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# TREE DOMATIC NUMBER IN GRAPHS

**Abstract.** A dominating set $S$ in a graph $G$ is a tree dominating set of $G$ if the subgraph induced by $S$ is a tree. The tree domatic number of $G$ is the maximum number of pairwise disjoint tree dominating sets in $V(G)$. First, some exact values of and sharp bounds for the tree domatic number are given. Then, we establish a sharp lower bound for the number of edges in a connected graph of given order and given tree domatic number, and we characterize the extremal graphs. Finally, we show that a tree domatic number of a planar graph is at most 4 and give a characterization of planar graphs with the tree domatic number 3.

**Keywords:** tree domatic number, regular graph, planar graph, Cartesian product.

**Mathematics Subject Classification:** 05C69, 05C35.

## 1. INTRODUCTION

Let $G = (V, E)$ be a simple graph of order $n$. The *degree, neighborhood and closed neighborhood* of a vertex $v$ in the graph $G$ are denoted by $d(v)$, $N(v)$ and $N[v] = N(v) \cup \{v\}$, respectively. The *minimum degree and maximum degree* of the graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. The graph induced by $S \subseteq V$ is denoted by $\langle S \rangle$. Let $P_n$, $C_n$ and $K_n$ denote the *path, cycle and complete graph* with $n$ vertices, respectively. Let $K_{n,m}$ denote the *complete bipartite graph.*

A set of vertices $D$ in a graph $G = (V, E)$ is a *dominating set* if every vertex in $V - D$ has at least one neighbor in $D$. A dominating set $D$ is called a *tree dominating set* if the subgraph induced by $D$ is a tree. In what follows, we assume that all graphs are connected. The minimum number of vertices in a tree dominating set of $G$ is called the *tree domination number* of $G$, and is denoted by $\gamma_{tr}(G)$. A *tree domatic partition* of $G$ is a partition of the vertex set $V$ into pairwise disjoint tree dominating sets. If such a partition exists, the maximum number of subsets in such a partition is called the *tree domatic number* of $G$ and is denoted by $d_{tr}(G)$, otherwise, define $d_{tr}(G) = 0$.

Zelinka [1] studied the connected domatic number of a graph. Hartnell *et al.* [2] gave the connected domatic number of a planar graph. Chen *et al.* [3] studied tree dominating set of a graph. We gave some graphs having no tree dominating set. For example, take a complete graph $K_{3n}$, and partition the vertices of $K_{3n}$ into $n$ sets of three vertices each. To each set of three vertices, add a new vertex and join it to each of these three vertices. Then the resulting 3-connected graph does not have a tree dominating set. If $G$ has no tree dominating set, then we define $d_{tr}(G) = 0$.

A *planar graph* is a graph that can be embedded in the plane so that no two edges intersect geometrically except at a vertex to which both are incident. A planar graph $G$ is called *outerplanar* if the embedding can be chosen so that the boundary of one of the planar regions contains every vertex of $G$. A graph $G_2$ is called an *elementary contraction* of $G_1$ if there is an edge $uv$ of $G_1$ such that $G_2$ is obtained from $G_1$ by deleting the vertices $u$ and $v$ and appending a new vertex, denoted by $x$, that is adjacent to all the vertices of $G - u - v$ that were originally adjacent to either of $u$ or $v$. If the graph $H$ is isomorphic to $G$ or is obtainable from $G$ by a finite sequence of elementary contractions, then we say that $H$ is a *contraction* of $G$. Perhaps a more intuitive way to think of a contraction $H$ of a graph $G$ is to consider a partition of $V(G)$ into subsets each of which induces a connected subgraph of $G$. Each member of the partition corresponds to a vertex of $H$, and two vertices of $H$ are adjacent if the union of the corresponding subsets of $G$ induces a connected subgraph of $G$. In effect each member of the partition has been shrunk to a single vertex and multiple edges have been removed. It is clear that the property of being planar is preserved under contractions.

