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**APPROXIMATION OF EIGENVALUES  
OF SOME UNBOUNDED SELF-ADJOINT  
DISCRETE JACOBI MATRICES  
BY EIGENVALUES OF FINITE SUBMATRICES**

**Abstract.** We investigate the problem of approximation of eigenvalues of some self-adjoint operator in the Hilbert space  $l^2(\mathbb{N})$  by eigenvalues of suitably chosen principal finite submatrices of an infinite Jacobi matrix that defines the operator considered. We assume the Jacobi operator is bounded from below with compact resolvent. In our research we estimate the asymptotics (with  $n \rightarrow \infty$ ) of the joint error of approximation for the first  $n$  eigenvalues and eigenvectors of the operator by the eigenvalues and eigenvectors of the finite submatrix of order  $n \times n$ . The method applied in our research is based on the Rayleigh-Ritz method and Volkmer's results included in [7]. We extend the method to cover a class of infinite symmetric Jacobi matrices with three diagonals satisfying some polynomial growth estimates.

**Keywords:** self-adjoint unbounded Jacobi matrix, asymptotics, point spectrum, tridiagonal matrix, eigenvalue.

**Mathematics Subject Classification:** 47B25, 47B36.

## 1. INTRODUCTION

Approximation of eigenvalues of an operator in an infinite dimensional Hilbert space by eigenvalues of suitable finite matrices seems to be reasonable and useful. We consider the problem of approximation for eigenvalues and eigenvectors of some self-adjoint, bounded from below discrete operator in the Hilbert space  $l^2 = l^2(\mathbb{N})$  by eigenvalues and eigenvectors of properly chosen principal finite submatrices of an infinite Jacobi matrix that defines the operator.

The eigenvalues of a self-adjoint, bounded from below operator with compact resolvent may be arranged non-decreasingly. In [7], Volkmer estimated the error of approximation for the eigenvalue and eigenvector, whose number is fixed. In our research we estimate the asymptotics (with  $n \rightarrow \infty$ ) of the joint error of approximation

for the first  $n$  eigenvalues and eigenvectors of the infinite Jacobi matrix by the eigenvalues and eigenvectors of the principal finite submatrix of order  $n \times n$ . The method applied is based on Volkmer's results included in [7]. We consider a class of infinite matrices slightly different from that in [7] because we admit the situation in which all three diagonals are unbounded but satisfy some polynomial growth estimates.

Let us consider an operator  $J$  in the Hilbert space  $l^2$  with the canonical basis given by the tridiagonal symmetric Jacobi matrix

$$\begin{pmatrix} d_1 & c_1 & 0 & \cdots & \cdots \\ c_1 & d_2 & c_2 & 0 & \ddots \\ 0 & c_2 & d_3 & c_3 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \quad (1)$$

and assume that  $J$  acts on the maximum domain

$$D(J) = \{ \{f_n\}_{n=1}^\infty \in l^2 : \{c_{n-1}f_{n-1} + d_n f_n + c_n f_{n+1}\}_{n=1}^\infty \in l^2 \}$$

(we here take  $c_0 = f_0 = 0$ ).

We investigate a class of operators  $J$  such that

$$\begin{aligned} (C1) \quad & d_n = \delta n^\alpha (1 + \delta_n), \quad \delta_n \rightarrow 0, \quad \delta > 0; \\ & |c_n| \leq S n^\beta; \\ & d_n, c_n \in \mathbb{R} \quad \text{for all } n \geq 1; \\ (C2) \quad & \beta \geq 0, \quad \alpha > 2\beta + 1; \\ (C3) \quad & \delta_{n+1} - \delta_n = o(1/n). \end{aligned}$$

If (C1) and (C2) are satisfied then  $\lim_{n \rightarrow \infty} |d_n| = +\infty$  and  $\liminf_{n \rightarrow \infty} d_n^2 (c_n^2 + c_{n-1}^2)^{-1} > 2$ ; therefore, the operator  $J$  is self-adjoint, has a compact resolvent and its spectrum is discrete by Janas and Naboko's criterion ([2]). Moreover, if conditions (C1)–(C3) are satisfied then the spectrum of  $J$  consists of the eigenvalues  $\lambda_k \in \mathbb{R}$ ,  $k \geq 1$  and it may be proved (see [1, 3]) that

$$\lambda_k = d_k + O\left(\frac{1}{k^{\alpha-2\beta-1}}\right), \quad k \rightarrow \infty \quad (2)$$

and  $J$  is bounded from below.

