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OSCILLATORY PROPERTIES OF FOURTH ORDER NONLINEAR DIFFERENCE EQUATIONS WITH QUASIDIFFERENCES

Abstract. In this paper we present the oscillation criterion for a class of fourth order nonlinear difference equations with quasidifferences.

Keywords: nonlinear difference equation, oscillatory solution, nonoscillatory solution, fourth order.

Mathematics Subject Classification: 39A10

1. INTRODUCTION

Consider the difference equation

$$\Delta(a_n\Delta(b_n\Delta(c_n\Delta y_n))) + f(n, y_n) = 0, \quad n \in \mathcal{N} \tag{E}$$

where $\mathcal{N} = \{0, 1, 2, \ldots\}$, Δ is the forward difference operator defined by $\Delta y_n = y_{n+1} - y_n$, (a_n), (b_n) and (c_n) are sequences of positive real numbers.

Function $f : \mathcal{N} \times \mathcal{R} \to \mathcal{R}$.

By a solution of equation (E) we mean a sequence (y_n) which is defined for $n \in \mathcal{N}$ and satisfies equation (E) for n sufficiently large. We consider such solutions only which are nontrivial for all large n. A solution of equation (E) is called oscillatory if its terms are not eventually positive or eventually negative. Otherwise it is called nonoscillatory. Equation (E) is called oscillatory if each solution (y_n) of this equation is oscillatory. Equations of the form (E) are conveniently classified according to the nonlinearity of $f(k, y)$ with respect to y. Equation (E) is said to be *superlinear* if, for each fixed integer k, $\frac{f(k,y)}{y}$ is nondecreasing in y for $y > 0$ and nonincreasing in y for $y < 0$. Equation (E) is called *strongly superlinear* if there is a number $\alpha > 1$

such that, for each fixed integer k, $\frac{f(k,y)}{|y|^{\alpha} sgn\ y}$ is nondecreasing in y for $y > 0$ and nonincreasing in y for $y < 0$. Clearly, if equation (E) is superlinear, then $f(\cdot, y)$ is nondecreasing on $(0, \infty)$.

In the last few years there has been an increasing interest in the study of oscillatory and asymptotic behavior of solutions of difference equations (see the monographs by Agarwal [1], by Elaydi [3] and by Kelly and Peterson [6]). Compared with second order difference equations, the study of higher order equations, and in particular fourth order equations (see for example [2, 4, 5, 7–20]) has received considerably less attention. Results obtained here are motivated by some results obtained by Thandapani and Arockiasamy in [19], and by Migda and Schmeidel in [9].

The purpose of this paper is to establish a sufficient condition for equation (E) to be oscillatory.

Throughout the rest of our investigations, one or several of the following assumptions will be imposed:

(H1) $a_n \geq c_n$, for all large n, and (a_n) is bounded away from zero.

(H2) $\sum\limits_{i=1}^{\infty} \dfrac{1}{a_i} = \infty$, $\sum\limits_{i=1}^{\infty} \dfrac{1}{b_i} < \infty$.

(H3) $\sum\limits_{j=1}^{\infty} \dfrac{1}{c_j} \sum\limits_{i=j}^{\infty} \dfrac{1}{b_i} < \infty$.

(H4) $yf(n, y) > 0$ for all $y \neq 0$ and $n \in \mathcal{N}$.

Let us denote

$$\mu_{n,N} = \sum_{i=N}^{n-1} \frac{1}{c_i},$$

$$\rho_n = \sum_{j=n}^{\infty} \frac{1}{c_j} \sum_{i=j}^{\infty} \frac{1}{b_i}$$

and

$$\nu_{n,N} = \frac{1}{a_n} \sum_{i=N}^{n-1} \frac{1}{c_i}.$$

Conditions (H1) and (H2) imply $\lim\limits_{n \to \infty} \mu_{n,N} = \infty$.

2. COMMON LEMMAS

In [9] we can find the following lemmas which will be used in this paper.

Lemma 1. *Assume that* (H1)–(H4) *hold. Let* (y_n) *be an eventually positive solution of equation* (E), *then one of the following four cases holds:*

$$\begin{array}{llll}
\text{(I)} & c_n\Delta y_n > 0, & b_n\Delta(c_n\Delta y_n) > 0, & a_n\Delta(b_n\Delta(c_n\Delta y_n)) > 0, \\
\text{(II)} & c_n\Delta y_n > 0, & b_n\Delta(c_n\Delta y_n) < 0, & a_n\Delta(b_n\Delta(c_n\Delta y_n)) > 0, \\
\text{(III)} & c_n\Delta y_n > 0, & b_n\Delta(c_n\Delta y_n) < 0, & a_n\Delta(b_n\Delta(c_n\Delta y_n)) < 0, \\
\text{(IV)} & c_n\Delta y_n < 0, & b_n\Delta(c_n\Delta y_n) > 0, & a_n\Delta(b_n\Delta(c_n\Delta y_n)) > 0.
\end{array}$$

for large n.