For an arbitrary graph $G$, *the vertex connectivity* $\kappa(G)$ is the minimum number of vertices whose removal will disconnected $G$.

The *Cartesian product* of $G$ and $H$, denoted by $G \times H$, has the vertex set $V(G \times H) = \{(g,h)|g \in V(G), h \in V(H)\}$ and the edge set $E(G \times H) = \{(g_1,h_1)(g_2,h_2)|\text{either } g_1 = g_2 \text{ and } h_1h_2 \in E(H) \text{ or } g_1g_2 \in E(G) \text{ and } h_1 = h_2\}$. In particular, if $G$ is isomorphic to $P_r$ and $H$ is isomorphic to $P_s$, then $G \times H$ is called $r \times s$ grid graph, denoted by $G_{r \times s}$.

## 2. SOME EXACT VALUES OF AND BOUNDS FOR THE TREE DOMATIC NUMBER

First, we give some upper bounds for the tree domatic number in term of minimum degree, tree domination number or vertex connectivity.

**Theorem 1.** *Let $G$ be a connected graph. Then,*

(1) *If $G$ is a complete graph, then $d_{tr}(G) = \delta(G)+1$; otherwise, $d_{tr}(G) \leq \delta(G)$ and the bound is sharp.*
(2) *If $\gamma_{tr}(G) > 0$, then $d_{tr}(G) \leq \frac{n}{\gamma_{tr}(G)}$ and the bound is sharp.*
(3) *$d_{tr}(G) \leq \kappa(G)$ and the bound is sharp.*

*Proof.* (1) Assume $d_{tr}(G) = t$ and $(D_1, D_2, \ldots, D_t)$ is a partition of $V(G)$ into $t$ tree dominating sets. If $G$ is a complete graph, then it is obvious that $d_{tr}(G) = n = \delta(G) + 1$. If $G$ is not a complete graph, then let $v$ be a vertex of $G$ such that

$d(v) = \delta(G) < n-1$. Assume $v \in D_1$. If $|D_1| = 1$, then $D_1 = \{v\}$. Since each vertex in $D_i$ is dominated by $v$ for $i = 2, 3, \ldots, t$, it follows that $d(v) = n-1$, which is a contradiction. Hence, $|D_1| \geq 2$. Since $\langle D_1 \rangle$ is a tree and $v$ is dominated by at least one vertex of $D_i$ for $i = 2, 3, \ldots, t$, it follows that $d(v) - 1 \geq t - 1$. That is $t \leq d(v)$. Hence, $d_{tr}(G) \leq \delta(G)$.

(2) Assume $d_{tr}(G) = t$ and $(D_1, D_2, \ldots, D_t)$ is a partition of $V(G)$ into $t$ tree dominating sets. Since each $\langle D_i \rangle$ is a tree dominating set, it follows that $|D_i| \geq \gamma_{tr}(G)$ for $i = 1, 2, \ldots, t$. Hence, $n = \sum_{1 \leq i \leq t} |D_i| \geq t\gamma_{tr}(G)$. That is $d_{tr}(G) \leq \frac{n}{\gamma_{tr}(G)}$.

(3) Let $S$ denote the cut set with cardinality $\kappa(G)$. Then $\langle V(G) - S \rangle$ is disconnected. It is obvious that any tree dominating set must contain at least one vertex of $S$. So $G$ has at most $|S|$ pairwise disjoint tree dominating sets. Hence, $d_{tr}(G) \leq \kappa(G)$.

The sharpness of the bounds is obvious from the following two corollaries and Theorem 4. □

Now, we give some exact values of the tree domatic number for some classes of graphs.