Denote by  $x_k$  the eigenvector of  $J$  associated with the eigenvalue  $\lambda_k$ . We can assume that the system of chosen eigenvectors  $\{x_1, x_2, x_3, \dots\}$  is an orthonormal basis in  $l^2$ . So, if  $k \geq 1$  then we have that

$$x_k = \{x_{k,n}\}_{n=1}^\infty$$

and

$$\|x_k\|^2 = \sum_{n=1}^\infty |x_{k,n}|^2 = 1.$$

For  $n \geq 1$ , consider the finite matrix of order  $n \times n$

$$J_n = \begin{pmatrix} d_1 & c_1 & 0 & \cdots & \cdots & 0 \\ c_1 & d_2 & c_2 & 0 & \cdots & \\ 0 & c_2 & d_3 & c_3 & \ddots & \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & c_{n-2} & d_{n-1} & c_{n-1} \\ 0 & \cdots & \cdots & 0 & c_{n-1} & d_n \end{pmatrix}. \quad (3)$$

Let us denote by  $\mu_{i,n}$ ,  $1 \leq i \leq n$ , the eigenvalues of the matrix  $J_n$  and assume that  $\mu_{1,n} \leq \mu_{2,n} \leq \cdots \leq \mu_{n,n}$ . For a fixed  $i \in \{1, 2, \dots, n\}$  let  $y_{i,n} \in \mathbb{R}^n$  be an eigenvector of  $J_n$  associated with  $\mu_{i,n}$  and assume that  $\{y_{1,n}, y_{2,n}, \dots, y_{n,n}\}$  is an orthonormal basis in  $\mathbb{R}^n$  such that

$$(x_k, y_{k,n}) \geq 0 \quad \text{for } k = 1, \dots, n, \quad (4)$$

where  $(\cdot, \cdot)$  stands for the inner product in  $l^2$  and we here treat  $y_{k,n}$  as an infinite sequence whose elements are equal to 0 for the indices greater than  $n$ .

The main result of this work is as follows.

**Theorem 1.** *Let  $J$  be an operator in  $l^2$  given by Jacobi matrix (1) satisfying conditions (C1) – (C3) and  $J_n$  be the matrix given by (3). Assume that  $\{\lambda_1, \lambda_2, \dots\}$  is a non-decreasing sequence of the eigenvalues of  $J$  and  $\{\mu_{1,n}, \dots, \mu_{n,n}\}$  is a non-decreasing sequence of the eigenvalues of  $J_n$ . Then there exist  $C > 0$  and  $N \in \mathbb{N}$  such that*

$$\sup_{k \in \{1, 2, \dots, n\}} |\mu_{k,n} - \lambda_k| \leq C \frac{1}{n^{\alpha-2\beta-1}} \quad \text{for } n \geq N.$$

A proof of Theorem 1 is given in Section 3. In Section 4 we give an example in which the above estimate is sharp.

**Theorem 2.** *Let the assumptions of Theorem 1 be satisfied for the operator  $J$  and matrix  $J_n$ . Then there exist  $C > 0$  and  $N \in \mathbb{N}$  such that for  $n \geq N$*

$$\sup_{k \in \{1, 2, \dots, n\}} \|x_k - y_{k,n}\|_{l^2} \leq C \frac{1}{n^{\alpha-2\beta-1}},$$

where  $\{x_1, x_2, \dots\}$  is the set the eigenvectors of  $J$  and  $\{y_{1,n}, \dots, y_{n,n}\}$  is the set of the eigenvectors of  $J_n$  with the properties described above the statement of this theorem.

Also a proof of Theorem 2 is in Section 3.

Jacobi matrices satisfying (C1) – (C3) appear in many cases. For example the energy spectrum of a molecule in the homogeneous point electric field can be considered as the point spectrum of a special Jacobi matrix (see [3]) and some spectral problems of the Mathieu equation and the spheroidal wave equation have Jacobi matrix representations (see [7]).

## 2. PRELIMINARIES

This section sets the background for the Rayleigh-Ritz method for approximation of eigenvalues and Volkmer's results (see [7]) we are going to apply in our proofs of Theorem 1 and Theorem 2.

Let  $H$  be a Hilbert space with an inner product  $(\cdot, \cdot)$  and  $J$  be a linear self-adjoint operator in  $H$  with a domain  $D(J)$ . Assume that  $J$  has a compact resolvent and is bounded from below. If the spectrum of  $J$  consists of the eigenvalues  $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$  then by the minimum-maximum principle, for all  $k \in \mathbb{N}$ , there is

$$\lambda_k = \min_{E_k} \max\{(Jx, x) : x \in E_k, \|x\| = 1\},$$

where the minimum is taken over all linear subspaces  $E_k \subseteq D(J)$  of dimension  $k$ . Let  $\{x_1, x_2, x_3, \dots\}$  be an orthonormal basis of  $H$  consisting of eigenvectors ordered according to the order of the non-decreasing sequence of the eigenvalues.

