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INITIAL DATA GENERATING BOUNDED SOLUTIONS OF LINEAR DISCRETE EQUATIONS

Abstract. A lot of papers are devoted to the investigation of the problem of prescribed behavior of solutions of discrete equations and in numerous results sufficient conditions for existence of at least one solution of discrete equations having prescribed asymptotic behavior are indicated. Not so much attention has been paid to the problem of determining corresponding initial data generating such solutions. We fill this gap for the case of linear equations in this paper. The initial data mentioned are constructed with use of two convergent monotone sequences. An illustrative example is considered, too.

Keywords: linear discrete equation, bounded solutions, initial data.

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1. INTRODUCTION

We use the following notation: for integers s, q, $s \le q$, we define $\mathbb{Z}_s^q := \{s, s+1, \dots, q\}$, where $s = -\infty$ and $q = \infty$ are admitted, too. Throughout this paper, using notation $\mathbb{Z}_s^q$, perhaps with other sub- or superscripts, we suppose $s \le q$.

We consider a scalar discrete equation

$$\Delta u(k) = f(k, u(k)), \tag{1.1}$$

where $\Delta u(k) = u(k+1) - u(k)$, $f : \mathbb{Z}_a^\infty \times \mathbb{R} \to \mathbb{R}$ and a is a fixed integer. Together with discrete equation (1.1), we consider an initial problem: for a given $s \in \mathbb{Z}_a^\infty$ determine a solution $u = u(k)$ of equation (1.1) satisfying the initial condition

$$u(a + s) = u_s \in \mathbb{R} \tag{1.2}$$

with a prescribed constant u_s. A solution of initial problem (1.1), (1.2) is an infinite sequence of numbers $\{u^k\}_{k=0}^\infty$ with $u^k = u(a + s + k)$, i.e.,

$$u^0 = u_s = u(a + s), u^1 = u(a + s + 1), \dots, u^n = u(a + s + n), \dots$$

such that for any $k \in \mathbb{Z}_{a+s}^{\infty}$ equality (1.1) holds. If f is continuous with respect to its second argument, then initial problem (1.1), (1.2) depends continuously on initial data.

Let $b(k), c(k)$ be real functions defined on $\mathbb{Z}_a^{\infty}$, such that $b(k) < c(k)$ for every $k \in \mathbb{Z}_a^{\infty}$. We define

$$\omega := \{(k,u) : k \in \mathbb{Z}_a^{\infty},\, u \in \omega(k)\} \quad \text{with} \quad \omega(k) := \{u : b(k) < u < c(k)\}$$

and the closure

$$\overline{\omega} := \{(k,u) : k \in \mathbb{Z}_a^{\infty},\, u \in \overline{\omega}(k)\} \quad \text{with} \quad \overline{\omega}(k) := \{u : b(k) \le u \le c(k)\}.$$

Define, moreover, $B := B_1 \cup B_2$ with

$$B_1 := \{(k,u) : k \in \mathbb{Z}_a^{\infty},\, u = b(k)\},\ \ B_2 := \{(k,u) : k \in \mathbb{Z}_a^{\infty},\, u = c(k)\}.$$

The following theorem concerning asymptotic behavior of solutions of equation (1.1) is a particular case of a more general result in [1, Theorem 2].

Theorem 1.1. *Let us suppose that* $f : \overline{\omega} \to \mathbb{R}$ *is continuous with respect to its second argument. If, moreover,*

$$f(k, b(k)) - b(k+1) + b(k) < 0$$

in the case of $(k,u) \in B_1$*, and*

$$f(k, c(k)) - c(k+1) + c(k) > 0$$

in the case of $(k,u) \in B_2$*, then there exists an initial condition*

$$u^*(a) = u^* \in \omega(a) \tag{1.3}$$

such that the corresponding solution $u = u^*(k)$ *satisfies the relation*

$$u^*(k) \in \omega(k) \tag{1.4}$$

for every $k \in \mathbb{Z}_a^{\infty}$*.*

Analysing the result given by Theorem 1.1, we conclude that the existence of (at least one) solution of problem (1.1), (1.3) having the indicated (asymptotic) behavior characterized by relations (1.4) is stated *with no specific corresponding initial data* u^* *given.* In this paper we particularly try to fill this gap in the linear case. Note that the questions concerning behavior of solutions of discrete equations are considered, e.g., in [1]–[5]. Unfortunately, the problem concerning the determination of corresponding initial data has not been considered there.
Let us put

