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CONSTRUCTION OF ALGEBRAIC-ANALYTIC DISCRETE APPROXIMATIONS FOR LINEAR AND NONLINEAR HYPERBOLIC EQUATIONS IN $\mathbb{R}^2$.

PART I

Abstract. An algebraic-analytic method for constructing discrete approximations of linear hyperbolic equations based on a generalized d'Alembert formula of the Lytvyn and Riemann expressions for Cauchy data is proposed. The problem is reduced to some special case of the fixed point problem.

Keywords: algebraic-analytic approximation, d'Alembert type formula, Riemann functions, fixed point problem.

Mathematics Subject Classification: 35B05, 65F05.

1. INTRODUCTION. GENERALIZED D'ALEMBERT FORMULA

Let us consider the problem of finding a mapping $u \in C^n(\mathbb{R}^2;\mathbb{R})$, where $n \geqslant N+1 \in \mathbb{Z}_+$, satisfying the following conditions:

$$\frac{\partial^s u}{\partial y^s}\bigg|_{y=0} = u_s(x) \tag{1.1}$$

for all $s = \overline{0,N}$, $x \in \mathbb{R}$.

This problem has effectively been solved before by O.M. Lytvyn [1]. He proposed the following integral formula

$$\begin{aligned} u(x,y) = \sum_{i=0}^{N} \Delta_{N,i}^{-1} \Bigg\{ (-1)^N \prod_{\substack{\nu=0\\ \nu\neq i}}^{N} \beta_\nu u_0(x+\beta_i y) + \\ + \sum_{s=1}^{w} (-1)^{N-s} \sum_{\substack{0\le i_1\le\ldots\le i_{N-s}\le w\\ i_\nu\neq i;\ \nu=\overline{1,N-s}}} \prod_{\nu=1}^{N-s} \beta_{i_\nu} \int_0^{x+\beta_i y} u_s(\xi) \frac{(x+\beta_i y-\xi)^{s-1}}{(s-1)!} \mathrm{d}\xi + \\ + \int_0^y \mathrm{d}\eta \int_0^{x+\beta_i(y-\eta)} \mathrm{d}\xi \left[\prod_{\nu=0}^{N} \left(\frac{\partial}{\partial\eta} - \beta_\nu \frac{\partial}{\partial\xi} \right) u(\xi,\eta) \right] \frac{[x+\beta_i(y-\eta)-\xi]^{N-1}}{(N-1)!} \Bigg\}, \end{aligned} \tag{1.2}$$

which holds for all $N \in \mathbb{Z}_+$, $(x,y) \in \mathbb{R}^2$ and arbitrary numbers $\beta_i \neq \beta_j \in \mathbb{R}$, $i,j = \overline{0,N}$, and where we denoted $\Delta_{N,i} := \prod_{\substack{\nu=0\\ \nu\neq i}} (\beta_i - \beta_j)$.

A proof of (1.2) one can find in Lytvyn's paper [1]. One can simply note that (1.2) is an original generalization of the classical d'Alembert formula for a solution of the wave equation in $\mathbb{R}^2$. If we additionally assume that the function $u \in C^\infty(\mathbb{R}^2;\mathbb{R})$ satisfies the hyperbolic equation:

$$\overline{A}_{N+1} u(x,y) := \prod_{\nu=0}^{N} \left(\frac{\partial}{\partial y} - \beta_\nu \frac{\partial}{\partial x} \right) u(x,y) = f(x,y), \tag{1.3}$$

where $f \in L_1(\mathbb{R}^2;\mathbb{R})$, then formula (1.2) immediately leads to the following d'Alembert type formula:

$$\begin{aligned} u(x,y) = \sum_{i=0}^{N} \Delta_{N,i}^{-1} \Bigg\{ (-1)^N \prod_{\nu=0}^{N} \beta_\nu u_0(x+\beta_i y) + \\ + \sum_{s=1}^{w} (-1)^{N-s} \sum_{\substack{0\le i_1\le\ldots\le i_{N-s}\le w\\ i_\nu\neq i;\ \nu=\overline{1,N-s}}} \prod_{\nu=1}^{N-s} \beta_{i_\nu} \int_0^{x+\beta_i y} u_s(\xi) \frac{(x+\beta_i y-\xi)^{s-1}}{(s-1)!} \mathrm{d}\xi + \\ + \int_0^y \mathrm{d}\eta \int_0^{x+\beta_i(y-\eta)} \mathrm{d}\xi \, f(\xi,\eta) \frac{[x+\beta_i(y-\eta)-\xi]^{N-1}}{(N-1)!} \Bigg\}, \end{aligned} \tag{1.4}$$

which solves hyperbolic equation (1.3) exactly, provided "initial" conditions (1.1) are satisfied.