**Corollary 1.** (1) *Let $T$ denote a tree with $n \geq 3$. Then $d_{tr}(T) = 1$.*

$$(2)\quad d_{tr}(C_n) = \begin{cases} 3 & \text{for } n = 3, \\ 2 & \text{for } n = 4, \\ 0 & \text{for } n \geq 5. \end{cases}$$

(3) $d_{tr}(K_{n,m}) = \min(n, m)$ *for $n, m \geq 2$.*

*Proof.* (1) By Theorem 1, since $\kappa(T) = 1$, it follows that $d_{tr}(G) \leq 1$. It is obvious that $T$ is a tree dominating set for $T$. So, $d_{tr}(T) = 1$.

(2) Let $C_n = v_1v_2 \ldots v_nv_1$. Since $\gamma_{tr}(C_n) = n-2$, from Theorem 1 it follows that $d_{tr}(T) \leq \frac{n}{n-2} = 1 + \frac{2}{n-2}$.

If $n = 3$, then $d_{tr}(T) \leq 3$. It is obvious that each vertex is a tree dominating set for $C_3$. So, $d_{tr}(T) = 3$.

If $n = 4$, then $d_{tr}(T) \leq 2$. It is obvious that $\{v_1, v_2\}$ and $\{v_3, v_4\}$ are two pairwise disjoint tree dominating sets for $C_4$. So, $d_{tr}(T) = 2$.

If $n \geq 5$, then $d_{tr}(T) \leq 1$. Hence, $V(C_n)$ is not a tree dominating sets for $C_n$. So, $d_{tr}(T) = 0$.

(3) Let $U = \{u_1, \ldots, u_n\}$ and $W = \{w_1, \ldots, w_m\}$ form a two vertex partition of $K_{n,m}$. Without loss of generality, assume $\min(n, m) = n$. By Theorem 1, it follows that $d_{tr}(K_{n,m}) \leq \delta(K_{n,m}) = n$. It is obvious that $\{u_i, w_i\}$ for $i = 1, \ldots, n-1$ and $\{u_n, v_n, \ldots, v_m\}$ are $n$ pairwise disjoint tree dominating sets of $K_{n,m}$. So, $d_{tr}(K_{n,m}) = n$. □

**Corollary 2.** *Let $G$ be a connected graph with $\kappa(G) = 1$. If $G$ contains a cycle, then $d_{tr}(G) = 0$.*

Let $H_t$ denote the set of trees of order $t$. Assume $T_1, T_2, \ldots, T_k \in H_t$. Let $G(T_1, T_2, \ldots, T_k)$ be obtained from $T_1, T_2, \ldots, T_k$ by joining each vertex $v$ in $T_i$ to exactly one vertex of each $T_j$ for $j = 1, 2, \ldots, k$ and $j \neq i$, and for arbitrary two vertices $u, v \in T_i$, $N(u) \cap V(T_j) \neq N(v) \cap V(T_j)$. Let $G_t = \{G(T_1, T_2, \ldots, T_k) | T_1, T_2, \ldots, T_k \in H_t\}$.

**Theorem 2.** *For any $G(T_1, T_2, \ldots, T_k) \in G_t$, there is $d_{tr}(G(T_1, T_2, \ldots, T_k)) = k$.*

*Proof.* It is clear that $T_1, T_2, \ldots, T_k$ is a tree domatic partition of $G(T_1, T_2, \ldots, T_k)$. Hence, $d_{tr}(G(T_1, T_2, \ldots, T_k)) \geq k$. Since $\delta(G(T_1, T_2, \ldots, T_k)) = k$, it follows that $d_{tr}(G(T_1, T_2, \ldots, T_k)) \leq k$ by Theorem 1. So, $d_{tr}(G(T_1, T_2, \ldots, T_k)) = k$. □

**Theorem 3.** *Let $G$ be a connected graph of order $n$ with tree domatic number $k \geq 1$. Then $G$ must have at least $\frac{(k+1)n}{2} - k$ edges. Furthermore, $G$ has exactly $\frac{(k+1)n}{2} - k$ edges if and only if $n \equiv 0(mod\ k)$ and $G \in G_{\frac{n}{k}}$.*

