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NUMERICAL SIMULATION OF LIQUID MOTION IN A PARTLY FILLED TANK

Abstract. The paper presents the problem of liquid motion in a 2D partly filled tank. It is assumed that the flow of liquid in tank is a potential, hence it can be described by Laplace equations with appropriate boundary conditions. The problem is solved using the boundary element method. The developed numerical algorithm makes it possible to determine the free surface elevation, the velocity field and the pressure field during the liquid motion in the tank. The area occupied by liquid is represented by a mesh changing in time. Numerical computations are performed for translatory and rotational motion of the tank. The results of numerical computations are verified by experiment.

Keywords: nonlinear boundary value problems, linear elliptic equations, sloshing, free-surface potential flows.

Mathematics Subject Classification: 35J65, 76B10, 76B07.

1. INTRODUCTION

A ship moving in waves generates motion of liquid in partially filled tanks of the ship. The motion induces change of liquid particle velocity in time and liquid particle acceleration, pressure field and domain occupied by the liquid. The liquid surface striking the tank walls can damage the tank structure. This physical phenomenon, called sloshing, is an important factor in the design of ship structures.

The solution of the problem comprises the determination of:

— the domain occupied by liquid (the position and shape of free surface),
— the velocity and pressure fields in this domain

at each instant of time.

In order to determine the velocity and pressure field in a domain Ω occupied by a liquid, the following methods are applied:

— The Finite Element Method (FEM), which is based on the division of liquid domain into finite elements. The approximate solution of the problem considered is determined on the finite element division and therefore this method requires the updating of the liquid domain division at each time step instant, [2].

— The Finite Difference Method (FDM) applied to Euler equations - this method requires a fine mesh and small time increment. Determination of the grid on the free surface changing in time and imposing the boundary conditions on the grid make problems in this method [1].
— The Boundary Element Method (BEM) – this method is applied to potential flows, represented by a harmonic function, which has its boundary integral representation (the values of a harmonic function is determined by its values on the boundary). This feature enables the problem of seeking the solution inside the liquid domain to be replaced by finding the solution on its boundary [5].

Proper determination of the position of the free surface significantly influences the accuracy of determining the liquid flow. Various methods have been worked out to determine the free surface changing in time, including:

— The Lagrangian Grid Method, which describes the motion of each liquid particle. Lagrangian grid moves with the fluid and automatically tracks free surface. This method cannot track surface that breaks apart or intersects.
— The Arbitrary Lagrangian-Eulerian Method, which is a modified Lagrangian method including regridding techniques. This method is applied for large amplitude surface motion.
— The Surface Height Method - this method is applied to determine low amplitude, shallow water waves and other motions, in which the surface can be described by height H. Time evolution is described by the following equation:

$$\frac{\partial H}{\partial t} = u_z - u_x \frac{\partial H}{\partial x} - u_y \frac{\partial H}{\partial y}. \tag{1}$$

— The Marker and Cell method (MAC), which is based on a fixed Eulerian grid. The location of fluid within the grid is determined by a set of marked particles moving with the liquid. The MAC method has been applied for two-dimensional simulation, because it requires considerable memory, [3].
— The Surface Marked Method – this method is based on marked particles kept only on the surface. This significantly reduces the computation time. The method requires additional conditions if surface breaks apart.
— The Volume of Fluid Method is based on the MAC method. One value only for pressure, velocity etc. is retained within each control volume, the fluid volume fraction is consistent with the resolution of the other flow quantities. Position of free surface is determined by cells partially filled by liquid, [4].

The last three methods require application of complicated numerical methods and normally simpler methods are applied.

This paper presents the formulation of the mathematical problem describing the phenomenon of liquid motion in partly filled tanks and numerical method of solving this problem. The method is based on the assumption that the flow is potential, which enables the Boundary Element Method to be applied to solve the problem comprising the Laplace equation and appropriate boundary conditions on the tank walls and on the free surface. An experiment has been carried out to verify the method used and the results obtained.