$$f(k, u(k)) := \varphi(k)u(k) + \delta(k)$$

in (1.1) with $\varphi(k), \delta(k) : \mathbb{Z}_a^\infty \to \mathbb{R}$ and consider the corresponding linear equation

$$\Delta u(k) = \varphi(k)u(k) + \delta(k) \tag{1.5}$$

together with an initial problem

$$u(a) = u^*. \tag{1.6}$$

It is easy to verify that in the linear case Theorem 1.1 takes the form:

Theorem 1.2. *Let the inequalities*

$$(1 + \varphi(k))b(k) + \delta(k) - b(k+1) \quad < \quad 0, \tag{1.7}$$

$$(1 + \varphi(k))c(k) + \delta(k) - c(k+1) \quad > \quad 0 \tag{1.8}$$

hold for every $k \in \mathbb{Z}_a^\infty$*. Then there exists an initial condition*

$$u^*(a) = u^* \in \omega(a), \tag{1.9}$$

such that the corresponding solution $u = u^*(k)$ *of equation* (1.5) *satisfies the inequalities*

$$b(k) < u^*(k) < c(k) \tag{1.10}$$

for every $k \in \mathbb{Z}_a^\infty$*.*

We note that, geometrically, inequalities (1.7), (1.8) express a strict egress property of solutions at the boundary of ω.

We consider the following problem:

Problem 1.3. *Determine at least one value* u^* *such that the corresponding solution* $u = u^*(k)$ *of linear problem* (1.5), (1.6) *satisfies the relations* $(k, u^*(k)) \in \omega$ *for every* $k \in \mathbb{Z}_a^\infty$*, i.e., it satisfies inequalities* (1.10) *for every* $k \in \mathbb{Z}_a^\infty$*.*

The idea of solving this problem goes as follows. We consider a sequence of points

$$(a + s, c(a + s)), \;\; s \in \mathbb{Z}_0^\infty$$

and for every fixed s we try to determine a solution $u = u_{cs}(a + k)$, $k \in \mathbb{Z}_0^\infty$ of equation (1.5) such that

$$(a + s, u_{cs}(a + s)) = (a + s, c(a + s)). \tag{1.11}$$

We consider the sequence $\{u_{cs}\}_{s=0}^\infty$ with $u_{cs} = u_{cs}(a)$. Due to (1.11), we can expect that, under some additional conditions, the limit $c^* := \lim_{s\to\infty} u_{cs}$ (provided it exists) defines one of the desired values u^*. Similarly, we consider a sequence of points

$$(a + s, b(a + s)), \;\; s \in \mathbb{Z}_0^\infty$$

and for every fixed s we try to determine a solution $u = u_{bs}(a + k)$, $k \in \mathbb{Z}_0^\infty$ of equation (1.5) such that

$$(a + s, u_{bs}(a + s)) = (a + s, b(a + s)). \tag{1.12}$$

We consider the sequence $\{u_{bs}\}_{s=0}^\infty$ with $u_{bs} = u_{bs}(a)$. Due to (1.12), we can expect that, under some additional conditions, the limit $b^* := \lim_{s\to\infty} u_{bs}(a)$ (provided it exists) defines one of the desired values u^* as well. Moreover, if $c^* \neq b^*$, then we can expect that every $u^* \in [b^*, c^*]$ solves Problem 1.3.