Formula (1.2) can be rewritten in the following generalized operator form

$$u(x,y) = g_0(x,y) + \sum_{i=0}^{N} K_i \left(\overline{A}_{N+1} u \right)(x,y) \tag{1.5}$$

for all $(x,y) \in \mathbb{R}^2$ with

$$g_0(x,y) := \sum_{i=0}^{N} \Delta_{N,i}^{-1} \Bigg\{ (-1)^N \prod_{\nu=0}^{N} \beta_\nu u_0(x+\beta_i y) + \\ + \sum_{s=1}^{w} (-1)^{N-s} \sum_{0 \le i_1 \le \ldots \le i_{N-s} \le w} \prod_{\nu=1}^{N-s} \beta_{i_\nu} \int\limits_{0}^{x+\beta_i y} u_s(\xi) \frac{(x+\beta_i y - \xi)^{s-1}}{(s-1)!} \mathrm{d}\xi \Bigg\}, \tag{1.6}$$

$$K_i \overline{A}_{N+1}(\ldots) := \Delta_{N,i}^{-1} \int_{\Omega_i^{(x,y)}} \mathrm{d}\eta \, \mathrm{d}\xi \frac{[x+\beta_i(y-\eta)-\xi]^{N-1}}{(N-1)!} \overline{A}_{N+1}(\ldots), \tag{1.6$'$}$$

where domains $\Omega_i^{(x,y)} \subset \mathbb{R}^2$, $i = \overline{0,N}$, are given as

$$\Omega_i^{(x,y)} := \left\{ (\xi,\eta) \in \mathbb{R}^2 : \eta \in (0,y), \xi \in (0,\xi_i(\eta)), \xi_i(\eta) := x + \beta_i(y-\eta) \right\}, \tag{1.7}$$

if $\beta_i < 0$ and $i = \overline{0,N}$, they have the shape shown in Figure 1, or in the opposite case $\beta_i > 0$, $i = \overline{0,N}$, as shown in Figure 2.

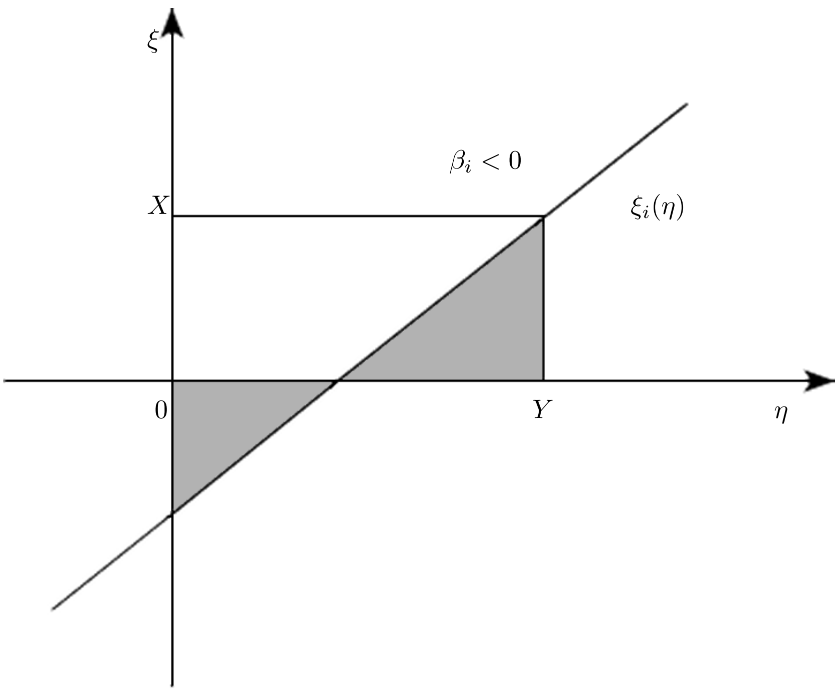

Fig. 1. Domain $\Omega_i^{(x,y)}$, $\beta_i < 0$

We pose the following problem for solving a hyperbolic equation of order $(N+1) \in \mathbb{Z}_+$ in $\Omega \subset \mathbb{R}^2$ of the form:

$$A_{N+1}u = f, \tag{1.8}$$

where $f \in L_{1,loc}(\Omega;\mathbb{R})$, the function $u \in C^2(\Omega;\mathbb{R})$ satisfies initial conditions (1.1) and the operator

$$A_{N+1} := \sum_{|\alpha|=0}^{N+1} c_\alpha(x,y)\frac{\partial^{|\alpha|}}{\partial x^{\alpha_1}\partial y^{\alpha_2}} \tag{1.9}$$

has the coefficients $c_\alpha \in C^{|\alpha|}(\Omega;\mathbb{R})$, $\alpha = (\alpha_1,\alpha_2) \in \mathbb{Z}_+^2$, $|\alpha| = \alpha_1 + \alpha_2 = \overline{0, N+1}$.