Let  $n \in \mathbb{N}$  and let us choose the linear subspace  $E \subseteq D(J)$  of dimension  $n$ . Let  $P$  denote the orthogonal projection in  $H$  onto  $E$  and  $Q = I - P$  be the orthogonal projection onto  $H \ominus E$ . Consider the operator  $PJ : E \rightarrow E$ . This finite dimensional operator has  $n$  eigenvalues:  $\mu_1 \leq \mu_2 \leq \dots \leq \mu_n$  and  $n$  eigenvectors  $y_1, y_2, \dots, y_n$  that form an orthonormal basis in  $E$  and can be chosen so that  $(x_k, y_k) \geq 0$  for  $k = 1, \dots, n$ . By virtue of the minimum-maximum principle,

$$\mu_k = \min_{E_k} \max\{(PJx, x) = (Jx, x) : x \in E_k, \|x\| = 1\},$$

where minimum is taken over all  $k$ -dimensional subspaces  $E_k \subseteq E$ , it is clear that

$$\mu_k - \lambda_k \geq 0$$

for  $k \in \{1, 2, \dots, n\}$ .

Now, fix  $k \in \{1, 2, \dots, n\}$  and define  $(k \times k)$ -dimensional matrices  $L = (L_{ij})$  and  $M = (M_{ij})$ , where  $L_{ij} = (Qx_i, x_j)$ ,  $M_{ij} = ((PJP - J)x_i, x_j)$  for  $i, j = 1, \dots, k$ . Using the minimum-maximum principle, Volkmer proved the following theorem.

**Theorem 2.1.** (Volkmer, see [7]). *If  $\|L\| < 1$  then*

$$\mu_k - \lambda_k \leq \frac{\|M + \lambda_k L\|}{1 - \|L\|},$$

where  $\|T\|$  stands for the operator norm of an operator  $T : \mathbb{R}^k \rightarrow \mathbb{R}^k$  given by the matrix  $T$ .

Moreover, if  $K = \|PJPx_k - PJx_k\|$  and  $\Delta = \max\{|\mu_i - \lambda_k|^{-1} : i \in \{1, \dots, n\} \setminus \{k\}\}$ , then

$$\begin{aligned} \|Px_k - y_k\|^2 &\leq \Delta^2 K^2 + (\|Qx_k\|^2 + \Delta^2 K^2)^2, \\ \|x_k - y_k\|^2 &\leq \|Qx_k\|^2 + \Delta^2 K^2 + (\|Qx_k\|^2 + \Delta^2 K^2)^2. \end{aligned}$$

Suppose that  $\{e_1, e_2, e_3, \dots\}$  is the canonical basis of  $l^2$ . Let now  $J$  be a Jacobi operator given in the canonical basis by matrix (1). Let  $E = \text{Lin}\{e_1, e_2, \dots, e_n\}$ ,  $P_n$  denote the orthogonal projection onto  $E$  and  $Q_n = I - P_n$ . Then the linear operator  $P_n J : E \rightarrow E$  has the matrix representation in the canonical basis given by the matrix  $J_n$  according to (3). Take  $k \in \{1, \dots, n\}$  and assume that  $L^{(k,n)} = (L_{i,j}^{(n)})_{i,j=1\dots k}$  and  $M^{(k,n)} = (M_{i,j}^{(n)})_{i,j=1\dots k}$  are  $k \times k$  matrices defined for  $P_n$  and  $Q_n$  according to the definitions of  $L$  and  $M$  respectively. Let  $K_{k,n} = \|P_n J P_n x_k - P_n J x_k\|$  and  $\Delta_{k,n} = \{|\mu_{i,n} - \lambda_k|^{-1} : i \in \{1, \dots, n\} \setminus \{k\}\}$  correspond to  $K$  and  $\Delta$  in this situation. Then the following lemma holds.

**Lemma 2.2.** (Volkmer, see [7]). *For fixed  $n \in \mathbb{N}$  and  $k \in \{1, 2, \dots, n\}$  the following estimates hold:*

$$\begin{aligned} |L_{i,j}^{(n)}| &\leq \|Q_n x_i\| \|Q_n x_j\|, \\ |M_{i,j}^{(n)} + \lambda_k L_{i,j}^{(n)}| &\leq |c_n| |x_{i,n+1}| |x_{j,n}| + |\lambda_k - \lambda_i| \|Q_n x_i\| \|Q_n x_j\| \end{aligned}$$

for  $i, j = 1, 2, \dots, k$  and

$$K_{k,n} \leq |c_n| |x_{k,n+1}|.$$

Using this lemma, we can easily obtain the following fact.

