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OSCILLATORY AND ASYMPTOTICALLY ZERO SOLUTIONS OF THIRD ORDER DIFFERENCE EQUATIONS WITH QUASIDIFFERENCES

Abstract. In this paper, third order difference equations are considered. We study the nonlinear third order difference equation with quasidifferences. Using Riccati transformation techniques, we establish some sufficient conditions for each solution of this equation to be either oscillatory or converging to zero. The result is illustrated with examples.

Keywords: linear, nonlinear, difference equations, third order, oscillatory solution, quasidifferences.

Mathematics Subject Classification: 39A10, 39A11.

1. INTRODUCTION

In this paper we consider the nonlinear difference equation of the form

$$\Delta(a_n\Delta(b_n\Delta y_n)) + p_n f(y_{n+l}) = 0, \tag{1}$$

where $l \in \{0, 1, \ldots\}$. Here Δ denotes the forward difference operator $\Delta x_n = x_{n+1} - x_n$ for $x : N \to R$. Sequences (a_n) and (b_n) are positive sequences such that

$$\sum_{n=1}^{\infty} \frac{1}{a_n} = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{b_n} = \infty. \tag{2}$$

Sequence (p_n) is positive, too. Function $f : R \to R$ is continuous and such that

$$uf(u) > 0 \quad \text{for} \quad u \neq 0, \tag{3}$$

and there exists a positive real number K such that

$$\frac{f(u)}{u} \geqslant K \quad \text{for} \quad u \geqslant \epsilon > 0. \tag{4}$$

By a solution of equation (1), we mean a nontrivial sequence (x_n) which satisfies equation (1) for n sufficiently large. A solution of equation (1) is said to be oscillatory if for every positive integer M there exists $n \geqslant M$ such that $x_n x_{n+1} \leqslant 0$. Otherwise, it is called nonoscillatory.

Let $r^{(i)}$ $(i = 1, 2, \ldots, m)$ be positive real sequences. For any real sequence x we denote

$$L_0(x_n) = x_n,$$
$$L_i(x_n) = r_n^{(i)} \Delta L_{i-1}(x_n), \qquad i = 1, 2, \ldots, m, \quad n \in N.$$

Following above definition, we can say that we consider a third order difference equation with quasidifferences.

In recent years, the study of the oscillatory and asymptotic properties of solutions of nonlinear difference equations and their applications has been a subject of great interest; see for example monographs by Agarwal [1], Elaydi [4] and Kelley and Peterson [6]. The study of third order difference equations has also received much attention. Third order linear difference equations were studied in Saker [11], Smith [12], [13], Smith and Taylor [14], and nonlinear ones were studied by Andruch-Sobiło and Migda [2], Došla and Kobza [3], Graef and Thandapani [5], Kobza [7], Migda, Schmeidel and Drozdowicz [9], Popenda and Schmeidel [10], and by Thandapani and Mahalingam [15].

2. PRELIMINARY RESULTS

We start with generalized Knaster Theorem proved by Migda in [8].

Theorem 1. *Suppose that*

$$\sum_{n=1}^{\infty} \frac{1}{r_n^{(i)}} = \infty \quad \textit{for all} \quad i = 1, 2, \ldots, m.$$

Let $x : N \to R \backslash \{0\}$ be a sequence of a constant sign. If $L_m(x_n)$ is of a constant sign and not identically zero for $n > M$ and for some $j \in \{1, 2\}$

$$(-1)^j x_n L_m(x_n) \geqslant 0 \quad \textit{for} \quad n \geqslant M,$$

then there exists an integer $l \in \{0, 1, \ldots, m\}$ with $m + l + j$ even, such that

$$x_n L_i(x_n) > 0 \quad \textit{for large} \quad n \quad \textit{and} \quad i = 0, 1, \ldots, l;$$
$$(-1)^{l+i} x_n L_i(x_n) > 0 \quad \textit{for all} \quad n \geqslant M, \quad i = l+1, l+2, \ldots, m.$$

Suppose that y is a nonoscillatory solution of equation (1) and condition (4) holds. Because p is a positive sequence, then quasidifference $\Delta(a_n\Delta(b_n\Delta y_n))$ is of a constant sign and not identically zero for sufficiently large n. Using above and putting, in the Theorem 1:

$$m = 3, \quad r^{(1)} = b, \quad r^{(2)} = a \quad \text{and} \quad r^{(3)} \equiv 1,$$

we obtain the following special case of Theorem 1.