2. TWO AUXILIARY SEQUENCES

Based on the described idea, we are going to define elements of two auxiliary sequences $\{u_{cs}\}_{s=0}^{\infty}$ and $\{u_{bs}\}_{s=0}^{\infty}$ explicitly. We consider equation (1.5). The solution passing through the initial point $(a, u(a))$ has the form

$$u(a+s) = u(a) \cdot \prod_{i=0}^{s-1}(1+\varphi(a+i)) + \sum_{i=0}^{s-1}\delta(a+i) \cdot \prod_{p=i+1}^{s-1}(1+\varphi(a+p)). \tag{2.1}$$

Supposing $\varphi(k) \neq -1$, $k \in \mathbb{Z}_a^{\infty}$, $u(a+s) = c(a+s)$ (in accordance with (1.11)) and letting $u(a) = u_{cs}$, $p = j - a$ in (2.1), we get

$$u_{cs} = \left[c(a+s) - \sum_{i=0}^{s-1}\delta(a+i) \prod_{j=a+i+1}^{a+s-1}(1+\varphi(j))\right] \cdot \left[\prod_{i=0}^{s-1}(1+\varphi(a+i))\right]^{-1}. \tag{2.2}$$

Elements of the sequence $\{u_{cs}\}_{s=0}^{\infty}$ are defined by formulas (2.2). Supposing $\varphi(k) \neq -1$, $k \in \mathbb{Z}_a^{\infty}$, $u(a+s) = b(a+s)$ (in accordance with (1.12)) and letting $u(a) = u_{bs}$, $p = j - a$ in (2.1), we obtain

$$u_{bs} = \left[b(a+s) - \sum_{i=0}^{s-1}\delta(a+i) \prod_{j=a+i+1}^{a+s-1}(1+\varphi(j))\right] \cdot \left[\prod_{i=0}^{s-1}(1+\varphi(a+i))\right]^{-1}. \tag{2.3}$$

Elements of the sequence $\{u_{bs}\}_{s=0}^{\infty}$ are defined by formulas (2.3). The following lemma gives an important property of both sequences.

Lemma 2.1. *Let $\varphi(k) \neq -1$, $k \in \mathbb{Z}_a^{\infty}$. Then elements of the sequence $\{u_{cs}\}_{s=0}^{\infty}$ satisfy the difference equation*

$$u_{c,s+1} - u_{cs} = \frac{c(a+s+1) - c(a+s) - \varphi(a+s)c(a+s) - \delta(a+s)}{\prod\limits_{i=0}^{s}(1+\varphi(a+i))} \tag{2.4}$$

and elements of the sequence $\{u_{bs}\}_{s=0}^{\infty}$ satisfy the difference equation

$$u_{b,s+1} - u_{bs} = \frac{b(a+s+1) - b(a+s) - \varphi(a+s)b(a+s) - \delta(a+s)}{\prod\limits_{i=0}^{s}(1+\varphi(a+i))}. \tag{2.5}$$

Proof. Relation (2.4) holds because in view of (2.2)

$$u_{c,s+1} - u_{cs} = \frac{c(a+s+1) - \sum\limits_{i=0}^{s} \delta(a+i) \prod\limits_{j=a+i+1}^{a+s} (1+\varphi(j))}{\prod\limits_{i=0}^{s} (1+\varphi(a+i))} -$$

$$- \frac{c(a+s) - \sum\limits_{i=0}^{s-1} \delta(a+i) \prod\limits_{j=a+i+1}^{a+s-1} (1+\varphi(j))}{\prod\limits_{i=0}^{s-1} (1+\varphi(a+i))} =$$

$$= \frac{c(a+s+1) - \delta(a+s) - \sum\limits_{i=0}^{s-1} \delta(a+i) \prod\limits_{j=a+i+1}^{a+s} (1+\varphi(j))}{\prod\limits_{i=0}^{s} (1+\varphi(a+i))} -$$

$$- \frac{\left[c(a+s) - \sum\limits_{i=0}^{s-1} \delta(a+i) \prod\limits_{j=a+i+1}^{a+s-1} (1+\varphi(j))\right] \cdot (1+\varphi(a+s))}{\prod\limits_{i=0}^{s} (1+\varphi(a+i))} =$$

$$= \frac{c(a+s+1) - c(a+s) - c(a+s)\delta(a+s) - \delta(a+s)}{\prod\limits_{i=0}^{s} (1+\varphi(a+i))} .$$

Relation (2.5) can be proved similarly with use of (2.3). □

Lemma 2.2. *Let $\varphi(k) > -1$ and inequalities (1.7), (1.8) hold for every $k \in \mathbb{Z}_a^\infty$. Then the sequence $\{u_{cs}\}_{s=0}^\infty$ is decreasing convergent and the sequence $\{u_{bs}\}_{s=0}^\infty$ is increasing convergent. Moreover, $u_{cs} > u_{bs}$ holds for every $s \in \mathbb{Z}_0^\infty$ and for the limits c^*, b^*, where*

$$c^* = \lim_{s\to\infty} u_{cs}, \;\; b^* = \lim_{s\to\infty} u_{bs}, \tag{2.6}$$

the inequality $c^* \geq b^*$ holds.