**Lemma 2.3.** *If  $n \in \mathbb{N}$  and  $k \in \{1, 2, \dots, n\}$  then*

$$\begin{aligned} \|L^{(k,n)}\| &\leq \sum_{i=1}^k \|Q_n x_i\|^2; \\ \|M^{(k,n)} + \lambda_k L^{(k,n)}\| &\leq |c_n| \left( \sum_{i=1}^k |x_{i,n+1}|^2 \right)^{1/2} \left( \sum_{j=1}^k |x_{j,n}|^2 \right)^{1/2} + \\ &\quad + \left( \sum_{i=1}^k |\lambda_k - \lambda_i|^2 \|Q_n x_i\|^2 \right)^{1/2} \left( \sum_{j=1}^k \|Q_n x_j\|^2 \right)^{1/2}. \end{aligned}$$

### 3. PROOF OF THE MAIN RESULT

In this section we assume (C1) – (C3) hold. Therefore, there exists  $n_0 \geq 1$  such that for  $n \geq n_0$  the following facts are true:

$$(F1) \quad \text{if } k \in \{1, 2, \dots, n\} \text{ then } d_k \leq d_n,$$

$$(F2) \quad \delta_n - \delta_{n-1} = \frac{1}{n} \epsilon_n \quad \text{and} \quad |\epsilon_n| \leq \frac{1}{4},$$

$$(F3) \quad |\delta_n| \leq \frac{1}{4}.$$

Moreover; there exists  $M > 0$  such that

$$(F4) \quad |\lambda_k - d_k| \leq \frac{M}{k^{\alpha-2\beta-1}} \leq M \quad \text{for all } k \geq 1,$$

because of (2).

Put

$$\begin{aligned} S_1 &= \delta \sup\{|1 + \delta_n| : n \geq 1\}; \\ R &= \sup\{|d_k - d_i| : i, k \leq n_0\}. \end{aligned} \quad (5)$$

For  $n > i \geq 1$ , define

$$f_{i,n} = \frac{|c_{n-1}|}{d_n - d_i - M - |c_n|}. \quad (6)$$

Let  $n \geq n_0 + 1$  and  $i < n$ ; then we can write the following estimates:

$$\begin{aligned} d_n - d_i &\geq d_n - d_{n-1} = \\ &= \delta n^\alpha (1 + \delta_n - (\frac{n-1}{n})^\alpha (1 + \delta_{n-1})) \geq \\ &\geq \delta n^\alpha (1 + \delta_n - \frac{n-1}{n} (1 + \delta_{n-1})) = \\ &= \delta n^\alpha (\frac{1}{n} + \delta_n - \delta_{n-1} + \frac{\delta_{n-1}}{n}) = \\ &= \delta n^{\alpha-1} (1 + \epsilon_n + \delta_{n-1}) \geq \frac{1}{2} \delta n^{\alpha-1}. \end{aligned}$$

Therefore, we obtain

$$0 \leq f_{i,n} \leq \frac{S(n-1)^\beta}{\frac{1}{2} \delta n^{\alpha-1} - M - S n^\beta} \leq \frac{a}{n^{\alpha-\beta-1}} \leq \frac{1}{2} < 1 \quad (7)$$

for  $n \geq N_0$  and  $1 \leq i \leq n-1$ , where  $N_0 \geq n_0 + 1$  is large enough and  $a = \frac{3S}{\delta}$ .

As in the previous sections,  $x_k = \{x_{k,n}\}_{n=1}^\infty \in l^2$  is the eigenvector associated with the eigenvalue  $\lambda_k$ . We still assume  $\{x_1, x_2, \dots\}$  is an orthonormal basis in  $l^2$  and  $\|x_k\|^2 = \sum_{n=1}^\infty |x_{k,n}|^2 = 1$ . Then [cf. Volkmer (see [7])] the following two lemmas hold true.

**Lemma 3.1.** *If  $n \geq N_0$  and  $1 \leq i \leq n-1$ , then  $|x_{i,n}| \leq f_{i,n} |x_{i,n-1}|$ .*

*Proof.* Let  $n \geq N_0$  and  $i \in \{1, \dots, n-1\}$  be fixed. Then there exists  $k \geq n \geq i+1$  such that  $|x_{i,k+1}| < |x_{i,k}|$ , because  $x_i \in l^2$ . Then

$$c_{k-1} x_{i,k-1} + (d_k - \lambda_i) x_{i,k} + c_k x_{i,k+1} = 0,$$

because  $Jx_i = \lambda_i x_i$ . This implies

$$|c_{k-1} x_{i,k-1}| \geq |d_k - \lambda_i| |x_{i,k}| - |c_k| |x_{i,k+1}|.$$

Therefore,

$$|c_{k-1}| |x_{i,k-1}| > (|d_k - \lambda_i| - |c_k|) |x_{i,k}| \geq (d_k - d_i - M - |c_k|) |x_{i,k}| > 0.$$

