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ASYMPTOTIC STABILITY OF A NEUTRAL INTEGRO-DIFFERENTIAL EQUATION

Abstract. The global stability behavior of a non-autonomous neutral functional integro-differential equation is studied. A sufficient condition for every solution of this equation to tend to zero is given.

Keywords: asymptotic behavior, nonlinear neutral integro-differential equation.

Mathematics Subject Classification: 34K10, 34C25.

1. INTRODUCTION

The following delay equation

$$x'(t) + a(t)x(t-\tau) = 0,\ t \geq 0,$$

where a is a continuous function on $[0, \infty)$ and τ is a nonnegative number, is well known in population models, and numerous its properties have been investigated. In particular, it has been shown that if $\sup_{t>0} \int_{t-\tau}^{t} a(s)ds \leq 3/2$, then the zero solution is uniformly stable [1]. There are now several extensions and/or variations of this result. For instance, in [2], the authors have obtained the global attractivity properties of integro-differential equations of the form

$$x'(t) = -\int_{t-r(t)}^{t} \sum_{i=1}^{n} f_i(t, x(s))\, d\mu_i(t, s). \tag{1}$$

To the best of our knowledge, however, the more general integro-differential equation with a neutral term

$$(x(t) + cx(t-\sigma))' = -\int_{t-r(t)}^{t} \sum_{i=1}^{n} f_i(t, x(s))\, d\mu_i(t, s),\ t \geq 0, \tag{2}$$

where $c \in (-1, 0]$ and $\sigma > 0$, has not been considered. Such an equation is a meaningful mathematical model since the term $cx'(t-\sigma)$ stands for the depletion rate of the state variable at time $t - \sigma$.

In this note, we will study this equation under the conditions that the real valued functions $f_1, \ldots, f_n$ and r are continuous, while $\mu_1, \ldots, \mu_n$ are continuous with respect to their first variables and nondecreasing with respect to their second variables. The domain of f_i is taken to be $[0, \infty) \times R$, that of r is $[0, \infty)$ and that of μ_i is R^2. As in [2], we additionally assume that

(H_1) each $f_i(t, x)$ is odd with respect to x, $xf_i(t, x) \geqslant 0$ and $\sum_{i=1}^n f_i(t, x) = 0$ if and only if $x = 0$;
(H_2) $r(0) \geq \sigma$, $r(t) > 0$, $t - r(t)$ is nondecreasing in t, and $t - r(t) \to \infty$ as $t \to \infty$;
(H_3) $\mu_i(t, t) > \mu_i(t, t - r(t))$.

The definitions of a solution, eventually positive solution, eventually negative solution, oscillatory solution and nonoscillatory solution are similar to those in [2] or [3], and hence omitted. Our main result is the following theorem.

Theorem 1. *Assume that each $f_i(t, x)$ is nondecreasing with respect to x and $|f_i(t, x)|$ is nondecreasing with respect to $|x|$, and*

$$|f_i(t, x)| \leq a_i(t)\,|x| \quad \text{for } t \geq 0 \text{ and } x \in R, \tag{3}$$

where each a_i is a nonnegative continuous function on $[0, \infty)$. If

$$\mu \equiv \limsup_{t \to \infty} \int_{t-r(t)}^{t} \sum_{i=1}^{n} a_i(t)\left[\mu_i(\tau, \tau) - \mu_i(\tau, \tau - r(\tau))\right] d\tau < \frac{3}{2} + 3c, \tag{4}$$

then every solution of (2) tends to a constant. If in addition, for some $v \neq 0$,

$$\int_0^\infty \sum_{i=1}^n f_i(\tau, v)\left[\mu_i(\tau, \tau) - \mu_i(\tau, \tau - r(\tau))\right] d\tau = \infty, \tag{5}$$

then all solutions of (2) tend to zero as $t \to \infty$.

We first remark that Theorem 2.1 in [2] is our Theorem 1 in the case of $c = 0$. Furthermore, there are several other special cases that may be of interest. First, consider the case of $f_i(t, x(s)) = x(s)$. Then (2) becomes

$$(x(t) + cx(t-\sigma))' = -\int_{t-r(t)}^{t} x(s)\, d\mu(t, s), \tag{6}$$

where $\mu(t, s) = \sum_{i=1}^n \mu_i(t, s)$. Applying our Theorem 1, we obtain the following corollary.