Proof. The property $u_{cs} > u_{bs}$, $s \in \mathbb{Z}_0^\infty$, is obvious since $c(a+s) > b(a+s)$ for every $s \in \mathbb{Z}_0^\infty$. We show that the sequence $\{u_{cs}\}_{s=0}^\infty$ is decreasing. This is a direct consequence of relation (2.4), since the numerator is negative by assumption (1.8) and the denominator is obviously positive. So the inequality $u_{cs} > u_{c,s+1}$ holds for for every $s \in \mathbb{Z}_0^\infty$.

Inequality $u_{bs} < u_{b,s+1}$ for $s \in \mathbb{Z}_0^\infty$ can be proved similarly with use of relation (2.5) and condition (1.7). All remaining statements follow from the theory of number sequences. □

Lemma 2.3. *Suppose that $\varphi(k) > -1$ for every $k \in \mathbb{Z}_a^\infty$. Then solutions $u(k)$, $U(k)$, $k \in \mathbb{Z}_a^\infty$, of two initial problems for linear equation* (1.5)

$$u(a) = \alpha \quad \text{and} \quad U(a) = \beta$$

with $\alpha < \beta$ satisfy the inequality

$$u(k) < U(k)$$

for every $k \in \mathbb{Z}_a^\infty$.

Proof. We omit the proof since it is trivial. □

Lemma 2.4. *Let $\varphi(k) > -1$ and inequalities* (1.7), (1.8) *hold for every $k \in \mathbb{Z}_a^\infty$. Then:*

a) *Solution $u = u^*_{cs}(k)$, $k \in \mathbb{Z}_a^\infty$ of the problem*

$$u^*_{cs}(a) = u_{cs}, \;\; s \in \mathbb{Z}_0^\infty \tag{2.7}$$

for linear equation (1.5) *satisfies*

$$u^*_{cs}(k) \in \omega(k) \tag{2.8}$$

for $k \in \mathbb{Z}_a^{a+s-1}$ and

$$u^*_{cs}(a+s) = c(a+s). \tag{2.9}$$

Moreover,

$$u^*_{c,s+1}(k) < u^*_{cs}(k) \tag{2.10}$$

for $k \in \mathbb{Z}_a^{a+s}$.

b) *Solution $u^*_{bs}(k)$, $k \in \mathbb{Z}_a^\infty$ of the problem*

$$u^*_{bs}(a) = u_{bs}, \;\; s \in \mathbb{Z}_0^\infty \tag{2.11}$$

for linear equation (1.5) *satisfies*

$$u^*_{bs}(k) \in \omega(k) \tag{2.12}$$

for $k \in \mathbb{Z}_a^{a+s-1}$ and

$$u^*_{bs}(a+s) = b(a+s). \tag{2.13}$$

Moreover,

$$u^*_{b,s+1}(k) > u^*_{bs}(k) \tag{2.14}$$

for $k \in \mathbb{Z}_a^{a+s}$.

Proof. We remark that relations (2.9), (2.13) hold due to the definition of sequences $\{u_{cs}\}_{s=0}^\infty$, $\{u_{bs}\}_{s=0}^\infty$. By Lemma 2.2, $u_{cs} > u_{bs}$, $s \in \mathbb{Z}_0^\infty$. Then, by Lemma 2.3, the inequality $u^*_{cs}(k) > u^*_{bs}(k)$ holds for every $k \in \mathbb{Z}_a^\infty$. Since, by Lemma 2.2, $u_{cs} > u_{c,s+1}$,