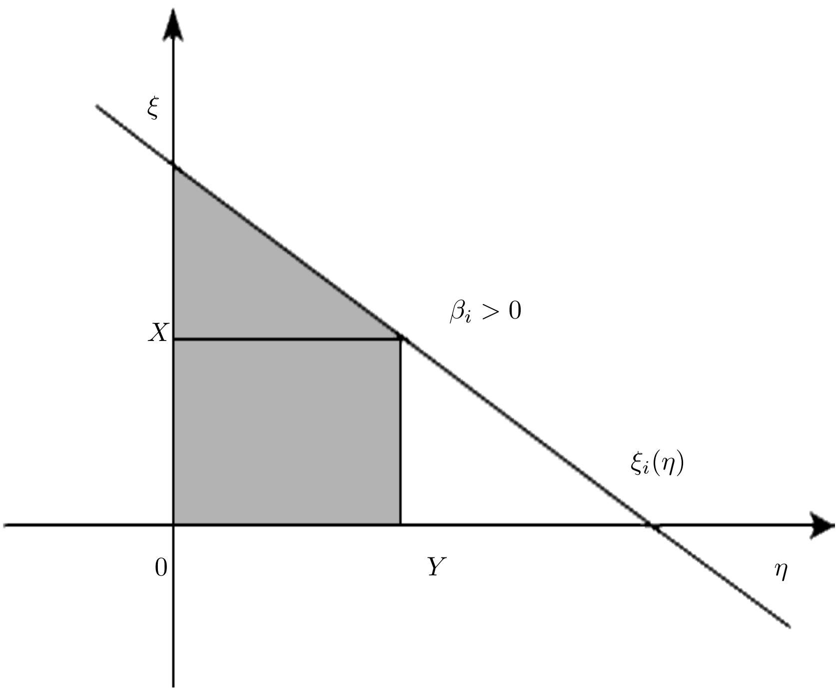

Fig. 2. Domain $\Omega_i^{(x,y)}$, $\beta_i > 0$

On the basis of previous results if $A_N = \overline{A}_N$ one can immediately find the solution to problem (1.8) in exact form (1.4). Otherwise, if the inequality $A_N \neq \overline{A}_N$ holds, one can use operator formula (1.5). Really, we obtain:

$$\begin{aligned} u &= g_0 + \sum_{i=0}^{N} \overline{K}_i\left(\overline{A}_{N+1} - A_{N+1}\right)u + \sum_{i=0}^{N} \overline{K}_i A_{N+1}u = \\ &= g_0 + \sum_{i=0}^{N} \overline{K}_i f + \sum_{i=0}^{N} \overline{K}_i \Delta A_{N+1}u := g_f + \sum_{i=0}^{N} \overline{K}_i \Delta A_{N+1}u, \end{aligned} \tag{1.10}$$

where

$$g_f := g_0 + \sum_{i=0}^{N} \overline{K}_i f, \qquad \Delta A_{N+1} := \overline{A}_{N+1} - A_{N+1}. \tag{1.11}$$

Problem (1.10) is equivalent to the ordinary problem of finding a fixed point for the mapping $Q : C(\Omega, \mathbb{R}) \to C(\Omega, \mathbb{R})$, where:

$$Qu := g_f + \sum_{i=0}^{N} \overline{K}_i \Delta A_{N+1} u = u. \tag{1.12}$$

Indeed, the following theorem is true.

Theorem 1.1. *Fixed-point problem* (1.12) *defined on each compact set* $\Omega \subset \mathbb{R}^2$*, for which* $\|\Delta Q\|_{C(\Omega,\mathbb{R})} < 1$*, possesses exactly one solution* $u \in C^{(N+1)}(\Omega; \mathbb{R})$*, satisfying equation* (1.8) *and initial conditions* (1.1).