Corollary 1. *Assume that*

$$\limsup_{t\to\infty} \int_{t-r(t)}^{t} \left[\mu(\tau,\tau) - \mu(\tau, \tau - r(\tau))\right] d\tau < \frac{3}{2} + 3c. \tag{7}$$

Then every solution of (6) *tends to a constant as* $t \to \infty$*. If in addition,*

$$\int_{0}^{\infty} \left[\mu(\tau,\tau) - \mu(\tau, \tau - r(\tau))\right] d\tau = \infty, \tag{8}$$

then every solution of (6) *tends to zero as* $t \to \infty$.

The special case of $c = 0$ in (6) was investigated in Haddock and Kuang [3]. Our Corollary 1 extends and improves their corresponding results.

Next, consider the special case

$$(x(t) + cx(t-\sigma))' = -\sum_{i=0}^{n} a_i(t)\, x(t - r_i(t)), \tag{9}$$

where each a_i is nonnegative and continuous on $[0,\infty)$, and, $r_0(t) = 0$ and $0 < r_i(t) < r_{i+1}(t) \le r(t)$ for $t \ge 0$ and $i = 1, 2, \ldots, n-1$.

Corollary 2. *Assume that*

$$\limsup_{t\to\infty} \sum_{i=0}^{n} \int_{t-r(t)}^{t} a_i(s)\, ds < \frac{3}{2} + 3c. \tag{10}$$

Then every solution of (9) *tends to a constant as* $t \to \infty$*. If, in addition,*

$$\int_{0}^{\infty} \sum_{i=0}^{n} a_i(t)\, dt = \infty, \tag{11}$$

then every solution of (9) *tends to zero as* $t \to \infty$.

The special case of $c = 0$ of (9) was investigated in [3]. Our Corollary 2 extends the corresponding results in [3].

2. PROOF

The proof of our main result will be follow easily from the following lemmas.

Lemma 1. *Let* $x(t)$ *be a nonoscillatory solution of* (2) *and* $u(t) = x(t) + cx(t-\sigma)$. *Then the limit*

$$\lim_{t\to\infty} u(t) = b \tag{12}$$

exists. Furthermore, if $x(t)$ *is eventually positive, then* $b \geqslant 0$*, while if* $x(t)$ *is eventually negative, then* $b \le 0$.

Proof. We may assume that $x(t)$ is an eventually positive solution of (2), since the other case can be proved similarly. Then in view of (2), we see that $u'(t) \le 0$ eventually. Thus $\lim_{t\to\infty} u(t) = b \in R$ or $\lim_{t\to\infty} u(t) = -\infty$. If $\lim_{t\to\infty} u(t) = -\infty$ or $b < 0$, then

$$x(t) + cx(t-\sigma) < 0 \tag{13}$$

eventually. We see that, for sufficiently large n,

$$0 < x(n\sigma) \le (-c)^n x(\sigma). \tag{14}$$

Thus, $\lim_{n\to\infty} x(n\sigma) = 0$. Since $u(n\sigma) = x(n\sigma) + cx((n-1)\sigma)$, we further see that $\lim_{i\to\infty} u(n\sigma) = 0$. This leads us to a contradiction. Thus $\lim_{t\to\infty} u(t) = b \geqslant 0$. The proof is complete. □

Lemma 2. *Let $x(t)$ be a nonoscillatory solution of* (2) *and $u(t) = x(t) + cx(t-\sigma)$. Then $\lim_{t\to\infty} x(t) = b/(1+c)$, where $b = \lim_{t\to\infty} u(t)$.*

Proof. We may assume that $x(t)$ is an eventually positive solution of (2). We assert that $x(t)$ is bounded. Otherwise, there would exist an integer sequence $\{t_i\}$ with $t_i \to \infty$ for $i \to \infty$ such that

$$\lim_{i\to\infty} x(t_i) = \infty$$

and

$$x(t) \le x(t_i), 0 < t \le t_i.$$

On the other hand, there is, eventually,

$$u(t_i) = x(t_i) + cx(t_i - \sigma) \ge (1+c)x_{t_i} \to \infty \text{ as } i \to \infty.$$

This is contradicts the assumption that $\lim_{t\to\infty} u(t) = b$. Thus $x(t)$ is bounded.