$s \in \mathbb{Z}_0^\infty$, and $u_{bs} < u_{b,s+1}$, $s \in \mathbb{Z}_0^\infty$, properties (2.10) and (2.14) are a consequence of Lemma 2.2 and Lemma 2.3. Let us prove relations (2.8), (2.12). Since

$$u_{bs}^*(a) < u_{cs}^*(a)$$

and

$$u_{b,s+1}^*(a) > u_{bs}^*(a), \quad u_{c,s+1}^*(a) < u_{cs}^*(a),$$

Lemma 2.3 gives

$$u_{bs}^*(k) < u_{b,s+1}^*(k) < u_{c,s+1}^*(k) < u_{cs}^*(k).$$

For $k = a + s$ we get

$$b(a+s) = u_{bs}^*(a+s) < u_{b,s+1}^*(a+s) < u_{c,s+1}^*(a+s) < u_{cs}^*(a+s) = c(a+s).$$

The last inequalities can be rewritten as

$$b(a+\tilde{s}) = u_{b\tilde{s}}^*(a+\tilde{s}) < u_{b,\tilde{s}+1}^*(a+\tilde{s}) < u_{c,\tilde{s}+1}^*(a+\tilde{s}) < u_{c\tilde{s}}^*(a+\tilde{s}) = c(a+\tilde{s}).$$

Consequently, letting $\tilde{s} = 0, 1, 2 \ldots, s-1$, we see that (2.8) and (2.12) hold, too. □

3. MAIN RESULTS

Theorem 3.1 (Main Result). *Let $\varphi(k) > -1$ and inequalities (1.7), (1.8) hold for every $k \in \mathbb{Z}_a^\infty$. Then every initial problem (1.9) with $u^* \in [b^*, c^*]$, where b^* and c^* are defined by (2.6), determines a solution satisfying inequalities (1.10).*

Proof. The proof is a straightforward consequence of Lemmas 2.2–2.4. For solutions of the problems $u(a) = b^*$, $U(a) = c^*$, there is

$$b(k) < u(k) \le U(k) < c(k)$$

for every $k \in \mathbb{Z}_a^\infty$ and

$$u(k) \le \tilde{u}(k) \le U(k)$$

for every $k \in \mathbb{Z}_a^\infty$ if starting point $\tilde{u}(a) \in [b^*, c^*]$. □

Remark 3.2. *From the statement of Theorem 3.1 and from the explicit form of problem (1.5), (1.9) it directly follows that the expression*

$$u(a+s) = u^* \cdot \prod_{i=0}^{s-1}(1+\varphi(a+i)) + \sum_{i=0}^{s-1}\delta(a+i) \cdot \prod_{p=i+1}^{s-1}(1+\varphi(a+p))$$

with fixed $u^* \in [b^*, c^*]$ is a solution of problem (1.5), (1.9), satisfying inequalities (1.10).

The following result can be proved easily. Therefore, we omit its proof.

Theorem 3.3. *Let $\varphi(k) > -1$ and inequalities (1.7), (1.8) hold for every $k \in \mathbb{Z}_a^\infty$. Then the initial problem*

$$u(a) = u^\nabla$$

with

$$u^{\nabla} \in [b(a), c(a)] \setminus [b^*, c^*]$$

generates a solution $u = u^{\nabla}(k)$ *of equation* (1.5) *not satisfying inequalities* (1.10) *for a* $k \in \mathbb{Z}_a^{\infty}$.

The following two corollaries follow from Theorems 3.1, 3.3.

Corollary 3.4. *Let* $\varphi(k) > -1$ *and inequalities* (1.7), (1.8) *hold for every* $k \in \mathbb{Z}_a^{\infty}$. *Then a solution* $u = u(k)$ *of* (1.5) *satisfies inequalities* (1.10) *for every* $k \in \mathbb{Z}_a^{\infty}$ *if and only if* $u(a) \in [b^*, c^*]$.