Proof. Proof of the theorem relies on the application of operator (1.9) to formula (1.10):

$$A_{N+1} u = A_{N+1} g_f + \sum_{i=0}^{N} A\overline{K}_i \Delta A_{N+1} u := \widetilde{f}, \tag{1.13}$$

the integration-by-parts formula and checking that the function $\widetilde{f} \equiv f \in L_{1,\text{loc}}(\Omega; \mathbb{R})$. Since the function $u \in C^{(N+1)}(\Omega; \mathbb{R})$, given in form (1.5), automatically satisfies initial conditions (1.1), it is sufficient to prove that the functional series

$$\sum_{k \in \mathbb{Z}_+} (\Delta Q)^k g_f = \widetilde{u} \tag{1.14}$$

converges in $C(\Omega, \mathbb{R})$, where

$$\Delta Q := \sum_{i=0}^{N} \overline{K}_i \Delta A_{N+1}. \tag{1.15}$$

Then, the function $\widetilde{u} \in C^{(N+1)}(\Omega; \mathbb{R})$ will be the solution of equation (1.8) owing to the condition $\|\Delta Q\|_{C(\Omega,\mathbb{R})} < 1$, that can be supplied by selecting the domain $\Omega \in \mathbb{R}^2$ and appropriate parameters $\beta_i \in \mathbb{R}$, $i = \overline{0, N}$. Thereby, we obtain the convergence of series (1.14) to the unique function $\widetilde{u} := u \in C^{(N+1)}(\Omega; \mathbb{R})$, which solves problem (1.12). □

2. THE LAGRANGE INTERPOLATION

Proceeding further similarly as it was proposed by Lytvyn [1], we are going to apply the Lagrange interpolation to the construction of the algebraic-analytic discrete approximation for equation (1.8) with initial conditions (1.1). Since expression (1.12) automatically satisfies initial conditions (1.1), one has only to solve the problem of discrete algebraically-analytic approximation for the operator $Q : C(\Omega_{(a,b)};\mathbb{R}) \to C(\Omega_{(a,b)};\mathbb{R})$ or, equivalently, for the operator

$$\sum_{i=0}^{N} \overline{K}_i \Delta A_{N+1} : C(\Omega_{(a,b)};\mathbb{R}) \to C(\Omega_{(a,b)};\mathbb{R}). \tag{2.1}$$

Therefore, we may define a function $u \in C^{(N+1)}(\Omega;\mathbb{R})$ on cube $\Omega_{(a,b)} = [0,a]\times[0,b] \subset \mathbb{R}^2$ chosen in such a way that $\Omega^{(x,y)} \subset \Omega_{(a,b)}$ for all $(x,y) \in \Omega_{(a,b)}$. Using now the Lagrange interpolation formula

$$u^{(n)}(x,y) = P_n u(x,y) = \sum_{i,j=1}^{n_x,n_y} u_{ij}^{(n)} l_i(x) \otimes l_j(y) \tag{2.2}$$

where $\otimes$ is the usual tensor product $(x,y) \in \Omega_{(a,b)}$, $n := (n_x,n_y) \in \mathbb{Z}_+^2$, $u^{(n)} \in \mathbb{R}^{n_x} \otimes \mathbb{R}^{n_y}$, and

$$l_i(x) := \prod_{k\neq i}^{n_x} \frac{x-x_k}{x_i-x_k}, \qquad l_j(y) := \prod_{k\neq j}^{n_y} \frac{y-y_k}{y_j-y_k}, \tag{2.3}$$

for $i = \overline{1,n_x}$, $j = \overline{1,n_y}$ are the fundamental Lagrangian polynomials on the cube $\Omega_{(a,b)} \subset \mathbb{R}^2$.

Now, one can use the algebraic-analytic method [6] developed before by Luśtyk and Bihun [4, 5] of constructing the discrete approximations for linear operators in suitable functional spaces, i.e., one can find a matrix quasi-representation of operator (2.1) in the space $C(\Omega_{(a,b)};\mathbb{R})$, using expression (1.6′). Thus, we obtain

$$\left(\overline{K}_i \Delta A_{N+1}\right)^{(n)} := P_n \overline{K}_i \Delta A_{N+1} P_n, \tag{2.4}$$

or equivalently, in the functional form:

$$\begin{aligned}
&\left\langle \left(\overline{K}_i \Delta A_{N+1}\right)^{(n)} u^{(n)}, l(x)\otimes l(y) \right\rangle = \\
&= P_n \Delta_{N,i}^{-1} \int\limits_{\Omega_i^{(x,y)}} \mathrm{d}\eta\, \mathrm{d}\xi \frac{[x+\beta_i(y-\eta)-\xi]^{N-1}}{(N-1)!} \left\langle \Delta A_{N+1}^{(n)} u^{(n)}, l(\xi)\otimes l(\eta) \right\rangle := \\
&:= P_n \left\langle \Delta A_{N+1}^{(n)} u^{(n)}, a_i(x,y) \otimes b_i(x,y) \right\rangle = \\
&= P_n \left\langle \Delta A_{N+1}^{(n)} u^{(n)}, a_i^{(n)} \otimes b_i^{(n)} l(x) \otimes l(y) \right\rangle = \\
&= \left\langle \left(a_i^{(n)} \otimes b_i^{(n)}\right) \Delta A_{N+1}^{(n)} u^{(n)},\ l(x)\otimes l(y) \right\rangle
\end{aligned} \tag{2.5}$$

for all $(x,y) \in \Omega_{(a,b)}$, where

$$a_i(x,y) \otimes b_i(x,y) := \Delta_{N,i}^{-1} \int\limits_{\Omega_i^{(x,y)}} \frac{[x+\beta_i(y-\eta)-\xi]^{N-1}}{(N-1)!} l(\xi)\otimes l(\eta) \mathrm{d}\eta\, \mathrm{d}\xi \tag{2.6}$$

and

$$a_i^{(n)} := a_i(X^{(n)}, Y^{(n)}), \qquad b_i^{(n)} := b_i(X^{(n)}, Y^{(n)}) \tag{2.7}$$

for $i = \overline{0, N}$. Whence we get

$$\left(\overline{K}_i \Delta A_{N+1}\right)^{(n)} = \left(a_i^{(n)} \otimes b_i^{(n)}\right) \Delta A_{N+1}^{(n)}, \tag{2.8}$$

where $i = \overline{0, N}$ and

$$\Delta A_{N+1}^{(n)} := \prod_{\nu=0}^{N} \left(Z_y^{(n)} - \beta_\nu Z_x^{(n)}\right) - \sum_{|\alpha|=0}^{N+1} c_\alpha \left(X^{(n)}, Y^{(n)}\right) \left(Z_x^{(n)}\right)^{\alpha_1} \left(Z_y^{(n)}\right)^{\alpha_2}, \tag{2.9}$$

is an appropriate quasi-representation of the differential expression $\Delta A_{N+1} = (\overline{A}_{N+1} - A_{N+1})$ in the space $C(\Omega_{(a,b)}; \mathbb{R})$.

Now fixed-point problem (1.12) can be rewritten in the approximation form as

$$g_f^{(n)} + \Delta Q^{(n)} u^{(n)} = u^{(n)}, \tag{2.10}$$

where $u^{(n)} \in \mathbb{R}^{n_x} \otimes \mathbb{R}^{n_y}$ and

$$\Delta Q^{(n)} := \sum_{i=0}^{N} \left(\overline{K}_i \Delta A_{N+1}\right)^{(n)}. \tag{2.11}$$

Solving discrete problem (2.10), making use of standard numerical methods, we obtain an approximate solution to problem (1.8) with initial conditions (1.1) in the form

$$u^{(n)}(x, y) = \left\langle u^{(n)}, l(x) \otimes l(y) \right\rangle \tag{2.12}$$

for all $n = (n_x, n_y) \in \mathbb{Z}_+^2$.

The above method of finding the approximate solution to problem (1.12) for linear hyperbolic operators of order $(N+1) \in \mathbb{Z}_+$ of form (1.9), based on the Lytvyn formula of d'Alembert type (1.2) and the Lagrange interpolation scheme, as is shown by examples, is very efficient from in terms of its convergence. Analogously, one can exploit the more accurate Hermite interpolation scheme, for which the convergence rate of the approximate solution will be better still. To the unfortunately, the expressions for quasi-representations of basic differential operators Z_x and Z_y in the cube $(x, y) \in \Omega_{(a,b)}$ are slightly complicated, as it was shown earlier [5].

The proposed algebraic-analytic method for finding approximate solutions to linear hyperbolic equations of order $(N+1) \in \mathbb{Z}_+$ in a cube $\Omega_{(a,b)} \subset \mathbb{R}^2$ can be applied, as usual, to quasi-linear and non-linear partial differential equations of order $(N+1) \in \mathbb{Z}_+$ that will not be covered within the framework of this paper.