2. GENERAL FORMULATION OF THE PROBLEM

The motion of liquid in a tank is described by the initial boundary-value Navier–Stokes or Euler problem. Currently, it is impossible to obtain a direct solution of the problem. The liquid flow in the tank is determined by applying numerical methods. This is possible after significant simplification of the phenomenon considered. Such phenomena as:

— flow separation during motion of liquid in tank,
— solubility of air in liquid moving in tank,
— occurrence of cushions when liquid strikes a tank wall,
— dynamic flow interaction with flexible tank structure,

are neglected in the description of liquid flow in the tank. Additionally, the following assumptions are introduced:

— the liquid density is constant during its motion,
— the pressure on the free surface is equal to the atmospheric pressure.

In this paper, the two-dimensional (2D) flow is considered. Such flow can be used to describe the flow in a three-dimensional tank in which the parameters of the flow are the same in all parallel cross sections of the tank and can be represented by a flow in a single 2D cross section. The tank is partially filled with liquid and the tank cross section can translate and rotate.

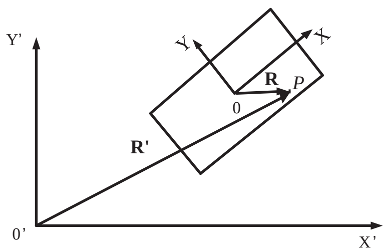

Fig. 1. Reference systems

Two reference systems presented in Figure 1 are introduced:

1) inertial reference system O'X'Y' – the tank motion is described in relation to the system, and
2) non-inertial reference system OXY – fixed to the tank cross section.

The velocity of liquid particle $\mathbf{u}'$ is called absolute velocity and determines the radius $\mathbf{R}'$ changing in time and describing the location of the liquid particle P in relation to the inertial system O'X'Y':

$$\mathbf{u}'(P(t),t) = \frac{d\mathbf{R}'(t)}{dt}, \quad P(t) \in \Omega(t), \quad t \in [t_0, T], \tag{2}$$

where Ω is the domain occupied by the liquid, t_0 is the initial time and T is the assumed period of liquid simulation. The velocity $\mathbf{u}$ is called the relative velocity and determines the radius $\mathbf{R}$ changing in time(location of particle in relation to the non-inertial system OXY):

$$\mathbf{u}(P(t),t) = \frac{d\mathbf{R}(t)}{dt}, \quad P(t) \in \Omega(t), \quad t \in [t_0, T]. \tag{3}$$

The absolute velocity $\mathbf{u}$ is the sum:

$$\mathbf{u}'(P(t),t) = \mathbf{u_e}(P(t),t) + \mathbf{u}(P(t),t), \quad P(t) \in \Omega(t), \quad t \in [t_0, T], \tag{4}$$

where $\mathbf{u_e}$ is velocity of transportation. The tank structure element velocity $\mathbf{u_e}$ is known and depends on the tank motion and position of the element P.

The flow in the tank is described by the following boundary-initial value problem:

— **Equation of liquid motion**. The Navier–Stokes equations describing liquid flow have the following form,

$$\frac{d\mathbf{u}'(P(t),t)}{dt} = \mathbf{U} - \frac{1}{\rho}\nabla\, p(P(t),t) + \nu\Delta\mathbf{u}', \quad P(t) \in \Omega(t), \quad t \in [t_0, T], \tag{5}$$

where $\mathbf{U}$ is the gravity acceleration vector, ρ is the liquid density, ν is the liquid viscosity. The experiments show that the viscosity forces acting in the liquid are small in relation to the inertial forces and can be neglected. This results in the following Euler equations:

$$\frac{d\mathbf{u}'(P(t),t)}{dt} = \mathbf{U} - \frac{1}{\rho}\nabla\, p(P(t),t), \quad P(t) \in \Omega(t), \quad t \in [t_0, T]. \tag{6}$$

— **The equation of mass conservation**. The assumption that the liquid is of constant density ρ leads to the following equation:

$$\nabla \cdot \mathbf{u}'(P(t),t) = 0, \quad P \in \Omega(t), \quad t \in [t_0, T]. \tag{7}$$