Corollary 3.5. *Let* $\varphi(k) > -1$ *and inequalities* (1.7), (1.8) *hold for every* $k \in \mathbb{Z}_a^{\infty}$. *Let, moreover,* $b^* = c^*$. *Then equation* (1.5) *has unique solution* $u = u^*(k)$ *satisfying inequalities* (1.10) *for every* $k \in \mathbb{Z}_a^{\infty}$. *This solution is determined by initial data*

$$u^*(a) = u^* = b^*.$$

We find sufficient conditions for the case $b^* = c^*$. Let $\varphi(k) > -1$ and inequalities (1.7), (1.8) hold for every $k \in \mathbb{Z}_a^{\infty}$. We denote

$$\Delta(s) := u_{cs} - u_{bs}, \;\; s \in \mathbb{Z}_0^{\infty}.$$

The length of the interval $[b^*, c^*]$ can be estimated (due to the monotonicity of sequences $\{u_{cs}\}_{s=0}^{\infty}$, $\{u_{bs}\}_{s=0}^{\infty}$) as

$$0 \le c^* - b^* < \Delta(s), \;\; s \in \mathbb{Z}_a^{\infty}.$$

Moreover,

$$\begin{aligned}\Delta(s) = u_{cs} - u_{bs} &= \frac{c(a+s) - \sum\limits_{i=0}^{s-1} \delta(a+i) \prod\limits_{j=a+i+1}^{a+s-1} (1+\varphi(j))}{\prod\limits_{i=0}^{s-1} (1+\varphi(a+i))} - \\ &\quad - \frac{b(a+s) - \sum\limits_{i=0}^{s-1} \delta(a+i) \prod\limits_{j=a+i+1}^{a+s-1} (1+\varphi(j))}{\prod\limits_{i=0}^{s-1} (1+\varphi(a+i))} = \\ &= \frac{c(a+s) - b(a+s)}{\prod\limits_{i=0}^{s-1} (1+\varphi(a+i))} \,.\end{aligned}$$

The following corollary is obvious.

Corollary 3.6. *Let* $\varphi(k) > -1$ *and inequalities* (1.7), (1.8) *hold for every* $k \in \mathbb{Z}_a^{\infty}$. *Then*

$$\begin{aligned}&0 < u_{cs} - c^* < \Delta(s), \;\; s \in \mathbb{Z}_0^{\infty},\\ &0 < b^* - u_{bs} < \Delta(s), \;\; s \in \mathbb{Z}_0^{\infty}.\end{aligned}$$

Theorem 3.7. *Let $\varphi(k) > -1$ and inequalities (1.7), (1.8) hold for every $k \in \mathbb{Z}_a^\infty$. Then $b^* = c^*$ if*

$$\lim_{s\to\infty} \Delta(s) = 0.$$

Proof. From Theorem 1.2, the existence of solution of problem (1.5), (1.9) follows. We get

$$c^* - b^* = \lim_{s\to\infty} \frac{c(a+s) - \sum\limits_{i=0}^{s-1} \delta(a+i) \prod\limits_{j=a+i+1}^{a+s-1} (1+\varphi(j))}{\prod\limits_{i=0}^{s-1} (1+\varphi(a+i))} -$$

$$- \lim_{s\to\infty} \frac{b(a+s) - \sum\limits_{i=0}^{s-1} \delta(a+i) \prod\limits_{j=a+i+1}^{a+s-1} (1+\varphi(j))}{\prod\limits_{i=0}^{s-1} (1+\varphi(a+i))} =$$

$$= \lim_{s\to\infty} \frac{c(a+s) - b(a+s)}{\prod\limits_{i=0}^{s-1} (1+\varphi(a+i))} = \lim_{s\to\infty} \Delta(s) = 0.$$

Consequently $c^* = b^*$. □

4. EXAMPLE

Let us consider a particular case of equation (1.5), with

$$\varphi(k) = 2/(k+1),\ \delta(k) = 2 + \exp(-k),\ a = 2,$$

i.e., the equation

$$\Delta u(k) = \frac{2}{k+1} u(k) + 2 + \exp(-k), \quad k \in \mathbb{Z}_2^\infty. \tag{4.1}$$

We are going to show that there exists the unique solution

$$u = u^*(k),\ \ k \in \mathbb{Z}_2^\infty$$

of (4.1) satisfying the inequality

$$k^2 < u^*(k) < (k+1)^2 \tag{4.2}$$

on $\mathbb{Z}_2^\infty$. Moreover, we are going to determine the initial value $u^* = u^*(2)$ generating it.