3. BOUNDARY VALUE PROBLEM FOR LINEAR HYPERBOLIC EQUATIONS OF THE SECOND ORDER ON $\mathbb{R}^2$

Consider the following boundary value problem for a linear hyperbolic equation of the second order in a cube $\Omega := [0, a] \times [0, b] \subset \mathbb{R}^2$:

$$A_2 u := \frac{\partial^2 u}{\partial x \partial y} + c_{(x)}(x, y)\frac{\partial u}{\partial x} + c_{(y)}(x, y)\frac{\partial u}{\partial y} + c_{(0)}(x, y)u = f, \tag{3.1}$$

where $f \in L_{1,\mathrm{loc}}(\Omega; \mathbb{R})$, $c_{(0)} \in C(\overline{\Omega}; \mathbb{R})$, $c_{(x)}, c_{(y)} \in C^{(1)}(\overline{\Omega}; \mathbb{R})$. The function $u \in C^{(2)}(\overline{\Omega}; \mathbb{R})$ has to satisfy the boundary conditions

$$u|_{\Gamma^{(x,y)}} = u_0, \qquad \frac{\partial u}{\partial n}\Big|_{\Gamma^{(x,y)}} = u_1 \tag{3.2}$$

on a smooth curve $\Gamma^{(x,y)} \subset \partial\Omega^{(x,y)}$, $\Gamma^{(x,y)} = \left\{\eta = \sigma(\xi), \sigma^{-1}(y) \le \xi \le x\right\}$ (see Fig. 3), where the functions $u_0(x, \sigma(x)) \in C^{(2)}([0, a]; \mathbb{R})$, $u_1(x, \sigma(x)) \in C^{(1)}([0, a]; \mathbb{R})$ are given.

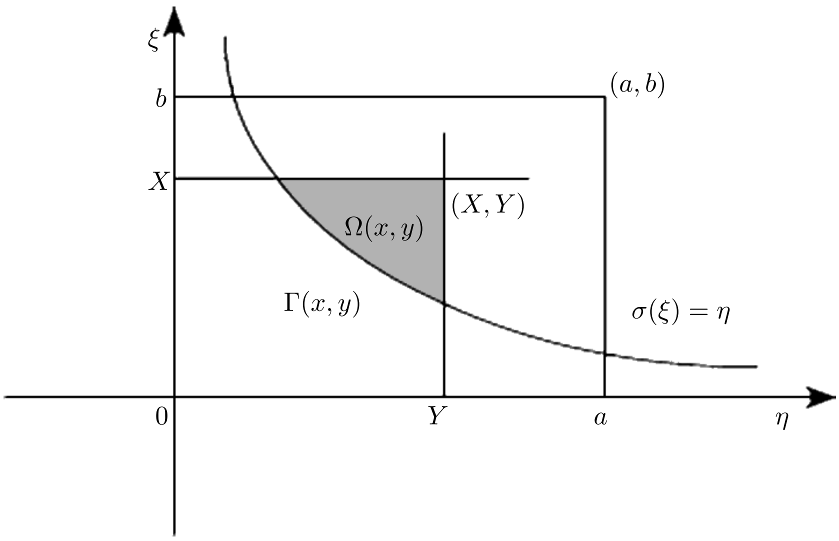

Fig. 3

We also need to assume that the curve $\Gamma^{(x,y)} \subset \partial\Omega^{(x,y)}$ does not intersect the characteristics of equation (3.1), i.e., $\sigma^{'}(\xi) < 0$ for all $\xi \in [0, a]$. Conditions (3.2) on the curve $\Gamma^{(x,y)}$ make it also possible to find easily the suitable expressions $u_x|_{\Gamma^{(x,y)}}$ and $u_y|_{\Gamma^{(x,y)}}$, solving the system of equations

$$\begin{cases} u_x|_{\Gamma^{(x,y)}} + u_y|_{\Gamma^{(x,y)}}\sigma' = u'_{0,x}|_{\Gamma^{(x,y)}}, \\ u_x|_{\Gamma^{(x,y)}}\dfrac{\sigma'}{\Delta} - u_y|_{\Gamma^{(x,y)}}\dfrac{1}{\Delta} = u_1|_{\Gamma^{(x,y)}} \end{cases} \tag{3.3}$$

for all $(\xi,\eta)\in\Gamma^{(x,y)}$, where $\Delta := \left[1+(\sigma'(\xi))^2\right]^{1/2}$.

The following theorem characterizes [3] solution to boundary problem (3.1) and (3.2).

