1 \\ \lceil \frac{n}{k} \rceil + 1, & \text{otherwise} \end{cases}, \quad k \geq 2.$

Example 3. $g_{2p}(K_2 \times P_n) = \begin{cases} n, & n \ \text{ is even} \\ n+1, & n \ \text{ is odd} \end{cases}.$

Proof. Let $\{v_{11}, v_{12}, \ldots, v_{1n}, v_{21}, v_{22}, \ldots, v_{2n}\}$ be the vertex set of $K_2 \times P_n$, where v_{ij} is adjacent to the vertices $v_{i(j+1)}$ and $v_{(i+1)j}$ whose first and second indices are modulo 2 and modulo n, respectively. Let S be a 2-perfect geodominating set for $K_2 \times P_n$. Then it is clear that $S \cap \{v_{11}, v_{21}\} \neq \emptyset$, and $S \cap \{v_{1n}, v_{2n}\} \neq \emptyset$. There is no integer j such that

$$\{v_{ij}, v_{i(j+1)}, v_{(i+1)j}, v_{(i+1)(j+1)}\} \cap S = \emptyset$$

for $i = 1, 2$. Moreover, if $S \cap \{v_{1j}, v_{2j}\} = \emptyset$ for some j, then

$$\{v_{1(j-1)}, v_{2(j-1)}, v_{1(j+1)}, v_{2(j+1)}\} \subseteq S.$$

So there is a map of S onto $\{1, 2, \ldots, n\}$, hence $|S| \geq n$. On the other hand, $\{v_{1(4k)}, v_{1(4k+1)}, v_{2(4k+2)}, v_{2(4k+3)} : k \geq 1\} \cup \{v_{11}, v_{22}, v_{23}\}$ is a 2-perfect geodominating set if n is even. Now let n be an odd number and S be a 2-perfect geodominating set of $K_2 \times P_n$. It is easy to see that there is no integer j such that $\{v_{ij}, v_{(i+1)(j+1)}, v_{(i+1)(j-1)}\} \subseteq S$ and $v_{(i+1)j} \notin S$ for $i = 1, 2$. Also, if $v_{11} \in S$, $v_{21} \notin S$, then $v_{22} \in S$ and if $v_{21} \in S$, $v_{11} \notin S$, then $v_{12} \in S$. A similar discussion holds for v_{1n}, v_{2n}. If $|S| = n$, then either there is an integer j such that $\{v_{ij}, v_{(i+1)(j+1)}, v_{(i+1)(j-1)}\} \subseteq S$, and $v_{(i+1)j} \notin S$ for $i = 1$ or 2, or there is an integer j such that $\{v_{ij}, v_{i(j+1)}, v_{i(j+2)}\} \subseteq S$, and $\{v_{(i+1)j}, v_{(i+1)(j+1)}, v_{(i+1)(j+2)}\} \cap S = \emptyset$, which in either case leads to a contradiction. So $|S| \geq n + 1$. On the other hand, considering the following 2-perfect geodominating sets:

$\{v_{1(4k)}, v_{1(4k+1)}, v_{2(4k+2)}, v_{2(4k+3)} : k \geq 1\} \cup \{v_{11}, v_{22}, v_{23}, v_{2(n-1)}\}$ for $n \overset{4}{\equiv} 1$,

$\{v_{1(4k)}, v_{1(4k+1)}, v_{2(4k+2)}, v_{2(4k+3)} : k \geq 1\} \cup \{v_{11}, v_{22}, v_{23}, v_{1(n-1)}\}$ for $n \overset{4}{\equiv} 3$ we verify the equality. □

4. RESULTS

In this section we prove some results about the perfect and k-perfect geodomination number of a graph. Let G be a connected graph with $g(G) \geq 3$. If G has some pendant vertices, then the neighbor of any pendant vertex of G belongs to any $g_p(G)$-set and the condition $g(G) \geq 3$ is necessary. To this end, see the path P_n. For trees we have the following proposition:

Proposition 4. *If a tree T has a proper perfect geodominating set, then T is a path.*

Proof. Let x, y and z be three pendant vertices of T and x', y' and z' be the adjacent vertices of x, y and z, respectively. Let S be a proper perfect geodominating set, then $\{x, y, z, x', y', z'\} \subseteq S$. If $w \in V(T)\backslash S$ is geodominated by a pair of vertices u, v in S, then clearly $\{u, v\} \cap \{x, y, z, x', y', z'\} = \emptyset$. But there is exactly one x-u geodesic containing w, so w is geodominated by x, u, which is a contradiction. □

If a tree T has more than two pendant vertices, then $g_p(T) = |V(T)|$, and so the inequality $g_p(G) \geq g(G)$ is strict.

Proposition 5. I) *For two positive integers k, n with $4 \leq k < n$, there exists a connected graph G of order n with $k-3$ pendant vertices such that $g_p(G) = k$.*

II) *For two positive integers a, b with $3 \leq a \leq b$ there exists a connected graph G with no pendant vertices such that $|V(G)| = b, g_p(G) = a$.*

III) *For two positive integers a, b with $b \geq (a-1)k+1$, there exists a connected graph G with $|V(G)| = b$ and $g_{kp}(G) = a$.*

Proof. I) Let x, y be two vertices of $K_{1,k-1}$ with $\deg(x) = \deg(y) = 1$. We add an ear x, w_i, y to $K_{1,k-1}$ for $i = 1, 2, \ldots, n-k$ to obtain a graph G. Then G has $k-3$ pendant vertices and $g_p(G) = k$.