Neglecting the flow separation and solubility of air in the liquid enables us to assume that the liquid boundary is described by a continuous surface S. The surface S satisfies the condition:

$$\forall_{t\in[t_0,T]}\ \frac{dS(P(t),t)}{dt} = 0, \quad P(t) \in S(t). \tag{8}$$

— **The boundary conditions**. The boundary $S = \partial\Omega$ of the domain Ω occupied by the liquid was divided into wetted surface S_C and free surface S_F. It is assumed that the liquid particle P slips on the boundary, which can be expressed in the form:

$$\frac{\partial \mathbf{u}(P(t),t)}{\partial \mathbf{n}} = 0, \quad P(t) \in S_C(t), \quad t \in [t_0, T], \tag{9}$$

(the normal relative velocity of the liquid particle equals zero).
The pressure p on the free surface S_F is equal to atmospheric pressure p_a:

$$p(P(t),t) = p_a, \quad P(t) \in S_F(t), \quad t \in [t_0, T]. \tag{10}$$

— **The initial conditions**. It is assumed that the initial relative velocity of fluid is equal to zero and the pressure field is equal to the hydrostatic pressure. The free surface is not disturbed and the pressure on the free surface is equal to the atmospheric pressure p_a.

3. DIFFERENTIAL PROBLEM DESCRIBING THE POTENTIAL LIQUID FLOW IN A PARTLY FILLED TANK

Due to the complexity of the problem presented in Chapter 2, additionally it is assumed that the flow is potential. This means that there exists a potential ϕ' of the absolute velocity $\mathbf{u}'$ of the liquid particle P:

$$\nabla\phi'(P(t),t) \stackrel{def}{=} \mathbf{u}'(P(t),t), \quad P \in \Omega(t), \quad t \in [t_0, T]. \tag{11}$$

For the potential flow equation (6) of liquid motion takes the following form:

$$\frac{\partial \phi'(P(t),t)}{\partial t} + \frac{1}{2}|\mathbf{u}'(P(t),t)|^2 - \mathbf{U} \cdot \mathbf{R}'(t) + \frac{p(P(t),t) - p_a}{\rho} = 0,$$
$$P \in \Omega(t), \quad t \in [t_0, T]. \tag{12}$$

The equation of mass conservation (7) and equations (9), (12) with (10) and (3) constitute the following boundary-value problem with the Laplace equation as the governing equation:

$$\Delta\phi'(P(t),t) = 0, \quad P \in \Omega(t), \quad t \in [t_0, T], \tag{13}$$

$$\frac{\partial \phi'(P(t),t)}{\partial \mathbf{n}} = \mathbf{n}(P(t),t) \cdot \mathbf{u_e}(P(t),t), \quad P(t) \in S_C(t),$$
$$\frac{\partial \phi'(P(t),t)}{\partial t} + \frac{1}{2}|\mathbf{u}'(P(t),t)|^2 - \mathbf{U} \cdot \mathbf{R}'(t) = 0, \quad P(t) \in S_F(t),$$
$$\frac{d\mathbf{R}(P(t))}{dt} = \mathbf{u}(P(t),t), \quad P(t) \in S_F(t).$$

The last condition is used to determine the moving free surface [8] in numerical simulation.

During the simulation the following condition:

$$\int_{\Omega(t)} d\Omega = \text{const} \tag{14}$$

is used to control the correctness of the numerical solution of boundary-value problem (13). Multiplying both sides of the last condition of problem (13) by the normal vector of the free surface and applying formula (4), the following condition of Neumann type is obtained:

$$\frac{\partial \phi'(P)}{\partial \mathbf{n}} = \mathbf{n}(P) \cdot \mathbf{u_e}(P) + \mathbf{n}(P) \cdot \frac{d\mathbf{R}}{dt}, \quad P \in S_F. \tag{15}$$