Lemma 2. *Assume that* (H1)–(H4) *and Case* (IV) *of Lemma 1 hold. Then there are a constant* k *and an integer* N *such that the following inequalities hold*

$$y_n \geq b_n\Delta(c_n\Delta y_n)\rho_n, \tag{1}$$

$$y_n \geq k\Delta(b_n\Delta(c_n\Delta y_n))\rho_n \sum_{i=N}^{n-1} \frac{1}{a_i}, \ \textit{for } n \geq N.$$

Remark 1. *Assume that* (H1)–(H4) *and Case* (III) *of Lemma 4 hold. Then there are a constant* k *and an integer* N *such that the following inequalities hold*

$$y_n \geq -b_n\Delta(c_n\Delta y_n)\rho_n,$$

$$y_n \geq -k\Delta(b_n\Delta(c_n\Delta y_n))\rho_n \sum_{i=N}^{n-1} \frac{1}{a_i}, \ \textit{for } n \geq N. \tag{2}$$

Lemma 3. *Assume that* (H1)–(H4) *hold. Let* (y_n) *be an eventually positive solution of equation* (E). *Then there exist positive constants* k^* *and* k^{**} *such that*

$$k^*\rho_n \leq y_n \leq k^{**}\mu_{n,N}, \ \textit{for large } n. \tag{3}$$

We present an oscillation criterion for equation (E) in strongly superlinear cases.

Theorem 1. *Assume that equation* (E) *is strongly superlinear,*

$$\textit{sequence } (c_n) \textit{ is nondecreasing,} \tag{4}$$

$$\lim_{n\to\infty} \nu_{n,N} > 0 \tag{5}$$

and conditions (H1)–(H4) *hold. If there exists a positive constant* M *such that*

$$a_n \leqslant Mc_{n-1} \textit{ for large } n, \tag{6}$$

and

$$\sum_{n=1}^{\infty} \nu_{n,N}|f(n, c\rho_n)| = \infty \tag{7}$$

for all $c \neq 0$, *then equation* (E) *is oscillatory.*

Proof. Suppose for contrary that $y_n > 0$ for large n. (The proof in the case $y_n < 0$ is analogous and hence omitted.) Chose $N \in \mathcal{N}$ so large that Case (i) (where successively i= (I), (II), (III), (IV)) from Lemma 1, (1), (2), (3), (6), $\rho_n \leqslant 1$ and $\mu_{n,N} \geqslant 1$ hold for $n \geqslant N$.

Case (I)
Since $c_n\Delta y_n > c_N\Delta y_N$ for $n \geqslant N$, by summation we get

$$y_n > y_N + c_N\Delta y_N \sum_{i=N}^{n-1} \frac{1}{c_i}.$$

Hence $y_n > c^* \sum_{i=N}^{n-1} \frac{1}{c_i}$ where $c^* = c_N\Delta y_N > 0$. So,

$$y_n \geqslant c^* \mu_{n,N}. \tag{8}$$

From (H3) and equation (E)

$$\Delta(a_n\Delta(b_n\Delta(c_n\Delta y_n))) < 0.$$

Hence, by (I), $(a_n\Delta(b_n\Delta(c_n\Delta y_n)))$ is a positive decreasing sequence. Then limit of this sequence exists and it is finite. Set

$$\lim_{n\to\infty} a_n\Delta(b_n\Delta(c_n\Delta y_n)) = c^{**} \geqslant 0. \tag{9}$$

From equation (E),

$$f(n, y_n) = -\Delta(a_n\Delta(b_n\Delta(c_n\Delta y_n))).$$

Summing the above equation over i from N to $n-1$, we obtain

$$\sum_{i=N}^{n-1} f(i, y_i) = -a_n\Delta(b_n\Delta(c_n\Delta y_n)) + xa_N\Delta(b_N\Delta(c_N\Delta y_N)).$$

Letting $n \to \infty$, by (9) we get

$$\sum_{i=N}^{\infty} f(i, y_i) = -c^{**} + a_N\Delta(b_N\Delta(c_N\Delta y_N)) < \infty. \tag{10}$$