*Proof.* Let $D_1, D_2, \ldots, D_k$ be a tree domatic partition of $G$ and let $G_i = \langle D_i \rangle$. Without loss of generality, we assume that $G_i$ has order $n_i$ for $i = 1, 2, \ldots, k$. Let $E(D_i, V(G) - D_i)$ denote such a set of edges that for any edge one of its endpoints belongs to $D_i$ and the other belongs to $V(G) - D_i$. Since $D_i$ is a tree dominating set for $G$, it follows that $|E(D_i, V(G) - D_i)| \geq n - n_i$. So,

$$\begin{aligned}|E(G)| &\geq \sum_{1\leq i\leq k} |E(G_i)| + \frac{1}{2} \sum_{1\leq i\leq k} |E(D_i, V(G) - D_i)| \geq \\ &\geq \sum_{1\leq i\leq k} (n_i - 1) + \frac{1}{2} \sum_{1\leq i\leq k} (n - n_i) \geq \\ &\geq (n-k) + \frac{1}{2} n(k-1) = \\ &= \frac{(k+1)n}{2} - k.\end{aligned}$$

For each $G \in G_{\frac{n}{k}}$, there exist $T_1, T_2, \ldots, T_k$ that belong to $H_{\frac{n}{k}}$ such that $G = G(T_1, T_2, \ldots, T_k)$. By the definition of $G(T_1, T_2, \ldots, T_k)$, it is clear that $G$ has exactly $\frac{(k+1)n}{2} - k$ edges.

Conversely, if $G$ has exactly $\frac{(k+1)n}{2} - k$ edges, then for each $D_i$, $|E(D_i, V(G) - D_i)| = n - n_i$. Since for each $v \in V(G) - D_i$, $v$ is dominated by at least one vertex in $D_i$, it follows that $v$ is dominated by exactly one vertex in $D_i$. Hence, $|D_1| = |D_2| = \ldots = |D_k|$. Indeed, let us for the contrary assume that $|D_i| \neq |D_j|$. Without loss of generality, assume that $|D_i| > |D_j|$. Since each vertex $v \in D_i$ is dominated by exactly one vertex in $D_j$, it follows that there exists one vertex in $D_j$ which is dominated by at least two vertices in $D_i$, which is a contradiction. So, $k|D_1| = n$. That is $n \equiv 0(mod\ k)$. For arbitrary two vertices $u, v \in D_i$, it is clear that $N(u) \cap D_j \neq N(v) \cap D_j$. Let $T_i = \langle D_i \rangle$ for $i = 1, 2, \ldots, k$. It is clear that $G = G(T_1, T_2, \ldots, T_k)$. Hence, $G \in G_{\frac{n}{k}}$. □

## 3. CHARACTERIZATION OF REGULAR GRAPHS AND PLANAR GRAPHS ATTAINING THE UPPER BOUNDS

**Theorem 4.** *Let $G$ be an $r$-regular graph of order $n$, where $r < n-1$. Then $d_{tr}(G) = r$ if and only if $n = 2r$ and there exists a perfect matching $M = \{e_1, e_2, \ldots, e_r\}$ of*

*$G$. If $e_i = u_i v_i$ for $i = 1, 2, \ldots, r$, then for each $u_i$ there is $|N(u_i) \cap e_j| = 1$ and $N(u_i) \cap e_j \neq N(v_i) \cap e_j$.*