This results in the following boundary value problem of Neumann type for each instant of time t:

$$\triangle \phi'(P) = 0, \quad P \in \Omega, \tag{16}$$

$$\frac{\partial \phi'(P)}{\partial \mathbf{n}} = \mathbf{n}(P) \cdot \mathbf{u_e}(P), \quad P \in S_C,$$
$$\frac{\partial \phi'(P)}{\partial \mathbf{n}} = \mathbf{n}(P) \cdot \left(\mathbf{u_e}(P) + \frac{d\mathbf{R}}{dt}\right), \quad P \in S_F.$$

The internal Neumann boundary-value problem has solutions which differ by a constant value [6], but the velocity field **u'**=$\nabla\phi$ ' is uniquely determined.

Problem (13) was solved numerically according to the following algorithm:

A1. Determine the shape of the free surface in the fixed coordinate system from the equations,

$$\frac{dx(t)}{dt} = \frac{\partial \phi'(P(t),t)}{\partial x} - u_{ex}(P(t),t), \tag{17}$$

$$\frac{dy(t)}{dt} = \frac{\partial \phi'(P(t),t)}{\partial y} - u_{ey}(P(t),t), \quad P(t) \in S_F(t),$$

where (x, y) are the coordinates of the liquid particle P in the reference system OXY fixed to the tank.

A2. Determine the potential ϕ_F on the free surface from equation (12),

$$\frac{d\phi_F(P(t),t)}{dt} = \frac{1}{2}\left(\frac{\partial \phi_F(P(t),t)}{\partial x'}\right)^2 + \frac{1}{2}\left(\frac{\partial \phi_F(P(t),t)}{\partial y'}\right)^2 + \mathbf{U} \cdot \mathbf{R}'(t), \tag{18}$$

where ϕ_F is the value of velocity potential ϕ' on the free surface S_F.

A3. Solve the Laplace problem for the new liquid domain Ω, determined in step A1, and for a new value ϕ'_F of potential ϕ' on free surface, determined in step A2:

$$\Delta\phi'(P) = 0, \quad P \in \Omega, \tag{19}$$

$$\frac{\partial\phi'(P)}{\partial\mathbf{n}} = \mathbf{n}(P)\cdot\mathbf{u_e}(P), \quad P \in S_C,$$
$$\phi'(P) = \phi_F(P), \quad P \in S_F.$$

A4. Determine the velocity field in the liquid domain Ω according to formula (11).
A5. Determine the pressure field from equation (12).

Steps A1 and A2 are computed using the Runge–Kutta method. The new position of free surface determines the new shape of domain Ω occupied by the liquid and new coordinates of the grid (Fig. 2).

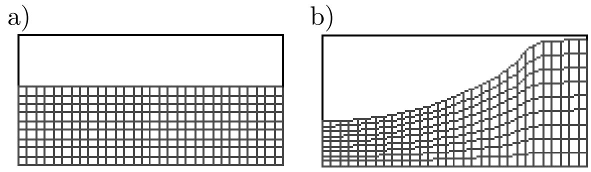

Fig. 2. The mesh approximating the liquid domain at the initial time (a) and during the simulation(b)

The numerical simulation of the free surface motion is stable, provided the Courant–Friedrichs–Lewy condition is satisfied, [7]. This condition says that the particle velocity is smaller than the velocity of the mesh $\Delta x/\Delta t$, where Δx denotes the distance between nodes of the mesh and Δt is the time increment.

In step A3, Laplace problem (19) is solved after translating problem (16) into the following Fredholm equation of second kind:

$$\mu(X_k)\phi'(X_k) = -\int\limits_{Y\in S}\phi'(Y)\frac{\partial\ln|X_k - Y|}{\partial n_y}dl + \int\limits_{Y\in S}\frac{\partial\phi'(Y)}{\partial n_y}\ln|X_k - Y|dl,$$
$$Y \in S = \bigcup_{l=1..m}[X_l, X_{l+1}], \quad X_k \in S, \quad k = 1..m, \tag{20}$$

where X_k are endpoints of polyline approximating the boundary S of liquid domain and m is the number of endpoints.