Notice that for  $k \geq N_0$  and  $i \leq k-1$ , there is  $|d_k - \lambda_i| \geq d_k - d_i - M$  and  $d_k - d_i - M - |c_k| > 0$ . Thus

$$|x_{i,k}| < \frac{|c_{k-1}|}{d_k - d_i - M - |c_k|} |x_{i,k-1}| \leq f_{i,k} |x_{i,k-1}| < |x_{i,k-1}|.$$

By induction we may prove that  $|x_{i,n}| \leq f_{i,n} |x_{i,n-1}|$ .  $\square$

**Lemma 3.2.** *Let  $n \geq N_0$ ,  $1 \leq i \leq n-1$  and denote  $i_o = \max\{N_0, i+1\}$ . Then*

$$|x_{i,n}| \leq f_{i,n} \cdot f_{i,n-1} \cdots \cdots f_{i,i_o}, \quad (8)$$

$$\|Q_n x_i\| \leq \frac{2}{\sqrt{3}} f_{i,n+1} \cdot f_{i,n} \cdots \cdots f_{i,i_o}. \quad (9)$$

*Proof.* Inequality (8) follows from lemma 3.1.

To prove (9), we apply (8) and (7) and notice that

$$\begin{aligned} \|Q_n x_i\|^2 &= \sum_{k=n+1}^{\infty} |x_{i,k}|^2 \leq \sum_{k=1}^{\infty} (f_{i,n+k} \cdot f_{i,n+k-1} \cdots \cdots f_{i,i_o})^2 = \\ &= (f_{i,n+1} \cdot f_{i,n} \cdots \cdots f_{i,i_o})^2 (1 + f_{i,n+2}^2 + f_{i,n+2}^2 f_{i,n+3}^2 + \cdots) \leq \\ &\leq (f_{i,n+1} \cdot f_{i,n} \cdots \cdots f_{i,i_o})^2 (1 + \frac{1}{4} + (\frac{1}{4})^2 + \cdots) = \\ &= \frac{4}{3} (f_{i,n+1} \cdot f_{i,n} \cdots \cdots f_{i,i_o})^2. \quad \square \end{aligned}$$

*Proof of Theorem 1.* Notice that (C2) implies the existence of a  $\gamma \geq 1$  such that

$$\alpha \geq \beta + \frac{\gamma+1}{\gamma}. \quad (10)$$

Then for such  $\gamma$  we may choose a constant  $r \in \mathbb{N}$  such that

$$r \geq \frac{3\gamma+6}{2}. \quad (11)$$

Conditions (10) and (11) imply

$$\alpha \geq \beta + \frac{2r-3}{2r-6},$$

but this condition, together with (C2), yields

$$2r(\alpha - \beta - 1) - 2\alpha - 1 \geq 2(\alpha - \beta - 1) \quad (12)$$

and

$$2r(\alpha - \beta - 1) - 1 \geq 2(\alpha - \beta - 1). \quad (13)$$

Now, let  $n \geq N_0 + r$  and  $k \in \{1, \dots, n\}$ . By (9) in Lemma 3.2 and (7), there is

$$\begin{aligned} \|L^{(k,n)}\| &\leq \sum_{i=1}^k \|Q_n x_i\|^2 \leq \sum_{i=1}^n \|Q_n x_i\|^2 \leq \sum_{i=1}^{n-r+1} \|Q_n x_i\|^2 + \sum_{i=n-r+2}^n \|Q_n x_i\|^2 \leq \\ &\leq \sum_{i=1}^{n-r+1} 4/3 f_{i,n+1}^2 f_{i,n}^2 \cdots \cdots f_{i,n-r+2}^2 + 4/3 \sum_{i=n-r+2}^n f_{i,n+1}^2 \leq \\ &\leq 4/3(n-r+1) \left(\frac{a}{(n+1)^{\alpha-\beta-1}}\right)^2 \left(\frac{a}{n^{\alpha-\beta-1}}\right)^2 \cdots \cdots \left(\frac{a}{(n-r+2)^{\alpha-\beta-1}}\right)^2 + \\ &\quad + 4/3(r-1) \left(\frac{a}{(n+1)^{\alpha-\beta-1}}\right)^2 \leq \\ &\leq A(n-r+1) \left(\frac{1}{n^{\alpha-\beta-1}}\right)^{2r} + (r-1)B \left(\frac{1}{n^{\alpha-\beta-1}}\right)^2 \leq \frac{C_0}{n^{2\alpha-2\beta-2}} \end{aligned}$$

for some  $C_0 > 0$ , where the last inequality holds because of the choice of  $r$  (see (11) and (13)).