*Proof.* Let $G$ be an $r$-regular graph with $d_{tr}(G) = r$ and let $D_1, D_2, \ldots, D_r$ be a tree domatic partition of $G$. By Theorem 3, it follows that $|E(G)| \geq \frac{(r+1)n}{2} - r$. That is $\frac{rn}{2} \geq \frac{(r+1)n}{2} - r$. So, $n \leq 2r$. If $n < 2r$, then there exists a $D_i$ such that $|D_i| = 1$. Let $D_1 = \{v\}$. Since each vertex in $D_j$ is dominated by $v$ for $j = 2, \ldots, r$, it follows that $d(v) = n - 1$ and $G$ is a complete graph, which is a contradiction. Hence, $n = 2r$ and $|D_i| = 2$ for $i = 1, 2, \ldots, r$. Let $M = \{\langle D_1 \rangle, \langle D_2 \rangle, \ldots, \langle D_r \rangle\}$. Then $M$ is a perfect matching of $G$. Assume $\langle D_i \rangle = e_i = u_i v_i$ for $i = 1, 2, \ldots, r$. Since $d(u_i) = r$ and $u_i$ is dominated by at least one vertex in $D_j$ for $j = 1, 2, \ldots, r$ and $j \neq i$, it follows that $|N(u_i) \cap e_j| = 1$. If $N(u_i) \cap e_j = N(v_i) \cap e_j$, that is both $u_i$ and $v_i$ are adjacent to the same vertex, say $u_j$, then $d_{\langle D_i \cup D_j \rangle}(u_j) = 3$. Hence, there exists a $D_k$ such that $u_j$ is not dominated by any vertex in $D_k$ for $k \in \{1, 2, \ldots, r\} \backslash \{i, j\}$, which is a contradiction.

Conversely, let $D_i = \{u_i, v_i\}$ for $i = 1, 2, \ldots, r$. It is obvious that $(D_1, D_2, \ldots, D_r)$ is a tree domatic partition of $G$. Hence, $d_{tr}(G) \geq r$. By Theorem 1, it follows that $d_{tr}(G) = r$. □

**Corollary 3.** *If $G$ is a cubic graph and $d_{tr}(G) = 3$, then $G$ is $K_{3,3}$ or $K_3 \times K_2$.*

**Lemma 1.** *A graph $G$ is planar(ourterplanar) if and only if neither $K_5$ nor $K_{3,3}$ ($K_4$ nor $K_{2,3}$) is a contraction of a subgraph of $G$.*

In a way similar to used in [3], we may prove the following theorem.

**Theorem 5.** *Let $G$ be a planar graph of order $n$. Then the tree domatic number of $G$ is at most 4 and $K_4$ is the only planar graph achieving this bound.*

*Proof.* Assume $G$ is a planar graph of order $n$ such that $d_{tr}(G) \geq 5$. By Theorem 3, it follows that $|E(G)| \geq 3n - 5$. But this contradicts the well known upper bound of $3n - 6$ for a planar graph of order $n$. Thus, $d_{tr}(G) \leq 4$.

Consider the case of $d_{tr}(G) = 4$ and $G$ not isomorphic to $K_4$. Let $D_1 \cup D_2 \cup D_3 \cup D_4$ be a tree domatic partition of $G$. Without loss of generality, assume $|D_1| \geq 2$. Let $a$ and $b$ be adjacent vertices in $D_1$. Let $H$ be the contraction of $G$ formed by identifying the vertices in each of $D_2$, $D_3$ and $D_4$ to a single vertex $x_2$, $x_3$ and $x_4$, respectively. $H$ is planar since it is a contraction of a planar graph. But since $D_2$ is a dominating set in $G$, it follows that $x_2$ is adjacent to each of $a, b, x_3, x_4$ in $H$. Similar statements hold for $x_3$ and $x_4$. But then $H$ contains the subgraph $\langle \{a, b, x_2, x_3, x_4\} \rangle$ which has been shown to be isomorphic to $K_5$, which is a contradiction. □

**Theorem 6.** *Let $G$ be a planar graph such that $d_{tr}(G) = 3$ and let $D_1 \cup D_2 \cup D_3$ be any tree domatic partition of $G$. Then each of the induced subgraphs $\langle D_1 \rangle$, $\langle D_2 \rangle$ and $\langle D_3 \rangle$ is a path.*