Let $\limsup_{t\to\infty} x(t) = Q$ and $\liminf_{n\to\infty} x(t) = q$. Then $0 \le q \le Q < \infty$. Moreover, there exist $\{t_s\}$ and $\{\bar{t}_s\}$: $\lim_{s\to\infty} t_s = \infty, \lim_{s\to\infty} \bar{t}_s = \infty$ such that $\lim_{s\to\infty} x(t_s) = Q$ and $\lim_{s\to\infty} x(\bar{t}_s) = q$. Since

$$b = \lim_{s\to\infty} u(t_s) = \lim_{s\to\infty} (x(t_s) + cx(t_s - \sigma)) \ge \limsup_{s\to\infty} x(t_s) + \liminf_{s\to\infty} cx(t_s - \sigma) \ge Q + cQ,$$

and

$$b = \lim_{s\to\infty} u(\bar{t}_s) = \lim_{s\to\infty} \left(x(\bar{t}_s) + cx(\bar{t}_s - \sigma)\right) \le \lim_{s\to\infty} x(\bar{t}_s) + \lim_{s\to\infty} \sup cx\left(\bar{t}_s - \sigma\right) \le q + cq,$$

there follows $(1+c)q \ge (1+c)Q$. It follows that $q = Q = \lim_{t\to\infty} x(t)$. In view of $u(t) = x(t) + cx(t-\sigma)$ and $\lim_{t\to\infty} u(t) = b$, there is

$$\lim_{n\to\infty} x(t) = \frac{b}{1+c}.$$

The proof is complete. □

Lemma 3. *Every nonoscillatory solution $x(t)$ of (2) tends to a constant. If in addition, (3) and (5) hold, then every nonoscillatory solution of (2) tends to zero as $t \to \infty$.*

Proof. We may assume that $x(t)$ is eventually positive. Let $u(t) = x(t) + cx(t-\sigma)$. From Lemma 1 and Lemma 2, $\lim_{t\to\infty} u(t) = b \in R$ and $b \geqslant 0$ and $\lim_{t\to\infty} x(t) = b/(1+c)$. If, in addition, (3) and (5) hold, we may assert that $b = 0$. Otherwise, if $b > 0$, then by setting $\alpha = b/2(1+c)$, there exists $T > 0$ sufficiently large for the following inequalities to hold:

$$x(t) \geqslant \alpha \text{ and } x(t - r(t)) \geqslant \alpha \text{ for } t \geqslant T. \tag{15}$$

Thus,

$$f_i(t, x(s)) = |f_i(t, x(s))| \geqslant |f_i(t, \alpha)| \text{ for } s \geqslant T - r(T). \tag{16}$$

Substituting this into the right hand side of (2), we get

$$u'(t) \leq \sum_{i=1}^{n} |f_i(t,\alpha)| \left[\mu_i(t,t) - \mu_i(t, t - r(t))\right] \text{ for } t \geqslant T \tag{17}$$

which, together with (5), yield $\lim_{t\to\infty} u(t) = -\infty$. This contradiction shows that $b = 0$ and so $\lim_{t\to\infty} x(t) = b/(1+c) = 0$. The proof is complete. □

Lemma 4. *Assume that (3) and (4) hold. Then every oscillatory solution $x(t)$ of (2) tends to zero as $t \to \infty$.*

Proof. Our proof is modelled after that of Theorem 2.1 in [2], but there are sufficient differences to call for a careful presentation. Let $x(t)$ be an oscillatory solution of (2). Let $u(t) = x(t) + cx(t-\sigma)$. Note that $\mu \geq 0$ in (5) since $a_i(t) \geq 0$ and $\mu_i(t,t) > \mu_i(t, t-r(t))$. Thus $1 + 2c > \mu/3 \geq 0$ by (4).

We first prove that $x(t)$ is bounded. Suppose the contrary holds. Then there is a sufficiently $T > \sigma$ large such that for $t \geqslant T$,

$$\max_{\sigma \leq s \leq t} |x(s)| = \max_{0 \leq s \leq t} |x(s)|. \tag{18}$$

Note that for $s \geqslant 0$,

$$x(s) = u(s) - cx(s-\sigma), \tag{19}$$

Thus we have for $t \geqslant T$,

$$\begin{aligned} \max_{0 \leq s \leq t} |x(s)| = \max_{\sigma \leq s \leq t} |x(s)| &\leq \max_{\sigma \leq s \leq t} |u(s)| - c \max_{\sigma \leq s \leq t} |x(s-\sigma)| \leq \\ &\leq \max_{0 \leq s \leq t} |u(s)| - c \max_{0 \leq s \leq t} |x(s)|. \end{aligned} \tag{20}$$

It follows that

$$\max_{0 \leq s \leq t} |x(s)| \leq \frac{\max_{0 \leq s \leq t} |u(s)|}{1+c} \text{ for } t \geqslant T, \tag{21}$$

and hence $u(t)$ is unbounded. Let $\varepsilon > 0$ be such that $1 < \mu - 2c + \varepsilon < 3/2 + c$. Also let $T_1 > T$ such that $t - r(t) \geqslant T$ for $t \geqslant T_1$ and

$$\int_{t-r(t)}^{t} \sum_{i=1}^{n} a_i(t)\left[\mu_i(\tau,\tau) - \mu_i(\tau,\tau - r(\tau))\right] d\tau \leq \mu + \varepsilon,\ t \geqslant T_1. \tag{22}$$