This equation is solved using the boundary element method presented in [9]. There was a problem with uniquely determining the boundary condition at the points joining the free surface and wetted surface of the tank, while numerically solving of the Laplace problem with mixed boundary conditions.

The velocity potential ϕ' is a continuous function on the boundary, and the normal derivate of the potential is not continuous at the points joining the free surface S_F and wetted surface S_C.

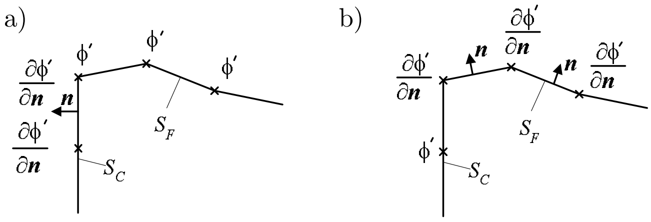

Fig. 3. The given (a) and unknown (b) value in the endpoints of polyline approximating the liquid boundary S

The values of the velocity potential are given on the free surface S_F, while the values of the normal derivatives of the potential are given on the sections of the polyline describing the wetted surface S_C (see Fig. 3). At the endpoints of the polyline describing the free surface, the value of normal derivate of the potential is the unknown value. The problem can be easily solved numerically using equation (20).

The values of the potential and its normal derivative at the liquid boundary S, determined from equation (20), enables the values of the potential to be computed inside the liquid domain Ω according to the formula:

$$2\pi\phi'(X) = -\int\limits_{Y\in S} \phi'(Y)\frac{\partial \ln|X-Y|}{\partial n_y}dl + \int\limits_{Y\in S} \frac{\partial\phi'(Y)}{\partial n_y}\ln|X-Y|dl,$$

$$Y \in S = \bigcup_{l=1..m} [X_l, X_{l+1}], \quad l = 1..m, \quad X \in \Omega\backslash S. \qquad (21)$$

4. EXPERIMENTAL VERIFICATION

The computer program TANK, developed based on the algorithm presented, enables the simulation of the liquid motion in a partially filled tank. The geometry of the tank, the extent to which it is filled and the tank's motion constitute the input data. The program simulates the moving free surface and the velocity and pressure fields changing in time. The algorithm and the program have been verified by experiment [8]. Examples of this verification are presented in Figures 4, 5 and 6.

The verification of simulation of the water motion in a tank filled up to 20 % of its height is presented in Figure 4. The tank is making horizontal harmonic motions with amplitude equal to 0,1 of the tank length and rotating motion with the amplitude equal to 2°. Experimental verification of the simulation of water motion is presented in Figure 5. The tank is performing horizontal harmonic motion with the amplitude equal to 0,0125 of the tank length. The tank is filled up to 60% of its height.

The comparison of pressure changing in time in the right bottom corner recorded during the experiment and obtained in simulations is presented in Figure 6.