We put

$$b(k) \equiv k^2,\ c(k) \equiv (k+1)^2,\ k \in \mathbb{Z}_2^\infty.$$

Then

$$(1+\varphi(k))b(k)+\delta(k)-b(k+1)=\left(1+\frac{2}{k+1}\right)k^2+2+\exp(-k)-(k+1)^2=$$
$$=\frac{1-k}{k+1}+\exp(-k)<0$$

for all $k\in\mathbb{Z}_2^\infty$ and inequality (1.7) holds for all $k\in\mathbb{Z}_2^\infty$. Moreover,

$$(1+\varphi(k))c(k)+\delta(k)-c(k+1)=\left(1+\frac{2}{k+1}\right)(k+1)^2+2+\exp(-k)-(k+2)^2=$$
$$=1+\exp(-k)>0.$$

and inequality (1.8) holds for all $k\in\mathbb{Z}_2^\infty$, too. Then, by virtue of Theorem 3.1, there exists at least one solution $u=u^*(k)$, $k\in\mathbb{Z}_2^\infty$ of (4.1) satisfying

$$b(k)<u^*(k)<c(k)$$

and property (4.2) holds. We determine the initial value $u^*=u^*(2)$. To this end, we compute c^* and b^*. Pursuant to (2.6), there is

$$c^*=\lim_{s\to+\infty}u_{cs}=\lim_{s\to+\infty}\frac{c(a+s)-\sum\limits_{i=0}^{s-1}\delta(a+i)\prod\limits_{j=a+i+1}^{a+s-1}(1+\varphi(j))}{\prod\limits_{i=0}^{s-1}(1+\varphi(a+i))}\ .$$

Therefore,

$$\begin{aligned}
c^*&=\lim_{s\to+\infty}\frac{(3+s)^2-\sum\limits_{i=0}^{s-1}(2+\mathrm{e}^{-2-i})\prod\limits_{j=3+i}^{s+1}\dfrac{j+3}{j+1}}{\prod\limits_{i=0}^{s-1}\dfrac{j+5}{j+3}}=\\
&=\lim_{s\to+\infty}\frac{(3+s)^2-\sum\limits_{i=0}^{s-1}(2+\mathrm{e}^{-2-i})\cdot\dfrac{i+6}{i+4}\cdot\dfrac{i+7}{i+5}\cdot\dots\cdot\dfrac{s+4}{s+2}}{\dfrac{5}{3}\cdot\dfrac{6}{4}\cdot\dots\cdot\dfrac{s+4}{s+2}}=\\
&=\lim_{s\to+\infty}\left[\frac{12(s+3)}{s+4}-24\sum_{i=0}^{s-1}\frac{1}{(i+4)(i+5)}-12\sum_{i=0}^{s-1}\mathrm{e}^{-2-i}\cdot\frac{1}{(i+4)(i+5)}\right]=\\
&=\lim_{s\to+\infty}\left[\frac{12(s+3)}{s+4}-24\sum_{i=0}^{s-1}\left(\frac{1}{i+4}-\frac{1}{i+5}\right)-12\sum_{i=0}^{s-1}\mathrm{e}^{-2-i}\cdot\frac{1}{(i+4)(i+5)}\right]=\\
&=12-24\cdot\frac{1}{4}-12\lim_{s\to+\infty}\sum_{i=0}^{s-1}\mathrm{e}^{-2-i}\cdot\frac{1}{(i+4)(i+5)}=\\
&=6-\frac{12}{\mathrm{e}^2}\sum_{i=0}^{\infty}\frac{1}{\mathrm{e}^i(i+4)(i+5)}\ .
\end{aligned}$$

Since

$$\lim_{s\to\infty}\Delta(s)=\lim_{s\to\infty}\frac{c(a+s)-b(a+s)}{\prod\limits_{i=0}^{s-1}(1+\varphi(a+i))}=$$

$$=\lim_{s\to\infty}\frac{(s+3)^2-(s+2)^2}{\frac{1}{12}\cdot(s+3)(s+4)}=\lim_{s\to\infty}\frac{12(2s+5)}{(s+3)(s+4)}=0,$$

then, by virtue of Theorem 3.7, $b^*=c^*$ and there exists only one solution of (4.1) with properties described above. It is defined by the initial value

$$u^*=6-\frac{12}{\mathrm{e}^2}\sum_{i=0}^{\infty}\frac{1}{\mathrm{e}^i(i+4)(i+5)}\ .$$

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