Set

$$A(t) = \sum_{i=1}^{n} a_i(t)\left[\mu_i(t,t) - \mu_i(t,t-r(t))\right]. \tag{23}$$

Then

$$\int_{t-r(t)}^{t} A(\tau)\, d\tau \leq \mu + \varepsilon \text{ for } t \geqslant T_1. \tag{24}$$

Since $x(t)$ is oscillatory, $u(t)$ cannot be eventually monotone. Indeed, if $u(t)$ is monotone, then $|u(t)|$ is eventually monotone. Since $u(t)$ is unbounded, $|u(t)|$ must then be eventually increasing and $\lim_{t\to\infty}|u(t)| = \infty$. Choose $t' > T$ such that $x(t') = 0$ and $x(t)$ is not identically vanishing on $[0,t')$ and $\max_{0\le s\le t'}|u(s)| = |u(t')|$. From (21), there follows

$$(1+c)\max_{0\le s\le t'}|x(s)| \le \max_{0\le s\le t'}|u(s)| = |u(t')| = |cx(t'-\sigma)| \le -c\max_{0\le s\le t'}|x(s)|, \tag{25}$$

which implies $1 + 2c \le 0$. This is a contradiction. Thus, $u(t)$ is not eventually monotone. But since $u(t)$ is unbounded, there must exist $t^* - r(t^*) > T_1$ such that $|u(t)| < |u(t^*)|$ for $-r(0) \le t < t^*$ and $u'(t^*) = 0$. Without loss of generality, we may assume that $u(t^*) = x(t^*) + cx(t^* - \sigma) > 0$. If $x(t^*) \le 0$, then $u(t^*) \le cx(t^* - \sigma)$ and using (18), we derive

$$u(t^*) \le -c\max_{0\le s\le t^*}|x(s)|. \tag{26}$$

By (21) and (26), there is

$$(1+c)\max_{0\le s\le t^*}|x(s)| \le u(t^*) \le -c\max_{0\le s\le t^*}|x(s)|, \tag{27}$$

hence that $1 + 2c \le 0$, which is impossible. Thus $x(t^*) > 0$. By (2),

$$\int_{t^*-r(t^*)}^{t^*} \sum_{i=1}^{n} f_i(t, x(s))\, d\mu_i(t,s) = 0,$$

which implies that there exists $t_0 \in (t^* - r(t^*), t^*)$ such that $x(t_0) = 0$ and $x(t) > 0$ for $(t_0, t^*]$. From (3), there follows

$$-f_i(t, x(s)) \le a_i(t)\max_{0\le s\le t^*}|x(s)| \text{ for } t > 0. \tag{28}$$

Thus, from (2), we derive

$$u'(t) = -\int_{t-r(t)}^{t} \sum_{i=1}^{n} f_i(t, x(s))\, d\mu_i(t,s) \le \max_{0\le s\le t^*} |x(s)| \int_{t-r(t)}^{t} \sum_{i=1}^{n} a_i(t)\, d\mu_i(t,s) =$$
$$= \max_{0\le s\le t^*} |x(s)| \sum_{i=1}^{n} a_i(t)\left[\mu_i(t,t) - \mu_i(t, t-r(t))\right] \le \frac{u(t^*)}{1+c} A(t), \tag{29}$$

for $t^* \geqslant t \geqslant T_1$. For $T_1 \le s \le t_0$, by integrating (29) from s to t_0, and using (18) and (21), we get

$$\begin{aligned} -x(s) &\le \frac{u(t^*)}{1+c}\int_s^{t_0} A(s)\,ds - cx(t_0-\sigma) + cx(s-\sigma) \le \\ &\le \frac{u(t^*)}{1+c}\int_s^{t_0} A(s)\,ds - 2c \max_{0\le v\le t_0-\sigma} |x(v)| \le \\ &\le \frac{u(t^*)}{1+c}\int_s^{t_0} A(s)\,ds - 2c \max_{0\le v\le t_0} |x(v)| \le \\ &\le \frac{u(t^*)}{1+c}\int_s^{t_0} A(s)\,ds - 2c \max_{0\le v\le t^*} |x(v)| \le \frac{u(t^*)}{1+c}\left[\int_s^{t_0} A(s)\,ds - 2c\right]. \end{aligned} \tag{30}$$