Theorem 3.1. *Let $\Gamma^{(x,y)}\subset\partial\Omega^{(x,y)}$ be a smooth curve of class $C^{(2)}$ and the coefficients of equation* (3.1) *satisfy the conditions mentioned above. Then there exists the unique solution of boundary value problem* (3.1) *and* (3.2) *in the Riemann form*

$$\begin{aligned} u(x,y) = {} & \frac{1}{2}u_0(\sigma^{-1}(y),y)\mathcal{R}(\sigma^{-1}(y),y;x,y)+\frac{1}{2}u_0(x,\sigma(x))\mathcal{R}(x,\sigma(x);x,y)+\\ &+\int\limits_{\Gamma^{(x,y)}}\left[\left(\frac{\mathcal{R}}{2}\frac{\partial u}{\partial\xi}-\frac{u_0}{2}\frac{\partial\mathcal{R}}{\partial\xi}+c_{(y)}u_0\mathcal{R}\right)\mathrm{d}\xi-\left(\frac{\mathcal{R}}{2}\frac{\partial u}{\partial\eta}-\frac{u_0}{2}\frac{\partial\mathcal{R}}{\partial\eta}+c_{(x)}u_0\mathcal{R}\right)\mathrm{d}\eta\right]+\\ &+\int\limits_{\Omega^{(x,y)}}\mathcal{R}f\mathrm{d}\xi\,\mathrm{d}\eta, \end{aligned} \tag{3.4}$$

where the domain $\Omega^{(x,y)}\subset\Omega$ and the curve $\Gamma^{(x,y)}\subset\Gamma$ are shown on Fig. 3, and the Riemann function $\mathcal{R}:\overline{\Omega}\times\overline{\Omega}\to\mathbb{R}$ satisfies the following conditions for all $(x,y)\in\overline{\Omega}$:

1. *functions $\mathcal{R}$, $\mathcal{R}_x$, $\mathcal{R}_y$ and $\mathcal{R}_{xy}\in C(\overline{\Omega}\times\overline{\Omega};\mathbb{R})$;*
2. *$A_2^*\mathcal{R}=0$;*
3. *the relationships*

$$\begin{aligned} &\mathcal{R}(x,y|\xi,\eta)|_{y=\eta}=c_{(y)}(x,\eta)\times\mathcal{R}(x,\eta;\xi,\eta),\\ &\mathcal{R}(x,y|\xi,\eta)|_{x=\xi}=c_{(x)}(\xi,y)\times\mathcal{R}(\xi,y;\xi,\eta),\\ &\mathcal{R}(x,y|\xi,\eta)\Big|_{\substack{x=\xi\\ y=\eta}}=1 \end{aligned} \tag{3.5}$$

hold for all $(\xi,\eta)\in\overline{\Omega}$.

Proof. A proof of the theorem is standard and relies on substitution of solution (3.4) into equation (3.1), using conditions (3.5) and the following property of the adjoint Riemann function $\mathcal{R}^*:\overline{\Omega}\times\overline{\Omega}\to\mathbb{R}$ for equation (3.1):

$$A_2\mathcal{R}^*=0,$$

where

$$\mathcal{R}^*(x,y|\xi,\eta)=\mathcal{R}(\xi,\eta|x,y),\ \mathcal{R}^*(\xi,\eta|\xi,\eta)=1$$

and

$$\begin{aligned} &\mathcal{R}^*_x(x,y|\xi,\eta)|_{y=\eta}=-c_{(y)}(x,\eta)\mathcal{R}^*(x,\eta;\xi,\eta),\\ &\mathcal{R}^*_y(x,y|\xi,\eta)|_{x=\xi}=-c_{(x)}(\xi,y)\mathcal{R}^*(\xi,y;\xi,\eta) \end{aligned} \tag{3.6}$$

for all $(x,y)\times(\xi,\eta)\in\overline{\Omega}\times\overline{\Omega}$. □

Expression (3.4) obtainded for solution to boundary value problem (3.1) and (3.2) is a consequence of the following Riemann identity for an arbitrary smooth function $u \in C^2(\Omega;\mathbb{R})$:

$$
\begin{aligned}
u(x,y) = {} & \frac{1}{2}u_0(\sigma^{-1}(y),y)\mathcal{R}(\sigma^{-1}(y),y;x,y) + \frac{1}{2}u_0(x,\sigma(x))\mathcal{R}(x,\sigma(x);x,y) + \\
& + \int\limits_{\Gamma^{(x,y)}} \left[\left(\frac{\mathcal{R}}{2}\frac{\partial u}{\partial \xi} - \frac{u_0}{2}\frac{\partial \mathcal{R}}{\partial \xi} + c_{(y)}u_0\mathcal{R}\right)\mathrm{d}\xi - \left(\frac{\mathcal{R}}{2}\frac{\partial u}{\partial \eta} - \frac{u_0}{2}\frac{\partial \mathcal{R}}{\partial \eta} + c_{(x)}u_0\mathcal{R}\right)\mathrm{d}\eta\right] + \\
& + \int\limits_{\Omega^{(x,y)}} \mathcal{R}(x,y;\xi,\eta)A_2 u(\xi,\eta)\mathrm{d}\xi\,\mathrm{d}\eta,
\end{aligned} \tag{3.7}
$$