By (3), and strong superlinearity of f, we get

$$\frac{f(i, k^*\rho_i)}{(k^*\rho_i)^\alpha} \leqslant \frac{f(i, y_i)}{(y_i)^\alpha}, \quad \text{where} \quad \alpha > 1. \tag{11}$$

Since $0 < \rho_i \leqslant 1$, there is $(\rho_i)^\alpha \leqslant 1$ and $(\frac{y_i}{\rho_i})^\alpha \geqslant (y_i)^\alpha$. So, by (8) and $\mu_{i,N} \geqslant 1$

$$\left(\frac{y_i}{\rho_i}\right)^\alpha \geqslant (c^*)^\alpha(\mu_{i,N})^\alpha \geqslant (c^*)^\alpha \mu_{i,N}.$$

From the above and (11)

$$f(i, y_i) \geqslant \frac{f(i, k^* \rho_i)}{(k^*)^\alpha (\rho_i)^\alpha} (y_i)^\alpha > \left(\frac{c^*}{k^*} \right)^\alpha \mu_{i,N} f(i, k^* \rho_i).$$

Summing the above, by (10) we get

$$\infty > \sum_{i=N}^{\infty} f(i, y_i) \geqslant \left(\frac{c^*}{k^*} \right)^\alpha \sum_{i=N}^{\infty} \mu_{i,N} f(i, k^* \rho_i)$$

for some positive constant k^*. By (H1) and the above

$$\sum_{n=N}^{\infty} \nu_{n,N} f(n, k^* \rho_n) < \infty.$$

This contradicts (7), which gives us the required result.

Case (II)

Multiplying equation (E) by $\mu_{n,N}$, we get

$$\mu_{n,N} f(n, y_n) = -\mu_{n,N} \Delta(a_n \Delta(b_n \Delta(c_n \Delta y_n))).$$

Summing the above equality from N to $n-1$, we derive

$$\begin{aligned}
\sum_{i=N}^{n-1} \mu_{i,N} f(i, y_i) &= -\sum_{i=N}^{n-1} \mu_{i,N} \Delta(a_i \Delta(b_i \Delta(c_i \Delta y_i))) = \\
&= -\sum_{i=N}^{n-1} \Delta(\mu_{i-1,N} a_i \Delta(b_i \Delta(c_i \Delta y_i))) + \sum_{i=N}^{n-1} \Delta(\mu_{i-1,N})(a_i \Delta(b_i \Delta(c_i \Delta y_i))) = \\
&= -\mu_{n-1,N} a_n \Delta(b_n \Delta(c_n \Delta y_n)) + \mu_{N-1,N} a_N \Delta(b_N \Delta(c_N \Delta y_N)) + \\
&\quad + \sum_{i=N}^{n-1} \frac{1}{c_{i-1}} a_i \Delta(b_i \Delta(c_i \Delta y_i)) = \\
&= -\mu_{n-1,N} a_n \Delta(b_n \Delta(c_n \Delta y_n)) + \sum_{i=N}^{n-1} \frac{a_i}{c_{i-1}} \Delta(b_i \Delta(c_i \Delta y_i)).
\end{aligned}$$

So, by (6)

$$\begin{aligned}
\sum_{i=N}^{n-1} \mu_{i,N} f(i, y_i) &\leqslant M \sum_{i=N}^{n-1} \Delta(b_i \Delta(c_i \Delta y_i)) - \mu_{n-1,N} a_n \Delta(b_n \Delta(c_n \Delta y_n)) = \\
&= M b_n \Delta(c_n \Delta y_n) - M b_N \Delta(c_N \Delta y_N) - \mu_{n-1,N} a_n \Delta(b_n \Delta(c_n \Delta y_n)) < \\
&< -M b_N \Delta(c_N \Delta y_N) - \mu_{n-1,N} a_n \Delta(b_n \Delta(c_n \Delta y_n)).
\end{aligned}$$

Hence

$$\sum_{i=1}^{\infty} \mu_{i,N} f(i, y_i) < \infty. \tag{12}$$

From (11)

$$f(n, y_n) \geqslant (k^*)^{-\alpha} \left(\frac{y_n}{\rho_n}\right)^{\alpha} f(n, k^*\rho_n).$$

Hence, by (3)

$$\sum_{n=N}^{\infty} \mu_{n,N} f(n, y_n) \geqslant (k^*)^{\alpha} \sum_{n=N}^{\infty} \left(\frac{y_n}{\rho_n}\right)^{\alpha} f(n, k^*\rho_n)\mu_{n,N} \geqslant \sum_{n=N}^{\infty} \mu_{n,N} f(n, k^*\rho_n).$$

By (12), $(H1)$ and the above

$$\sum_{n=N}^{\infty} \nu_{n,N} f(n, k^*\rho_n) < \infty.$$

This contradicts (7), which gives us the required result.