*Proof.* Let $G$ be a planar graph and $D_1 \cup D_2 \cup D_3$ be any tree domatic partition of $G$ such that $\langle D_1 \rangle$ has a vertex $a$ of degree at least three. Let $b, c, d$ be three of its neighbours in $D_1$ and let $u_2 \in D_2$ and $u_3 \in D_3$. Let $H$ be the planar graph obtained from $G$ by contracting $D_i$ into $u_i$ for $i = 2, 3$, and by removing any of the edges $bc, bd, cd$ which is present in $G$. Since each vertex in $D_1$ is dominated by at least one vertex in $D_2$, each of $a, b, c$ and $d$ is adjacent to $u_2$ in $H$. Consider any planar embedding of $H$. By Euler's formula, the induced subgraph $K = \langle\{a, b, c, d, u_2\}\rangle$ has four regions, and $u_2$ belongs to the boundary of each of these regions. Since, $K$ contains a subgraph isomorphic to $K_{2,3}$, it is not outerplanar, and so none of the four regions has a boundary which contains all of $a, b, c$ and $d$. Now $u_3$ lies in one of these regions and can be adjacent only to vertices on the boundary of this region. But then there exists in $\{a, b, c, d\}$ a vertex which it is not dominated by $u_3$, contradicting our assumption above. Therefore, the maximum degree of any of the induced subgraphs $\langle D_1\rangle$, $\langle D_1\rangle$ and $\langle D_3\rangle$ is no more than two. Hence, each of the induced subgraphs $\langle D_1\rangle$, $\langle D_1\rangle$ and $\langle D_3\rangle$ is a path. □

**Theorem 7.** *Suppose* $2 \leq r \leq s$. *If* $r = 2$, *then* $d_{tr}(G_{r,s}) = 2$; *otherwise,* $d_{tr}(G_{r,s}) = 0$.

*Proof.* Let $P_r$ be the path $v_1, v_2, \ldots, v_r$ and let $P_s$ be the path $w_1, w_2, \ldots, w_s$.

The case $r = 2$ is easy. Assume $r \geq 3$ and $\{D_1, D_2\}$ is a partition of $V(G_{r,s})$ into two tree dominating sets. For ease of reference, let $a, b, c, d, e, f$ be the vertices $(v_1, w_2)$, $(v_2, w_1)$, $(v_1, w_{s-1})$, $(v_2, w_s)$, $(v_{r-1}, w_s)$, $(v_r, w_{s-1})$, respectively. Note that if $r = 3$, then $d = e$. (If, in addition, $s = 3$, then $a = c$ as well). We may assume without loss of generality that $a \in D_1$ and $b \in D_2$. Since $D_1$ and $D_2$ are tree dominating sets of $G_{r,s}$, some neighbours of $(v_1, w_s)$ must belong to $D_2$, and some neighbours of $(v_r, w_s)$ must belong to $D_1$. Let $\{x\} = D_2 \cap \{c, d\}$ and let $\{y\} = D_1 \cap \{e, f\}$.

If $r = 3 = s$, then $f \in D_1$ and $d \in D_2$. But then $(v_2, w_2) \in D_1$ and so $\langle D_2\rangle$ is not connected, a contradiction. Therefore, assume $s \geq 4$. There are three cases to consider. If $r = 3$ and $y = e$, then it follows that $x = c$ and $f \in D_2$. Thus the vertices of any $a - y$ path form a cutset which separates vertices $b$ and $c$. If $r = 3$ and $y = f$, then the vertex set of any $a - y$ path separates vertices $b$ and $d$. If $r \geq 4$, then vertices $b$ and $x$ are separated by the vertex set of any $a - y$ path. In each case, we come to a contradiction with the assumption that $\langle D_2\rangle$ is connected. So, $d_{tr}(G_{r,s}) \leq 1$. Since $G_{r,s}$ is not a tree, it follows that $d_{tr}(G_{r,s}) = 0$. □

**Acknowledgments**
*Supported by National Natural Sciences Foundation of China(19871036).*

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*Received March 28, 2005.*