holding for all $(x,y) \in \Omega$. Looking at equation (3.7) one can easily notice that in the case of linear hyperbolic equations of the second order the expression of the d'Alembert type, obtained by Lytvyn [2], is a specific case of the Riemann type formula for a linear equation of the second order on a cube $\Omega \subset \mathbb{R}^2$. It means that by constructing a suitable Riemann function $\overline{\mathcal{R}}$ for linear equation (1.3) of order $(N+1) \in \mathbb{Z}_+$ with proper boundary conditions on a smooth curve $\Gamma^{(x,y)} \subset \partial\Omega^{(x,y)}$, one may similarly write down the following identity for $u \in C^{(N+1)}(\Omega;\mathbb{R})$:

$$
u(x,y) = g_0(x,y) + \int\limits_{\Omega^{(x,y)}} \overline{\mathcal{R}}(x,y;\xi,\eta)\overline{A}_{N+1}u(\xi,\eta)\,\mathrm{d}\xi\mathrm{d}\eta, \tag{3.8}
$$

where $(x,y) \in \Omega$ and $g_0 \in C^{(N+1)}(\Omega;\mathbb{R})$ is a suitable known function depending on boundary conditions on the curve $\Gamma^{(x,y)} \subset \partial\Omega^{(x,y)}$.

Having expressed (3.8) in the operator form

$$
u = g_0 + \overline{K}\left(\overline{A}_{N+1} - A_{N+1}\right)u + \overline{K}A_{N+1}u, \tag{3.9}
$$

where

$$
A_{N+1} := \sum_{|\alpha|=0}^{N+1} a_\alpha \frac{\partial^{|\alpha|}}{\partial x^{\alpha_1}\partial y^{\alpha_2}}, \tag{3.10}
$$

one can obtain a solution to the linear equation of the $(N+1)$-th order

$$
A_{N+1}u = f \tag{3.11}
$$

for $f \in L_{1,\mathrm{loc}}(\Omega;\mathbb{R})$ in the form of a fixed-point problem in the space $C^{(N+1)}(\Omega;\mathbb{R})$:

$$
u = Qu := g_f + \overline{K}\Delta A_{N+1}u. \tag{3.12}
$$

In the preceding formula

$$
\Delta A_{N+1} := \overline{A}_{N+1} - A_{N+1}, \qquad g_f := g_0 + \overline{K}f, \qquad \overline{K}f := \int\limits_{\Omega^{(x,y)}} \overline{\mathcal{R}}f\,\mathrm{d}\xi\mathrm{d}\eta \tag{3.13}
$$

for all $(x,y) \in \Omega$.

Since expression (3.12) automatically satisfies boundary conditions (3.2) on the smooth curve $\Gamma^{(x,y)} \subset \partial\Omega^{(x,y)}$, we can state that a solution to fixed-point problem (3.12) on the compact cube $\Omega \subset \mathbb{R}^2$ will be a solution to linear differential equation (3.11), satisfying the same boundary conditions on the curve $\Gamma^{(x,y)} \subset \partial\Omega^{(x,y)}$.

The preceding reasoning is a basis for our algebraic-analytical method for discrete approximations applied to equivalent fixed-point problem (3.12) in the space $C^{(N+1)}(\Omega;\mathbb{R})$. In the particular case of the second order equation (3.1) with boundary conditions (3.2), fixed-point problem (3.12) gets the standard form

$$u = g_f + \overline{K}\Delta A_2 u, \tag{3.14}$$

for which, as usual, we obtain the following discrete approximation:

$$u^{(n)} = g_f^{(n)} + \left(\overline{K}\Delta A_2\right)^{(n)} u^{(n)}, \tag{3.15}$$

where $u^{(n)} = \mathbb{R}^{n_x} \otimes \mathbb{R}^{n_y}$, $n := (n_x, n_y) \in \mathbb{Z}_+^2$.

Solving approximate problem (3.15) as a finite-dimensional fixed-point problem in the space $\mathbb{R}^{n_x} \otimes \mathbb{R}^{n_y}$ by means of available numerical methods, and next applying formula (2.12), we can obtain an approximate solution to boundary value problem (3.1) and (3.2).

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