On the other hand, for $t_0 \le t \le t^*$, from (H_1), (2) and (30), we conclude that

$$\begin{aligned} u'(t) &= -\int_{t-r(t)}^{t} \sum_{i=1}^{n} f_i(t,x(s))\,d\mu_i(t,s) = \\ &= -\int_{t_0}^{t} \sum_{i=1}^{n} f_i(t,x(s))\,d\mu_i(t,s) - \int_{t-r(t)}^{t_0} \sum_{i=1}^{n} f_i(t,x(s))\,d\mu_i(t,s) \le \\ &\le -\int_{t-r(t)}^{t_0} \sum_{i=1}^{n} f_i(t,x(s))\,d\mu_i(t,s) \le \\ &\le \frac{u(t^*)}{1+c}\left\{\int_{t-r(t)}^{t_0} \sum_{i=1}^{n} a_i(t)\left[\int_s^{t_0} A(s)\,ds - 2c\right] d\mu_i(t,s)\right\} = \\ &= \frac{u(t^*)}{1+c}\left\{\int_{t-r(t)}^{t_0} \left(\int_{t-r(t)}^{\tau} \sum_{i=1}^{n} a_i(t)\,d\mu_i(t,s)\right) A(\tau)\,d\tau - \right. \\ &\qquad \left. - 2c\int_{t-r(t)}^{t_0} \sum_{i=1}^{n} a_i(t)\,d\mu_i(t,s)\right\} \le \\ &\le \frac{u(t^*)}{1+c}\left\{\sum_{i=1}^{n} a_i(t)\left(\mu_i(t,t) - \mu_i(t,t-r(t))\right)\left(\int_{t-r(t)}^{t_0} A(\tau)\,d\tau - 2c\right)\right\} \le \\ &\le \frac{u(t^*)}{1+c} A(t)\left\{\int_{t-r(t)}^{t_0} A(\tau)\,d\tau - 2c\right\}. \end{aligned} \tag{31}$$

Thus, for $t_0 \leq t \leq t^*$, there holds

$$u'(t) \leq \frac{u(t^*)}{1+c} \min \left\{ A(t), A(t) \left\{ \frac{u(t^*)}{1+c} \left(\int_{t-r(t)}^{t_0} A(\tau)\, d\tau - 2c \right) \right\} \right\}. \tag{32}$$

There are two cases to consider:

Case 1. $\int_{t_0}^{t^*} A(t)\, dt \leq 1$. Then by (32),

$$\begin{aligned} u(t^*) &\leq \frac{u(t^*)}{1+c} \int_{t_0}^{t^*} A(t) \left\{ \int_{t-r(t)}^{t_0} A(\tau)\, d\tau - 2c \right\} dt = \\ &= \frac{u(t^*)}{1+c} \int_{t_0}^{t^*} A(t) \left\{ \int_{t-r(t)}^{t} A(\tau)\, d\tau - \int_{t_0}^{t} A(\tau)\, d\tau - 2c \right\} dt \leq \\ &\leq \frac{u(t^*)}{1+c} \int_{t_0}^{t^*} A(t) \left\{ \int_{t-r(t)}^{t} A(\tau)\, d\tau - \int_{t_0}^{t} A(\tau)\, d\tau - 2c \right\} dt \leq \\ &\leq \frac{u(t^*)}{1+c} \int_{t_0}^{t^*} A(t) \left\{ \mu - 2c + \varepsilon - \int_{t_0}^{t} A(\tau)\, d\tau \right\} dt \leq \\ &\leq \frac{u(t^*)}{1+c} \left\{ (\mu - 2c + \varepsilon) \int_{t_0}^{t^*} A(t)\, dt - \frac{1}{2} \left(\int_{t_0}^{t^*} A(t)\, dt \right)^2 \right\} \leq \\ &\leq \frac{u(t^*)}{1+c} \left(\mu - 2c + \varepsilon - \frac{1}{2} \right) < u(t^*), \end{aligned} \tag{33}$$

since the function $g(x) = (\mu - 2c + \varepsilon)x - \frac{1}{2}x^2$ is increasing for $x \leq \mu - 2c + \varepsilon$ and $\mu - 2c + \varepsilon > 1$. This yields a contradiction.

