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# CONSTRUCTION OF AN INTEGRAL MANIFOLD FOR LINEAR DIFFERENTIAL-DIFFERENCE EQUATIONS

**Abstract.** In this paper we establish sufficient conditions for the existence of an asymptotic integral manifold of solutions of a linear system of differential-difference equations with a small parameter. This integral manifold is described by a linear system of differential equations without deviating argument.

**Keywords:** system with deviating argument, integral manifold of solutions, fundamental matrix, exponential dichotomy.

**Mathematics Subject Classification:** 34K06.

## 1. INTRODUCTION AND PRELIMINARIES

The theory of linear differential equations with deviating argument is well established. There are numerous important papers on the subject. One of the classical works here is [2]. For a recent account of the theory, we refer the reader to [1] and the references given there.

We consider the linear system of differential equations with deviating argument

$$\begin{aligned} \frac{dX(t)}{dt} &= A(t)X(t) + \mu\sum_{k=1}^{n}(A_k(t)X(t+\tau_k(t)) + B_k(t)Y(t+\tau_k(t))) \\ \frac{dY(t)}{dt} &= B(t)Y(t) + \mu\sum_{k=1}^{n}(C_k(t)X(t+\tau_k(t)) + D_k(t)Y(t+\tau_k(t))), \end{aligned} \tag{1}$$

where $t \geq 0$, $\mu$ is a small parameter, $dim X(t) = p$, $dim Y(t) = q$,

$$|\tau_k(t)| \leq \tau \qquad (k = 1, \ldots, n;\ t \geq 0). \tag{2}$$

We assume that the matrices in System (1) are bounded:

$$\|A(t)\| \le \alpha_0, \quad \sum_{k=1}^{n} \|A_k(t)\| \le \alpha,$$
$$\sum_{k=1}^{n} \|B_k(t)\| \le \alpha, \quad \sum_{k=1}^{n} \|C_k(t)\| \le \alpha, \quad \sum_{k=1}^{n} \|D_k(t)\| \le \alpha, \tag{3}$$

If the parameter $\mu = 0$, System (1) decouples into two independent subsystems. Let the system

$$\frac{dX(t)}{dt} = A(t)X(t)$$

have a fundamental matrix of solutions $P(t,s)$ normalized at $t = s$ and satisfy the condition

$$\|P(t,s)\| \le c_1 e^{\varepsilon|t-s|} \ \ (c_1 \ge 1, \ \varepsilon > 0) \tag{4}$$

Let the system

$$\frac{dY(t)}{dt} = B(t)X(t)$$

have a fundamental matrix of solutions $Q(t,s)$ normalized at $t = s$ and satisfy the condition

$$\|Q(t,s)\| \le c_2 e^{\lambda|t-s|} \ \ (c_2 \ge 1, \ \lambda > \varepsilon) \tag{5}$$

Thus, for $\mu = 0$, System (1) possesses an exponential dichotomy with an exponent $\sigma$, where $-\lambda < \sigma < -\varepsilon$ (see [3]).
We will construct an integral manifold of solutions of System (1) in the form of ( [3,4])

$$\frac{dX(t)}{dt} = H(t,\mu)X(t), \ \ Y(t) = K(t,\mu)X(t). \tag{6}$$

Let a fundamental matrix of solutions of System (6) be denoted by $N(t,s,\mu)$, it follows that

$$X(t) = N(t,s,\mu)X(s). \tag{7}$$

From this we obtain

$$\begin{gathered} X(t+\tau_k(t)) = N(t+\tau_k(t), t.\mu)X(t), \\ Y(t+\tau_k(t)) = K(t+\tau_k(t),\mu)N(t+\tau_k(t),t,\mu)X(t). \end{gathered} \tag{8}$$

If the solutions of System (6) satisfy (1), then

$$\begin{aligned} H(t,\mu) = A(t) + \mu \sum_{k=1}^{n} (A_k(t)N(t+\tau_k(t),t,\mu) + \\ + B_k(t)K(t+\tau_k(t),\mu)N(t+\tau_k(t),t,\mu)) \end{aligned} \tag{9}$$

$$\begin{aligned}\frac{\partial K(t,\mu)}{\partial t} &+ K(t,\mu)(A(t) + \mu\sum_{k=1}^{n}(A_k(t)N(t+\tau_k(t),t,\mu)+ \\ &+ B_k(t)K(t+\tau_k(t),\mu)N(t+\tau_k(t),t,\mu))) = \\ &= B(t)K(t,\mu) + \mu\sum_{k=1}^{n}(C_k(t)N(t+\tau_k(t),t,\mu)+ \\ &+ D_k(t)K(t+\tau_k(t),\mu)N(t+\tau_k(t),t,\mu)).\end{aligned} \tag{10}$$