Let  $n \geq N_0 + r$  and  $k \in \{1, \dots, n\}$  then Lemma 3.2, (C1), (C2) and finally (13), yield the following estimate:

$$\begin{aligned}
|c_n| \left( \sum_{i=1}^k |x_{i,n+1}|^2 \right)^{1/2} &\leq S n^\beta \left( \sum_{i=1}^{n-r+1} |x_{i,n+1}|^2 + \sum_{i=n-r+2}^n |x_{i,n+1}|^2 \right)^{1/2} \leq \\
&\leq S n^\beta \left( \sum_{i=1}^{n-r+1} f_{i,n+1}^2 \cdots f_{i,n-r+2}^2 + \sum_{i=n-r+2}^n f_{i,n+1}^2 \right)^{1/2} \leq \\
&\leq S n^\beta \left( (n-r+1) \left( \frac{a}{(n+1)^{\alpha-\beta-1}} \right)^2 \cdots \left( \frac{a}{(n-r+2)^{\alpha-\beta-1}} \right)^2 + \right. \\
&\quad \left. + (r-1) \left( \frac{a}{n^{\alpha-\beta-1}} \right)^2 \right)^{1/2} \leq \\
&\leq S n^\beta \left( \tilde{A} (n-r+1) \left( \frac{1}{n^{\alpha-\beta-1}} \right)^{2r} + (r-1) \tilde{B} \left( \frac{1}{n^{\alpha-\beta-1}} \right)^2 \right)^{1/2} \leq \\
&\leq \frac{\tilde{C}}{n^{\alpha-2\beta-1}};
\end{aligned}$$

and with similar methods we may also prove:

$$\left( \sum_{j=1}^k |x_{j,n}|^2 \right)^{1/2} \leq \tilde{C} + 1.$$

To estimate  $\sum_{i=1}^k |\lambda_k - \lambda_i|^2 \|Q_n x_i\|^2$  we notice that

$$|\lambda_k - \lambda_i| \leq C_1 + d_n - d_i$$

for  $1 \leq i \leq k \leq n$  and  $C_1 = 2M + R$ . Indeed, (F4) implies

$$|\lambda_k - \lambda_i| \leq 2M + |d_k - d_i|.$$

Then

$$|\lambda_k - \lambda_i| \leq 2M + R \quad \text{for } 1 \leq i \leq k \leq n_0$$

and

$$|\lambda_k - \lambda_i| \leq d_k - d_i + 2M \leq d_n - d_i + 2M$$

for  $n \geq k \geq n_0$  and  $i \in \{1, \dots, k\}$ , owing to (F1).

Now, let us take  $n \geq N_0 + r$  and  $1 \leq k \leq n$ ; we may now write the inequalities:

$$\sum_{i=1}^k |\lambda_k - \lambda_i|^2 \|Q_n x_i\|^2 \leq \sum_{i=1}^n (d_n - d_i + C_1)^2 \|Q_n x_i\|^2 =$$



$$= \underbrace{\sum_{i=1}^{n-r+1} (d_n - d_i + C_1)^2 \|Q_n x_i\|^2}_I + \underbrace{\sum_{i=n-r+2}^{n-1} (d_n - d_i + C_1)^2 \|Q_n x_i\|^2}_{II} + \underbrace{C_1^2 \|Q_n x_n\|^2}_{III}.$$

For addend  $I$  above, the following estimates hold:

$$\begin{aligned} I &\leq \sum_{i=1}^{n-r+1} 4/3(2S_1 n^\alpha + C_1)^2 f_{i,n+1}^2 f_{i,n}^2 \cdots f_{i,n-r+2}^2 \leq \\ &\leq 4/3(2S_1 n^\alpha + C_1)^2 (n-r+1) \left(\frac{a}{(n+1)^{\alpha-\beta-1}}\right)^2 \left(\frac{a}{n^{\alpha-\beta-1}}\right)^2 \cdots \left(\frac{a}{(n-r+2)^{\alpha-\beta-1}}\right)^2 \leq \\ &\leq K_1 \frac{1}{n^{2r(\alpha-\beta-1)-2\alpha-1}} \leq K_1 \frac{1}{n^{2(\alpha-\beta-1)}} \end{aligned}$$

for some  $K_1 > 0$  independent on  $n$  and the last inequality is justified by (12).

If  $n-r+2 \leq i \leq n-1$  then using (F1), (C1) and (C3) we may write

$$\begin{aligned} |d_n - d_i| &\leq \delta n^\alpha (1 + \delta_n) - \delta (n-r+2)^\alpha (1 + \delta_{n-r+2}) \leq \\ &\leq \delta n^\alpha \left(1 + \delta_n - \left(1 - \frac{r-2}{n}\right)^\alpha (1 + \delta_{n-r+2})\right) \leq \\ &\leq \delta n^\alpha \left(1 + \delta_n - \left(1 - \frac{(r-2)\alpha}{n}\right) (1 + \delta_{n-r+2})\right) \leq \\ &\leq \delta n^\alpha (\delta_n - \delta_{n-r+2}) + \frac{(r-2)\alpha}{n} (1 + \delta_{n-r+2}) \leq C_2 n^{\alpha-1} \end{aligned}$$

for some  $C_2 > 0$ . Thus we derive

$$II \leq 4/3(r-1)(C_2 n^{\alpha-1} + C_1)^2 \left(\frac{a}{(n+1)^{\alpha-\beta-1}}\right)^2 \left(\frac{a}{n^{\alpha-\beta-1}}\right)^2 \leq K_2 \frac{1}{n^{2\alpha-4\beta-2}},$$

where  $K_2 > 0$  is an appropriately chosen constant.