Case 2. $\int_{t_0}^{t^*} A(t)\, dt > 1$. Then there exists $\bar{t} \in (t_0, t^*)$ such that

$$\int_{\bar{t}}^{t^*} A(t)\, dt = 1 \tag{34}$$

and

$$u'(t) \leq \frac{u(t^*)}{1+c} \min \left\{ A(t), A(t) \left\{ \frac{u(t^*)}{1+c} \int_{t-r(t)}^{t_0} A(\tau)\, d\tau - 2c \right\} \right\}.$$

Thus,

$$
\begin{aligned}
u(t^*) &\le \frac{u(t^*)}{1+c}\left\{\int_{t_0}^{\bar t} A(t)\,dt + \int_{\bar t}^{t^*} A(t)\left\{\int_{t-r(t)}^{t_0} A(\tau)\,d\tau - 2c\right\}dt\right\} = \\
&= \frac{u(t^*)}{1+c}\left\{\int_{\bar t}^{t^*} A(t)\int_{t_0}^{\bar t} A(\tau)\,d\tau dt + \int_{\bar t}^{t^*} A(t)\left\{\int_{t-r(t)}^{t_0} A(\tau)\,d\tau - 2c\right\}dt\right\} = \\
&= \frac{u(t^*)}{1+c}\left\{\int_{\bar t}^{t^*} A(t)\int_{t-r(t)}^{\bar t} A(\tau)\,d\tau dt - 2c\int_{\bar t}^{t^*} A(t)\,dt\right\} \le \\
&\le \frac{u(t^*)}{1+c}\left\{\int_{\bar t}^{t^*} A(t)\left(\mu+\varepsilon-\int_{\bar t}^{t} A(\tau)\,d\tau\right)dt - 2c\int_{\bar t}^{t^*} A(t)\,dt\right\} = \qquad (35) \\
&= \frac{u(t^*)}{1+c}\left\{(\mu+\varepsilon-2c)\int_{\bar t}^{t^*} A(t)\,dt - \int_{\bar t}^{t^*} A(t)\int_{\bar t}^{t} A(\tau)\,d\tau dt\right\} \le \\
&\le \frac{u(t^*)}{1+c}\left\{(\mu-2c+\varepsilon)\int_{t_0}^{t^*} A(t)\,dt - \frac{1}{2}\left(\int_{\bar t}^{t^*} A(t)\,dt\right)^2\right\} \le \\
&\le \frac{u(t^*)}{1+c}\left(\mu-2c+\varepsilon-\frac{1}{2}\right) < u(t^*),
\end{aligned}
$$

which is again a contradiction.

Hence $x(t)$ is bounded, and so $u(t) = x(t) + cx(t-\sigma)$ is bounded. Hence if we let $\lambda = \limsup_{n\to\infty}|x(t)|$ and $\nu = \limsup_{n\to\infty}|u(t)|$, then $0 \le \lambda, \nu < \infty$ and from $x(t) = u(t) - cx(t-\sigma)$,

$$\lambda \le \frac{\nu}{1+c}. \qquad (36)$$

Next, we show that $\lim_{t\to\infty} x(t) = 0$. If suffices to show that $\lambda = 0$. Suppose to the contrary that $\lambda > 0$; then there is $S > 0$ such that

$$|x(t-r(t))| < \lambda + \eta \text{ and } |x(t-\sigma)| < \lambda + \eta \text{ for } t \geqslant S, \qquad (37)$$

where η is some positive number. Since $x(t)$ is oscillatory, we may assert that $u(t)$ is not eventually monotone. Otherwise, $|u(t)|$ would eventually be monotone and $\lim_{t\to\infty}|u(t)| = \nu$. Let $\{t'_n\}$ be an increasing infinite sequence such that $\lim_{n\to\infty} t'_n = \infty$ and $x(t'_n) = 0$. Then

$$\nu = \lim_{n\to\infty}|u(t'_n)| = \lim_{n\to\infty}|cx(t'_n-\sigma)| \le -c\limsup_{n\to\infty}|x(t)| = -c\lambda. \qquad (38)$$

It is easy to see from (36) and (38) that $1+2c \le 0$, which is impossible. Thus, $u(t)$ is not eventually monotone. Since $u(t)$ is not eventually monotone, there is an increasing infinite sequence $\{t_n\}$ such that $\lim_{n\to\infty} t_n = \infty$, $t_n - r(t_n) > S$, $|u(t_n)| \to \nu$ as $n \to \infty$, and $u'(t_n) = 0$. Without loss of generality, we assume $u(t_n) > 0$ for $n \ge 1$.