Case (III)
By the mean value theorem

$$-\Delta\left([a_n\Delta(-b_n\Delta(c_n\Delta y_n))]^{1-\alpha}\right) = (1-\alpha)\xi^{-\alpha}\Delta(a_n\Delta(b_n\Delta(c_n\Delta y_n))),$$

where $\alpha > 1$ and $a_n\Delta(-b_n\Delta(c_n\Delta y_n)) < \xi < a_{n+1}\Delta(-b_{n+1}\Delta(c_{n+1}\Delta y_{n+1}))$.

Therefore, using equation (E), (3), (2) and the strong superlinearity of equation (E) in the above, we get

$$\begin{aligned}
&-\Delta([a_n\Delta(-b_n\Delta(c_n\Delta y_n)]^{1-\alpha}) \geqslant (\alpha-1)[a_n\Delta(-b_n\Delta(c_n\Delta y_n))]^{-\alpha} f(n, y_n) = \\
&\qquad = (\alpha-1)[a_n\Delta(-b_n\Delta(c_n\Delta y_n))]^{-\alpha} \frac{f(n, y_n)}{(y_n)^{\alpha}} (y_n)^{\alpha} \geqslant \\
&\geqslant (\alpha-1)[a_n\Delta(-b_n\Delta(c_n\Delta y_n))]^{-\alpha} \frac{f(n, k^*\rho_n)}{(k^*\rho_n)^{\alpha}} \left(-k\Delta(b_n\Delta(c_n\Delta y_n))\rho_n \sum_{i=N}^{n-1} \frac{1}{a_i}\right)^{\alpha} = \\
&\qquad = (\alpha-1)\left(\frac{k}{k^*}\right)^{\alpha} (a_n)^{-\alpha} \left(\sum_{i=N}^{n-1} \frac{1}{a_i}\right)^{\alpha} f(n, k^*\rho_n),
\end{aligned}$$

for $n \geqslant N$. Summing the above, we get

$$\begin{aligned}
&-[a_n\Delta(-b_n\Delta(c_n\Delta y_n))]^{1-\alpha} + [a_N\Delta(-b_N\Delta(c_N\Delta y_N))]^{1-\alpha} \geqslant \\
&\qquad \geqslant (\alpha-1)\left(\frac{k}{k^*}\right)^{\alpha} \sum_{i=N}^{n-1} \frac{1}{(a_i)^{\alpha}} \left(\sum_{j=N}^{i-1} \frac{1}{a_j}\right)^{\alpha} f(i, k^*\rho_i).
\end{aligned}$$

Since (4), by (6) there is $a_j \leqslant Mc_j$. Hence

$$\begin{aligned}
[a_N\Delta(-b_N\Delta(c_N\Delta y_N))]^{1-\alpha} &\geqslant (\alpha-1)\left(\frac{k}{k^*}\right)^{\alpha} \sum_{i=N}^{n-1} \frac{1}{(a_i)^{\alpha}} \left(\sum_{j=N}^{i-1} \frac{1}{a_j}\right)^{\alpha} f(i, k^*\rho_i) \geqslant \\
&\geqslant (\alpha-1)\left(\frac{kM^{\alpha}}{k^*}\right)^{\alpha} \sum_{i=N}^{n-1} (\nu_{i,N})^{\alpha} f(i, k^*\rho_i).
\end{aligned}$$

So

$$\sum_{i=N}^{\infty} (\nu_{i,N})^{\alpha} f(i, k^* \rho_i) < \infty.$$

Since (5) then

$$\sum_{i=N}^{\infty} \nu_{i,N} f(i, k^* \rho_i) < \infty.$$

This contradicts (7), which gives us the required result.