Therefore, there exists a  $K_3 > 0$  such that

$$\begin{aligned} \sum_{i=1}^k |\lambda_k - \lambda_i|^2 \|Q_n x_i\|^2 &\leq K_1 \frac{1}{n^{2\alpha-2\beta-2}} + K_2 \frac{1}{n^{2\alpha-4\beta-2}} + 4/3a^2 C_1^2 \frac{1}{n^{2\alpha-2\beta-2}} \leq \\ &\leq K_3 \frac{1}{n^{2\alpha-4\beta-2}} \end{aligned}$$

for  $n \geq N_0 + r$  and  $k \in \{1, \dots, n\}$ .

The above estimates and Lemma 2.3 enable us to estimate the following matrix norm for  $n \geq N_0 + r$  and  $k \in \{1, \dots, n\}$

$$\begin{aligned} \|M^{(k,n)} + \lambda_k L^{(k,n)}\| &\leq \tilde{C}(\tilde{C} + 1) \frac{1}{n^{\alpha-2\beta-1}} + \left(K_3 \frac{1}{n^{2\alpha-4\beta-2}}\right)^{1/2} \left(\frac{C_0}{n^{2(\alpha-\beta-1)}}\right)^{1/2} \leq \\ &\leq K_4 \frac{1}{n^{\alpha-2\beta-1}}, \end{aligned}$$

where  $K_4 > 0$  is a suitable constant independent of  $n$ .

Choose  $N_1 \geq N_0 + r$  large enough for the following inequality

$$\|L^{(k,n)}\| \leq \frac{C_0}{n^{2(\alpha-\beta-1)}} \leq 1/2$$

to hold for  $n \geq N_1$ . By Theorem 2.1, we obtain

$$\mu_{k,n} - \lambda_k \leq K_4 \frac{1}{n^{\alpha-2\beta-1}} \cdot (1 - 1/2)^{-1} = \frac{2K_4}{n^{\alpha-2\beta-1}}$$

for  $n \geq N_1$  and  $k \in \{1, \dots, n\}$ . The proof of Theorem 1 is complete.  $\square$

*Proof of Theorem 2.* Without losing generality, we may assume  $c_n \neq 0$  for  $n \geq 1$ . Then the multiplicity of every eigenvalue of  $J$  is one. Because the asymptotics given in (2) is satisfied, then

$$\inf\{|\lambda_i - \lambda_k| : i, k \geq 1, i \neq k\} = \rho > 0$$

and farther

$$|\mu_{i,n} - \lambda_k| \geq |\lambda_i - \lambda_k| - |\mu_{i,n} - \lambda_i| \geq \rho - \frac{C}{n^{\alpha-2\beta-1}} \geq \frac{1}{2}\rho$$

for  $i \neq k, i, k \in \{1, \dots, n\}, n \geq N_2$ , with  $N_2$  large enough and existing by Theorem 1. Thus,

$$\Delta_{k,n} = \max_{i \in \{1, \dots, n\}, i \neq k} (|\mu_{i,n} - \lambda_k|^{-1}) \leq \frac{2}{\rho}$$

for  $k \in \{1, \dots, n\}$ , where  $n \geq N_2$ .

Therefore, by Lemma 3.2 and Lemma 2.2 there following estimates holds for  $k \in \{1, \dots, n\}$  and  $n \geq N_2$  :

$$\begin{aligned} \|Q_n x_k\|^2 + \Delta_{k,n}^2 K_{k,n}^2 &\leq 4/3 f_{k,n+1}^2 + (2/\rho)^2 c_n^2 |x_{k,n+1}|^2 \leq \\ &\leq 4/3 f_{k,n+1}^2 + (2/\rho)^2 S^2 n^{2\beta} f_{k,n+1}^2 \leq \\ &\leq C_1 (n^\beta f_{k,n+1})^2 \leq C_1 a^2 \left(\frac{1}{n^{\alpha-2\beta-1}}\right)^2. \end{aligned}$$