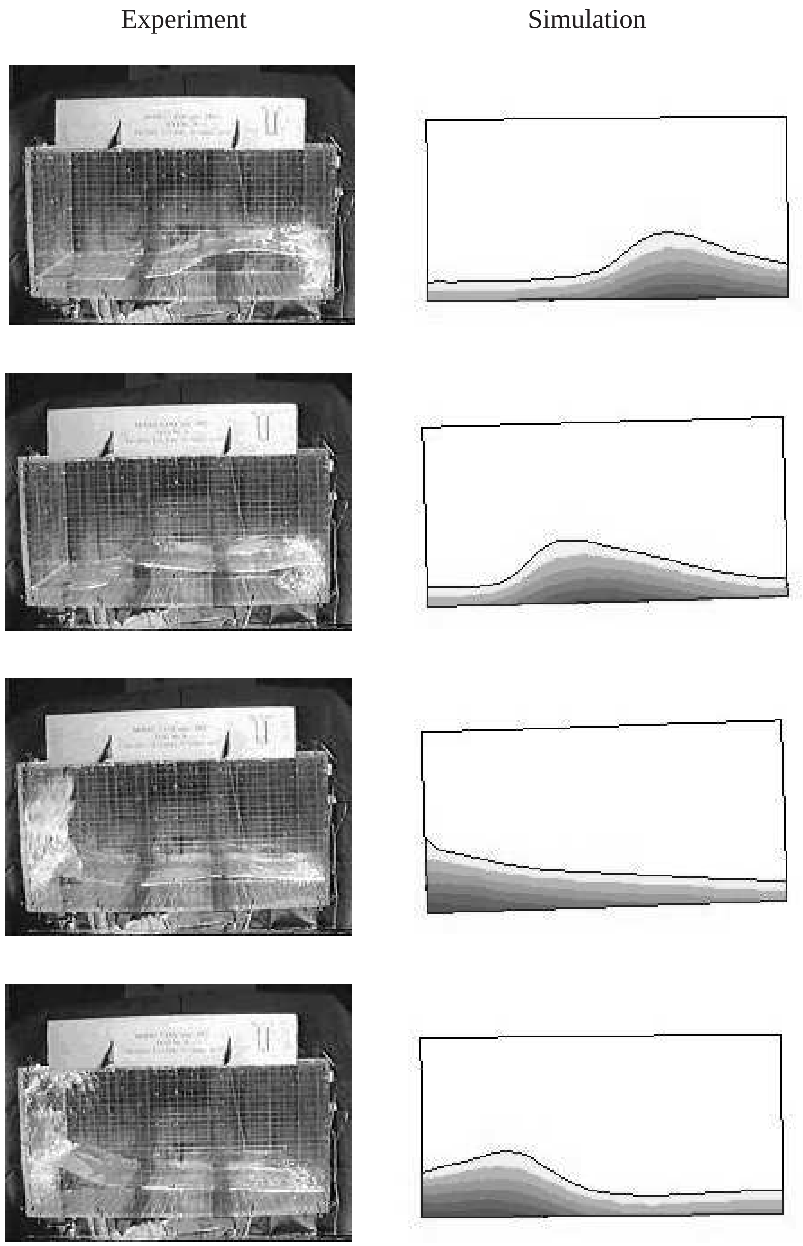

Fig. 4. Verification of water motion in a tank filled up to 20% of its height

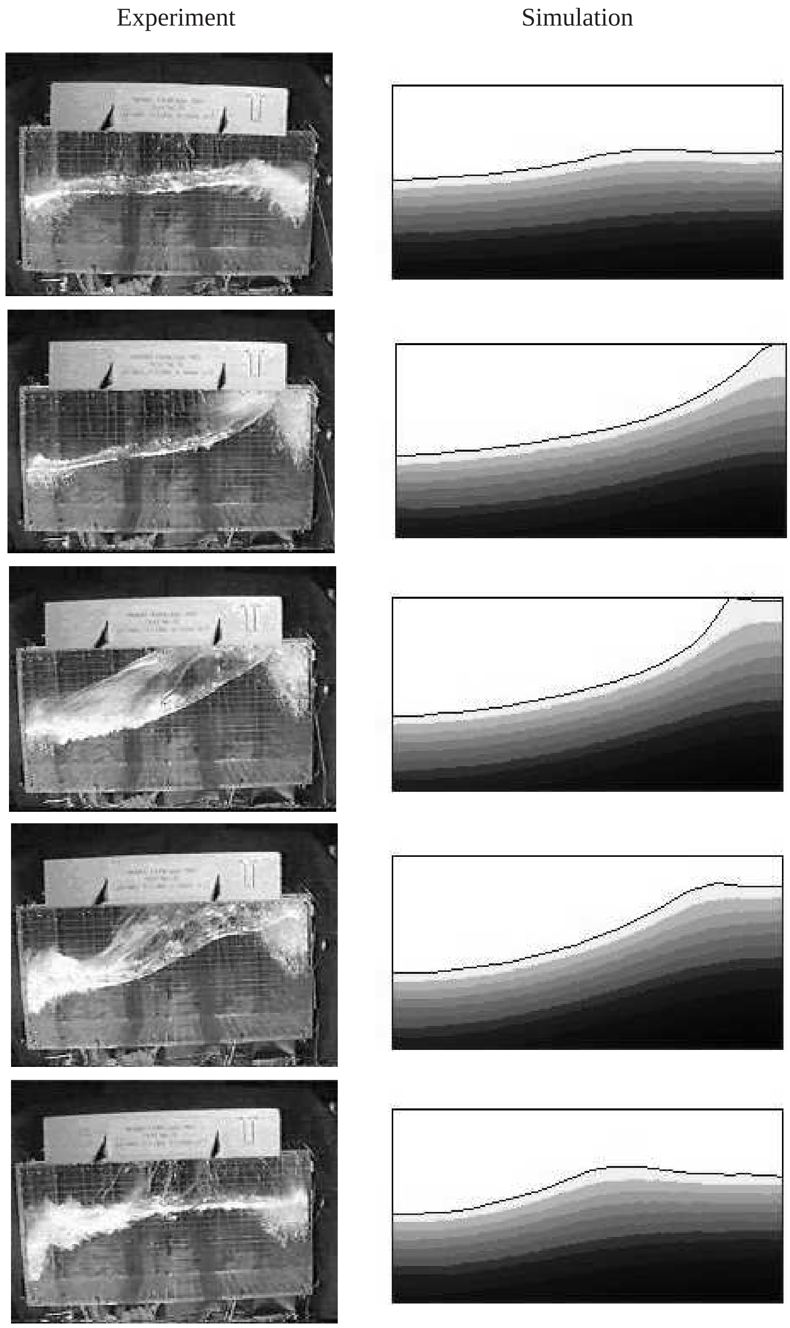

Fig. 5. Verification of water motion in a tank filled up to 60% of its height

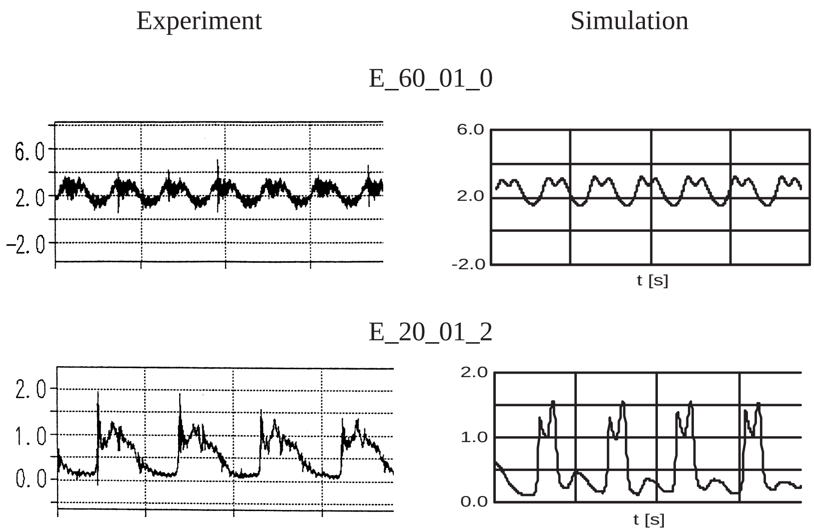

Fig. 6. Comparison of pressure recorded in the experiment and obtained in simulations

5. CONCLUSIONS

The problem of determining liquid motion in a partly filled tank has been solved using a numerical method. This method is based on the precise determination of the liquid boundary and velocity potential values and its normal derivatives at the boundary changing in time. The values on the boundary determine the velocity field of liquid particles and the pressure field. The assumption that the flow is potential means that the well established boundary element method can be applied to solve the problem.

A two dimensional problem has been considered. The experiment conducted to verify the presented method of liquid motion simulation in a partly filled tank shows that the free surface flow and pressure inside the tank are correctly simulated. The simulation of free surface flow is very sensitive to the algorithm used and numerical method applied to solve the problem considered.

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