Case (IV)
Summing equation (E) from n to ∞, we get

$$a_n \Delta(b_n \Delta(c_n \Delta y_n)) \geqslant \sum_{i=n}^{\infty} f(i, y_i),$$

and by (6)

$$\Delta(b_n \Delta(c_n \Delta y_n)) \geqslant \frac{1}{a_n} \sum_{i=n}^{\infty} f(i, y_i) \geqslant \frac{1}{M} \frac{1}{c_{n-1}} \sum_{i=n}^{\infty} f(i, y_i).$$

Summing the above from N to $n-1$, we obtain

$$\begin{aligned} b_n \Delta(c_n \Delta y_n) - b_N \Delta(c_N \Delta y_N) &\geqslant \\ &\geqslant \frac{1}{M} \sum_{i=N}^{n-1} \frac{1}{c_{i-1}} \sum_{j=i}^{\infty} f(j, y_j) \geqslant \frac{1}{M} \sum_{i=N}^{n-1} \frac{1}{c_{i-1}} \sum_{j=i}^{n-1} f(j, y_j) = \\ &= \frac{1}{M} \sum_{i=N}^{n-1} \left(f(i, y_i) \sum_{j=N}^{i} \frac{1}{c_{j-1}} \right) \geqslant \frac{1}{M} \sum_{i=N}^{n-1} \left(f(i, y_i) \sum_{j=N+1}^{i} \frac{1}{c_{j-1}} \right) = \\ &= \frac{1}{M} \sum_{i=N}^{n-1} f(i, y_i) \mu_{i,N}. \end{aligned}$$

Hence

$$b_n \Delta(c_n \Delta y_n) \geqslant \frac{1}{M} \sum_{i=N}^{n-1} \mu_{i,N} f(i, y_i). \tag{13}$$

The strong superlinearity of equation (E), (3), (1) and $\rho_n \leqslant 1$ imply

$$f(i, y_i) \geqslant \frac{f(i, k^* \rho_i)}{(k^* \rho_i)^{\alpha}} [(b_i \Delta(c_i \Delta y_i)) \rho_i]^{\alpha} \geqslant (k^*)^{-\alpha} [b_i \Delta(c_i \Delta y_i)]^{\alpha} f(i, k^* \rho_i).$$

From (13) and the above

$$[b_n\Delta(c_n\Delta y_n)]^{-\alpha} \leqslant \left(\frac{1}{M}\sum_{i=N}^{n-1}\mu_{i,N}f(i,y_i)\right)^{-\alpha} \leqslant$$
$$\leqslant \left(\frac{1}{M}\sum_{i=N}^{n-1}\mu_{i,N}(k^*)^{-\alpha}[b_i\Delta(c_i\Delta y_i)]^{\alpha}f(i,k^*\rho_i)\right)^{-\alpha}.$$

Multiplying the last inequality by

$$\mu_{N,n}[b_n\Delta(c_n\Delta y_n)]^{\alpha}f(n,k^*\rho_n)$$

we get

$$\mu_{N,n}f(n,k^*\rho_n) \leqslant$$
$$\leqslant \frac{1}{(k^*M)^{\alpha}}\left(\sum_{i=N}^{n-1}\mu_{i,N}[b_i\Delta(c_i\Delta y_i)]^{\alpha}f(i,k^*\rho_i)\right)^{-\alpha}\mu_{N,n}(b_n\Delta(c_n\Delta y_n))^{\alpha}f(n,k^*\rho_n) \leqslant$$
$$\leqslant \frac{1}{(k^*M)^{\alpha}}\frac{\mu_{n,N}(b_n\Delta(c_n\Delta y_n))^{\alpha}f(n,k^*\rho_n)}{\left(\sum\limits_{i=N}^{n-1}\mu_{i,N}[b_i\Delta(c_i\Delta y_i)]^{\alpha}f(i,k^*\rho_i)\right)^{\alpha}}.$$

Summing the above from N to $n-1$, we get

$$\sum_{i=N}^{n-1}\mu_{N,i}f(i,k^*\rho_i) \leqslant \frac{1}{(k^*M)^{\alpha}}\sum_{i=N}^{n-1}\frac{\mu_{i,N}(b_i\Delta(c_i\Delta y_i))^{\alpha}f(i,k^*\rho_i)}{\left(\sum\limits_{j=N}^{i-1}\mu_{j,N}[b_j\Delta(c_j\Delta y_j)]^{\alpha}f(j,k^*\rho_j)\right)^{\alpha}} \leqslant$$
$$\leqslant \frac{1}{(k^*M)^{\alpha}}\sum_{i=N}^{\infty}\frac{\Delta z_i}{(z_i)^{\alpha}} < \infty$$

where $z_i = \sum\limits_{j=N}^{i-1}\mu_{j,N}[b_j\Delta(c_j\Delta y_j)]^{\alpha}f(j,k^*\rho_j)$. Hence, by (12), (H1)

$$\sum_{n=N}^{\infty}\nu_{n,N}f(n,k^*\rho_n) < \infty.$$

This contradicts (7), which gives us the required result.

This completes the proof of this theorem. □

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