We may then assert that there is a t_m such that $x(t_m) > 0$. Suppose to the contrary that $x(t_n) \le 0$ for $n \ge 1$. Then

$$u(t_n) = cx(t_n - \sigma) \le -c|x(t_n - \sigma)| < -c(\lambda + \eta) \quad \text{for } n \ge 1. \tag{39}$$

It follows that

$$\nu \le -c(\lambda + \eta). \tag{40}$$

By letting $\eta \to 0$, we get

$$\nu \le -c\lambda. \tag{41}$$

From (36) and (41), we see that $1 + 2c \le 0$, which is impossible. Thus there is a t_m such that $x(t_m) > 0$. By (2), there follows

$$\int_{t_m - r(t_m)}^{t_m} \sum_{i=1}^{n} f_i(t, x(s))\, d\mu_i(t, s) = 0, \tag{42}$$

and, hence, there exists $\xi_m \in (t_m - r(t_m), t_m)$ such that $x(\xi_m) = 0$ and $x(t) > 0$ for $t \in (\xi_m, t_m]$. For $S \le s \le \xi_m$, from (3) there follows

$$-f_i(t, x(s)) \le a_i(t)|x(s)|, \ t \geqslant 0, s \geqslant S. \tag{43}$$

Thus, from (2), for $t \geqslant S$, there is

$$\begin{aligned} u'(t) &= -\int_{t-r(t)}^{t} \sum_{i=1}^{n} f_i(t, x(s))\, d\mu_i(t, s) \le \\ &\le (\lambda + \eta) \int_{t-r(t)}^{t} \sum_{i=1}^{n} a_i(t)\, d\mu_i(t, s) = \\ &= (\lambda + \eta) \sum_{i=1}^{n} a_i(t)\left[\mu_i(t, t) - \mu_i(t, t - r(t))\right] \le \\ &\le (\lambda + \eta) A(t). \end{aligned} \tag{44}$$

For $S \le s \le \xi_m$, by integrating (44) from s to ξ_m, we get

$$\begin{aligned} -x(s) &\le (\lambda + \eta) \int_{s}^{\xi_m} A(s)\, ds - cx(\xi_m - \sigma) + cx(s - \sigma) \le \\ &\le (\lambda + \eta) \left(\int_{s}^{\xi_m} A(s)\, ds - 2c \right). \end{aligned} \tag{45}$$

On the other hand, for $\xi_m \le t \le t_m$, from (H$_1$), (2) and (45), we conclude

$$\begin{aligned}
u'(t) &= -\int_{t-r(t)}^{t} \sum_{i=1}^{n} f_i(t, x(s))\, d\mu_i(t,s) = -\int_{\xi_m}^{t} \sum_{i=1}^{n} f_i(t, x(s))\, d\mu_i(t,s) - \\
&\quad - \int_{t-r(t)}^{\xi_m} \sum_{i=1}^{n} f_i(t, x(s))\, d\mu_i(t,s) \le -\int_{t-r(t)}^{\xi_m} \sum_{i=1}^{n} f_i(t, x(s))\, d\mu_i(t,s) \le \\
&\le (\lambda+\eta) \left\{ \int_{t-r(t)}^{\xi_m} \sum_{i=1}^{n} a_i(t) \left[\int_{s}^{\xi_m} A(s)\, ds - 2c \right] d\mu_i(t,s) \right\} = \\
&= (\lambda+\eta) \left\{ \int\limits_{t-r(t)}^{\xi_m} \left(\int\limits_{t-r(t)}^{\tau} \sum_{i=1}^{n} a_i(t)\, d\mu_i(t,s) \right) A(\tau)\, d\tau - 2c \int\limits_{t-r(t)}^{\xi_m} \sum_{i=1}^{n} a_i(t)\, d\mu_i(t,s) \right\} \le \\
&\le (\lambda+\eta) \left\{ \sum_{i=1}^{n} a_i(t) \left(\mu_i(t,t) - \mu_i(t, t-r(t))\right) \left(\int_{t-r(t)}^{\xi_m} A(\tau)\, d\tau - 2c \right) \right\} \le \\
&\le (\lambda+\eta) A(t) \left\{ \int_{t-r(t)}^{\xi_m} A(\tau)\, d\tau - 2c \right\}.
\end{aligned} \tag{46}$$

Thus, for $\xi_m \le t \le t_m$,

$$u'(t) \le (\lambda+\eta) \min \left\{ A(t), A(t) \left\{ \left(\int_{t-r(t)}^{\xi_m} A(\tau)\, d\tau - 2c \right) \right\} \right\}. \tag{47}$$

There are two cases to consider:

Case 1. $\int_{\xi_m}^{t_m} A(t)\, dt \le 1$. Then by (36), (47) and the fact that the function $g(x) = (\mu - 2c + \varepsilon) x - \frac{1}{2} x^2$ is increasing for $x \le \mu - 2c + \varepsilon$ and $\mu - 2c + \varepsilon > 1$, there is

$$\begin{aligned}
u(t_m) &\le (\lambda+\eta) \int_{\xi_m}^{t_m} A(t) \left\{ \int_{t-r(t)}^{\xi_m} A(\tau)\, d\tau - 2c \right\} dt = \\
&= (\lambda+\eta) \int_{\xi_m}^{t_m} A(t) \left\{ \int_{t-r(t)}^{t} A(\tau)\, d\tau - \int_{\xi_m}^{t} A(\tau)\, d\tau - 2c \right\} dt \le \\
&\le (\lambda+\eta) \int_{\xi_m}^{t_m} A(t) \left\{ \int_{t-r(t)}^{t} A(\tau)\, d\tau - \int_{\xi_m}^{t} A(\tau)\, d\tau - 2c \right\} dt \le \\
&\le (\lambda+\eta) \int_{\xi_m}^{t_m} A(t) \left\{ \mu - 2c + \varepsilon - \int_{\xi_m}^{t} A(\tau)\, d\tau \right\} dt \le \\
&\le (\lambda+\eta) \left\{ (\mu - 2c + \varepsilon) \int_{\xi_m}^{t_m} A(t)\, dt - \frac{1}{2} \left(\int_{\xi_m}^{t_m} A(t)\, dt \right)^2 \right\} \le \\
&\le (\lambda+\eta) \left(\mu - 2c + \varepsilon - \frac{1}{2} \right) \le \frac{1}{1+c} \left(\nu + \eta(1+c)\right) \left(\mu - 2c + \varepsilon - \frac{1}{2} \right).
\end{aligned} \tag{48}$$

By letting $m \to \infty$ and $\eta \to 0$, we get

$$\nu \leq \frac{\nu}{1+c}\left(\mu - 2c + \varepsilon - \frac{1}{2}\right). \tag{49}$$

Since $\left(\mu - 2c + \varepsilon - \frac{1}{2}\right)/(1+c) < 1$, we see that $\nu = 0$. It follows from (36) that $\lambda = 0$.

Case 2. $\int_{\xi_m}^{t_m} A(t)\,dt > 1$. Then there exists $\eta_m \in (\xi_m, t_m)$ such that

$$\int_{\eta_m}^{t_m} A(t)\,dt = 1. \tag{50}$$

Thus,

$$\begin{aligned}
u(t_m) &\leq (\lambda+\eta)\left\{\int_{\xi_m}^{\eta_m} A(t)\,dt + \int_{\bar{t}}^{t_m} A(t)\left\{\int_{t-r(t)}^{\xi_m} A(\tau)\,d\tau - 2c\right\}dt\right\} = \\
&= (\lambda+\eta)\left\{\int_{\eta_m}^{t_m} A(t)\int_{t_0}^{\eta_m} A(\tau)\,d\tau dt + \int_{\eta_m}^{t_m} A(t)\left\{\int_{t-r(t)}^{\xi_m} A(\tau)\,d\tau - 2c\right\}dt\right\} = \\
&= (\lambda+\eta)\left\{\int_{\eta_m}^{t_m} A(t)\int_{t-r(t)}^{\eta_m} A(\tau)\,d\tau dt - 2c\int_{\eta_m}^{t_m} A(t)\,dt\right\} \leq \\
&\leq (\lambda+\eta)\left\{\int_{\eta_m}^{t_m} A(t)\left(\mu + \varepsilon - \int_{\eta_m}^{t} A(\tau)\,d\tau\right)dt - 2c\int_{\eta_m}^{t_m} A(t)\,dt\right\} = \\
&= (\lambda+\eta)\left\{(\mu+\varepsilon-2c)\int_{\eta_m}^{t_m} A(t)\,dt - \int_{\eta_m}^{t_m} A(t)\int_{\eta_m}^{t} A(\tau)\,d\tau dt\right\} \leq \\
&\leq (\lambda+\eta)\left\{(\mu-2c+\varepsilon)\int_{\eta_m}^{t_m} A(t)\,dt - \frac{1}{2}\left(\int_{\eta_m}^{t_m} A(t)\,dt\right)^2\right\} \leq \\
&\leq (\lambda+\eta)\left(\mu - 2c + \varepsilon - \frac{1}{2}\right) \leq \\
&\leq \frac{1}{1+c}\left(\nu + \eta(1+c)\right)\left(\mu - 2c + \varepsilon - \frac{1}{2}\right),
\end{aligned} \tag{51}$$

By letting $m \to \infty$ and $\eta \to 0$, we get

$$\nu \leq \frac{\nu}{1+c}\left(\mu - 2c + \varepsilon - \frac{1}{2}\right), \tag{52}$$

which implies $\nu = 0$ again. It follows from (36) that $\lambda = 0$. □

In view of our previous Lemmas, Theorem 1 is true.

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