We now proceed to considering the auxiliary matrix differential equation

$$\frac{\partial K(t,\mu)}{\partial t} = B(t)K(t,\mu) - K(t,\mu)A(t) + F(t), \tag{11}$$

where $\|F(t)\| \leq b$ for $t \geq 0$. It is easy to check that the matrix

$$K(t,\mu) = \int_0^t Q(t,s)F(t)P(s,t)ds \tag{12}$$

is a solution of (11). In addition, under conditions (4) and (5), we obtain

$$\|K(t,\mu)\| = c_1c_2\int_0^t e^{-\lambda(t-s)} sup\,\|F(t)\|\, e^{\varepsilon(t-s)} \leq \frac{c_1c_2}{\lambda-\varepsilon} sup\,\|F(t)\|\,, \ \ t \geq 0. \tag{13}$$

Application of (11) and (12) enables us to write System (10) in the form

$$\begin{aligned}K(t,\mu) = \mu\int_0^t Q(t,s)&\sum_{k=1}^{n}(C_k(s) + D_k(s)K(s+\tau_k(s),\mu)- \\ &- K(s,\mu)A_k(s) - K(s,\mu)B_k(s)K(s+\tau_k(s),\mu))N(s+\tau_k(s),t,\mu)P(s,t)ds.\end{aligned} \tag{14}$$

Our purpose here is to give a proof that an integral manifold of solutions of System (1) exists in form (6).

## 2. SUCCESSIVE APPROXIMATIONS

System (9),(14) of matrix equations defines the matrices $H(t,\mu)$, $K(t,\mu)$ and it can be solved by the method of successive approximations. We start this process by letting $H_0(t,\mu) = 0$, $K_0(t,\mu) = 0$ and

$$\begin{aligned}H_{j+1}(t,\mu) &= A(t) + \mu\sum_{k=1}^{n}(A_k(t)N_j(t+\tau_k(t),t,\mu)+ \\ &\quad + B_k(t)K_j(t+\tau_k(t),\mu)N_j(t+\tau_k(t),t,\mu)), \\ K_{j+1}(t,\mu) &= \mu\int_0^t Q(t,s)\sum_{k=1}^{n}(C_k(s) + D_k(s)K_j(s+\tau_k(s),\mu)- \\ &\quad - K_j(s,\mu)A_k(s) - K_j(s,\mu)B_k(s)K_j(s+\tau_k(s),\mu))\times \\ &\quad \times N_j(s+\tau_k(s),t,\mu)P(s,t)ds,\end{aligned} \tag{15}$$

for j=0,1,2,...

Let $N_j(t,s,\mu)$ be a fundamental matrix of solutions of the system

$$\frac{dX(t)}{dt} = H_j(t,\mu)X(t) \quad (j = 0,1,2,\ldots). \tag{16}$$

Supposing that the inequalities

$$\|H_j(t,\mu)\| \le h_j, \quad \|K_j(t,\mu)\| \le k_j,$$

take place, we find the estimates

$$\|N_j(t,s,\mu)\| \le e^{h_j|t-s|}, \quad \|N_j(t+\tau_k(t),t,\mu)\| \le e^{\tau h_j}.$$

And owing to System (15) it follows that

$$\begin{aligned} h_{j+1} &\le \alpha_0 + |\mu|\,\alpha(1+k_j)e^{\tau h_j}, \\ k_{j+1} &\le |\mu|\,\alpha\beta(1+k_j)^2 e^{\tau h_j}, \end{aligned} \tag{17}$$

where

$$\beta \equiv \frac{c_1 c_2}{\lambda - \varepsilon}.$$

In order for the sequences $\{h_j\}$ and $\{k_j\}$ to be bounded from below, i.e.

$$h_j \ge h > 0, \;\; k_j \ge k > 0, \quad j = 0,1,2,\ldots$$

it is necessary and sufficient that the system of nonlinear equations

$$\alpha_0 + |\mu|\,\alpha(1+k)e^{\tau h} = h, \quad |\mu|\,\alpha\beta(1+k)^2 e^{\tau h} = k \tag{18}$$

has a positive solution.
We need to find the largest value $\mu = \mu_0$ for which System (18) has multiple solutions. In this connection $h$ and $k$ achieve the maximum. We obtain

$$\mu_0 = \frac{2(\tau + \sqrt{\tau^2 + 4\beta^2}}{\alpha(\tau + 2\beta + \sqrt{\tau^2 + 4\beta^2}} \exp\left\{ -\tau(\alpha + \frac{2}{\tau + 2\beta + \sqrt{\tau^2 + 4\beta^2}}) \right\} \tag{19}$$

$$h = \alpha_0 + \frac{k}{\beta(1+k)}, \quad k = \frac{2\beta}{\tau + \sqrt{\tau^2 + 4\beta^2}}.$$

It follows that for $|\mu| \le \mu_0$ the sequences of matrices $H_j(t,\mu)$ and $K_j(t,\mu)$ (j=0,1,2,...) are well defined and bounded in norm for $t \ge 0$.