Finally, applying Theorem 2.1, we obtain

$$\|x_k - y_{k,n}\|_l^2 \leq C_1 a^2 \left(\frac{1}{n^{\alpha-2\beta-1}}\right)^2 + (C_1 a^2 \left(\frac{1}{n^{\alpha-2\beta-1}}\right)^2)^2 \leq \tilde{C} \left(\frac{1}{n^{\alpha-2\beta-1}}\right)^2$$

for  $n \geq N_2$ ; so the proof is complete.  $\square$

## 4. SHARPNESS OF THE ESTIMATIONS IN THE MAIN RESULTS

In this section we show the estimatie in Theorem 1 is sharp. Let us consider the sequences

$$d_n = n^2, \quad c_n = \begin{cases} 1 & \text{for } n \text{ odd} \\ \frac{1}{n^4} & \text{for } n \text{ even} \end{cases}, \quad n \geq 1.$$

Now consider the operator  $J$  in  $l^2$ , represented by the tridiagonal infinite matrix

$$J = \begin{pmatrix} 1^2 & 1 & 0 & 0 & \dots \\ 1 & 2^2 & c_2 & 0 & \\ 0 & c_2 & 3^2 & 1 & \\ 0 & 0 & 1 & 4^2 & \\ \vdots & & & & \ddots \end{pmatrix} = J_0 + \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & c_2 & 0 & \\ 0 & c_2 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \\ \vdots & & & & \ddots \end{pmatrix},$$

where

$$J_0 = \begin{pmatrix} 1^2 & 1 & 0 & 0 & \dots \\ 1 & 2^2 & 0 & 0 & \\ 0 & 0 & 3^2 & 1 & \\ 0 & 0 & 1 & 4^2 & \\ \vdots & & & & \ddots \end{pmatrix}.$$

Unfortunately, this example is not interesting in the context of Jacobi matrices, because the sequence  $\{c_n\}$  is rather strange. However, the sequences  $\{d_n\}$  and  $\{c_n\}$  satisfy conditions (C1) – (C3) with  $\alpha = 2$  and  $\beta = 0$ . The point spectrum of  $J_0$  is the set  $\sigma(J_0) = \{v_1, v_2, v_3, \dots\}$ , where

$$v_{2k-1} = (2k-1)^2 - \frac{2}{4k-1 + \sqrt{(4k-1)^2 + 4}},$$

$$v_{2k} = (2k)^2 + \frac{2}{4k-1 + \sqrt{(4k-1)^2 + 4}} \quad \text{for } k \geq 1.$$

Then, by the Rozenbljum theorem (see [4]),  $\sigma(J) = \{\lambda_1, \lambda_2, \lambda_3, \dots\}$ , with

$$\lambda_{2k-1} = (2k-1)^2 - \frac{2}{4k-1 + \sqrt{(4k-1)^2 + 4}} + \rho_{2k-1},$$

$$\lambda_{2k} = (2k)^2 + \frac{2}{4k-1 + \sqrt{(4k-1)^2 + 4}} + \rho_{2k} \quad \text{for } k \geq 1,$$

where

$$\rho_n = O\left(\frac{1}{n^4}\right), \quad n \rightarrow \infty.$$

Let  $n > 1$  and  $J_n$  be the principal  $n \times n$  submatrix of  $J$  and suppose that  $\sigma(J_n) = \{\mu_{1,n}, \mu_{2,n}, \dots, \mu_{n,n}\}$ .

Denote

$$r_n = \sup\{|\mu_{i,n} - \lambda_i| : i \in \{1, 2, \dots, n\}\}.$$

Theorem 1 implies that  $r_n = O\left(\frac{1}{n^{\alpha-2\beta-1}}\right) = O\left(\frac{1}{n}\right)$ .

If  $n = 2k - 1$  then

$$|\mu_{n,n} - n^2| \leq c_{n-1} = c_{2(k-1)} = \frac{1}{16(k-1)^4},$$

by virtue of the Weyl theorem (see [5]) or the Wilkinson theorem (see [6]). So, for the joint error of approximation of  $n$  eigenvalues of  $J$  by eigenvalues of  $J_n$  we obtain

$$\begin{aligned} r_n &\geq |\mu_{n,n} - \lambda_n| = |\mu_{n,n} - n^2 + \frac{2}{4k-1 + \sqrt{(4k-1)^2 + 4}} - \rho_{2k-1}| \geq \\ &\geq \frac{2}{4k-1 + \sqrt{(4k-1)^2 + 4}} - \frac{1}{16(k-1)^4} - o\left(\frac{1}{(2k-1)^4}\right) \geq \frac{1}{8(2k-1)} = \frac{1}{8n} \end{aligned}$$

for an odd  $n$  large enough.

This implies that

$$r_n = O\left(\frac{1}{n}\right) = O\left(\frac{1}{n^{\alpha-2\beta-1}}\right)$$

and the sequence  $r_n$  admits no better estimate.

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