II) Let x, y be two vertices of K_a. We delete the edge $\{x, y\}$ and add an ear x, w_i, y for $i = 1, 2, \ldots, b-a$ to K_a to obtain a graph G. Then $|V(G)| = b$ and $g_p(G) = a$.

III) Let $P_{(a-1)k+1}$ be the path with vertices $v_1, v_2, \ldots, v_{(a-1)k+1}$. We add an ear $v_1 w_i v_3$ for $i = 1, 2, \ldots, b-((a-1)k+1)$ to obtain a graph G. Then $|V(G)| = b$ and $g_{kp}(G) = a$. □

Proposition 6. I) *If a graph G with no pendant vertex has a proper perfect geodominating set, then $|V(G)| \geq 4$.*

II) *If a graph G with exactly one pendant vertex has a proper perfect geodominating set, then $|V(G)| \geq 5$.*

Proof. I) If S is a proper g_p-set in the graph G, then there is a vertex $v \in V(G)\backslash S$ which is geodominated by two vertices x and y of S. But $\deg(x) \geq 2$ and $\deg(y) \geq 2$, so $|V(G)| \geq 4$.

II) Let x be the pendant vertex of G and x' be the neighbor of x. Let S be a proper perfect geodominating set of G and $y \in V(G)\backslash S$. The following cases are possible:

1) If $x' \in S$, then it is clear that no vertex of S together with x or x' can perfectly geodominate y, so y is geodominated by two vertices of S other than x, x', thus $|V(G)| \geq 5$.
2) If $x' \notin S$, then y is geodominated by a pair of vertices of S with the degree of one of them at least 2. So $|V(G)| \geq 5$. □

The above bounds are best possible. Indeed, the graph C_4 has no pendant vertices, and by adding a pendant edge to C_4 we obtain a graph with five vertices. Similarly, if $g_p(G) = k \geq 2$ and G has an independent $g_p(G)$-set, then $|V(G)| \geq \left\lceil \frac{k^2+k-1}{2} \right\rceil$.

Vertices u, v in a graph G are *antipodal* if $d(u, v) = diam(G)$.

Proposition 7. *Let G be a connected graph of order $n \geq 3$, with $diam(G) \geq 3$ and $g(G) = 2$ and let $k \geq 1$ be an integer. Then $g_{kp}(G) = g(G)$ if and only if $k = diam(G)$.*

Proof. Let $diam(G) = d$ and $S = \{x, y\}$ be a $g(G)$-set. Then x and y are antipodal vertices, so S is a perfect d-geodominating set. Consequently, $g_{dp}(G) = 2$. For the converse, note that if $k = 1$ or $k > d$, then $g_{kp}(G) = |V(G)| \neq g(G)$. Suppose that $2 \leq k \leq d - 1$ and $g_{kp}(G) = g(G) = 2$. Then any minimum k-perfect geodominating set contains two antipodal vertices. But $k \leq d - 1$, which is a contradiction. □

There are graphs with a perfect geodominating set which is not essential. To see this, consider $S = \{v_1, v_{n+1}, v_{n+2}\}$ in the graph $C_{2n+1}, (n \geq 5)$, then for two vertices v_{n+1}, v_{n+2}, there is no vertex $w \neq v_{n+1}, v_{n+2}$ of G which would lie in a v_{n+1}-v_{n+2} geodesic. But for independent perfect geodominating set the following proposition in true:

Proposition 8. *Any independent perfect geodominating set of a graph G is essential.*

Proof. Let S be an independent perfect geodominating set and $x, y \in S$. Then there exists a vertex $v \notin \{x, y\}$ which is geodominated by x, y. Since S is a perfect geodominating set, then v lies in no $x' - y'$ geodesic for $x', y' \in S$ and $\{x', y'\} \neq \{x, y\}$. So S is essential. □

If a graph G has a uniform perfect geodominating set with a fixed number k, then $|V(G)| = \begin{pmatrix} g_p(G) \\ 2 \end{pmatrix} (k-1) + g_p(G)$. It is not true, either, that any k-geodominating set or any uniform geodominating set with a fixed number k in a graph G is a k-perfect geodominating set.

Let $K_k^{(k-1)}$ denote the multigraph of order k in which every two vertices are joined by $k-1$ edges and let $G_k = S(K_k^{(k-1)})$ be the subdivision graph of $K_k^{(k-1)}$. It was shown that $V(K_k^{(k-1)})$ is a uniform, essential minimum geodetic set for G_k and $g(G_k) = k$ (see [1]). It is easily seen that $g_k(G_k) = g(G_k) = k$ and $V(K_k^{(k-1)})$ is a uniform, essential minimum perfect geodominating set for G_k. So for each integer $k \geq 2$, there exists a connected graph G with $g_p(G) = k$ which contains a uniform, essential minimum perfect geodominating set.

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