## 3. CONVERGENCE

Let us turn to prove that the sequences of matrices $H_j(t,\mu)$ and $K_j(t,\mu)$ $(j = 0,1,2,\ldots)$ converge uniformly in $t$, for $t \ge 0$.
Let us introduce the notation

$$\|H_j(t,\mu) - H_{j-1}(t,\mu)\| \le u_j, \quad \|K_j(t,\mu) - K_{j-1}(t,\mu)\| \le v_j \;\; (j = 1,2,3,\ldots).$$

Then from System (15), for $\mu = \mu_0$ there follows:

$$\begin{aligned} u_{j+1} &\le \ell(1+k)\tau u_j + l v_j, \\ v_{j+1} &\le \ell\beta\tau(1+k)^2 u_j + 2\ell\beta(1+k)v_j, \end{aligned} \tag{20}$$

where

$$\ell \equiv \frac{1}{2\beta(1+k)+\tau}.$$

The matrix of coefficients of the expression on the right hand side of (20)

$$R(\mu) = \begin{pmatrix} \ell\tau(1+k) & \ell \\ \ell\beta\tau(1+k)^2 & 2\ell\beta \end{pmatrix}$$

has the largest eigenvalue (in terms of the absolute value)

$$\rho_{max} = \frac{(2\beta+\tau+\sqrt{\tau^2+4\beta^2})^2}{2(2\beta+\sqrt{\tau^2+4\beta^2})(\tau+\sqrt{\tau^2+4\beta^2})} \equiv 1.$$

For $|\mu| < \mu_0$, the absolute values of eigenvalues of the matrix $R(\mu)$ are less than 1 and, therefore, the terms of the series

$$H(t,\mu) = \sum_{j=0}^{\infty} \left(H_{j+1}(t,\mu) - H_j(t,\mu)\right), \tag{21}$$

$$K(t,\mu) = \sum_{j=0}^{\infty} \left(K_{j+1}(t,\mu) - K_j(t,\mu)\right). \tag{22}$$

are bounded from above by the terms of the decreasing geometric progression. It follows that series (21) and (22) converge uniformly for $|\mu| \le \mu_1 < \mu_0$.

We can now formulate our main results.

**Theorem 3.1.** *Let System* (1) *of differential-difference equations be dichotomic for* $\mu = 0$ *and* $t \ge 0$*, caused by inequalities* (4) *and* (5)*. If conditions* (2) *and* (3) *hold, then System* (1) *has an integral manifold of form* (6)*, where the matrices* $H$ *and* $K$ *depend analytically on* $\mu$*, provided* $|\mu| < \mu_0$*, where* $\mu_0$ *is given by* (19)*.*

The construction of integral manifolds in form (6) for systems of differential- difference equations can be used in the investigation of qualitative properties of this systems.

**Example 3.2.** *For the delay system*

$$\begin{aligned} \frac{dx(t)}{dt} &= \mu\cos t x(t) + \mu a y(t-\tau), \\ \frac{dy(t)}{dt} &= -y(t) + \mu b x(t-\tau), \end{aligned} \tag{23}$$

*we are interested in the construction of an integral manifold of solutions in form (6)*

$$\frac{dx(t)}{dt} = h(t,\mu)x(t), \quad y(t) = k(t,\mu)x(t).$$

*The functions k and h are defined by the system*

$$h(t,\mu) = \mu\cos t + \mu a k(t-\tau,\mu)\exp\int_t^{t-\tau} h(r,\mu)dr,$$
$$k(t,\mu) = \mu\int_0^t (e^{-(t-s)}(b-k(s,\mu))\cos s - \tag{24}$$
$$-ak(s,\mu)k(s-\tau,\mu))\exp\int_s^{s-\tau} h(r,\mu)dr)ds.$$

*Solving System (24) by the method of successive approximations, we find*

$$h(t,\mu) = \mu\cos t + \mu^2 ab + O(\mu^3).$$

*The zero solution of the differential equation*

$$\frac{dx(t)}{dt} = (\mu\cos t + \mu^2 ab + O(\mu^3))x(t)$$

*and also the zero solution of System (23) are asymptotically stable for a sufficiently small values of* $\mu > 0$ *and* $ab < 0$, *and unstable if* $ab > 0$.

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