Theorem 2. *Let (y_n) be a nonoscillatory solution of equation (1). Assume that conditions (2) and (3) hold. Then exactly one of the following cases holds for all sufficiently large n:*

$$y_n > 0,\ \Delta y_n > 0,\ \Delta(b_n\Delta y_n) > 0,\ (or\ \ y_n < 0, \Delta y_n < 0, \Delta(b_n\Delta y_n) < 0)$$
$$y_n > 0,\ \Delta y_n < 0,\ \Delta(b_n\Delta y_n) > 0,\ \ (or\ \ y_n < 0, \Delta y_n > 0, \Delta(b_n\Delta y_n) < 0).$$

3. MAIN RESULT

By using the Riccati transformation techniques we establish sufficient conditions for each solution to be oscillatory or tend to zero.

Theorem 3. *Assume conditions (2), (3) and (4) hold,*

$$a_n b_n \geqslant 1,\ \ for\ n \in N, \tag{5}$$

$$\liminf_{n\to\infty} \sum_{k=1}^{n-1} \frac{1}{b_k} \sum_{j=1}^{n-1} \frac{1}{a_j} \sum_{i=n}^{\infty} p_i = \infty, \tag{6}$$

and there exists a positive sequence ρ such that

$$\limsup_{n\to\infty} \sum_{i=n_1}^{n} \left[K\rho_i p_i - \frac{(\Delta\rho_i)^2}{4\rho_i(i-n_0)a_{i+1}b_{i+1}}\right] = \infty,\ \ for\ n_1 > n_0, \tag{7}$$

where K is given by (4). Then every solution y of equation (1) is oscillatory or tends to zero.

Proof. Let (y_n) be a nonoscillatory solution of (1). Without loss of generality, let us assume $y_n > 0$ eventually. Hence, by Theorem 2, one of the following cases

$$y_n > 0,\ \Delta y_n > 0,\ \Delta(b_n\Delta y_n) > 0, \tag{8}$$
$$y_n > 0,\ \Delta y_n < 0,\ \Delta(b_n\Delta y_n) > 0, \tag{9}$$

holds for all sufficiently large n.

We consider case (8) first. Let $n_0 \in N$ be so large that condition (8) holds, for $n > n_0$. Let ρ be a positive sequence. We define w_n by some modification of the *Riccati* substitution

$$w_n = \rho_n \frac{a_n\Delta(b_n\Delta y_n)}{y_{n+l}} = a_n\Delta(b_n\Delta y_n)\frac{\rho_n}{y_{n+l}}.$$

Thus w is a positive sequence, too. Hence

$$\Delta w_n = a_{n+1}\Delta(b_{n+1}\Delta y_{n+1})\Delta\left(\frac{\rho_n}{y_{n+l}}\right) + \frac{\rho_n}{y_{n+l}}\Delta(a_n\Delta(b_n\Delta y_n)).$$

Using equation (3), we obtain

$$\Delta w_n = a_{n+1}\Delta(b_{n+1}\Delta y_{n+1})\Delta\left(\frac{\rho_n}{y_{n+l}}\right) - \frac{\rho_n}{y_{n+l}}p_n f(y_{n+1}) =$$
$$= a_{n+1}\Delta(b_{n+1}\Delta y_{n+1})\left(\frac{y_{n+l}\Delta\rho_n - \rho_n\Delta y_{n+l}}{y_{n+l}y_{n+l+1}}\right) - \frac{\rho_n}{y_{n+l}}p_n f(y_{n+l}).$$

Condition (8) implies that $\lim\limits_{n\to\infty} y_n > 0$. Then there exists $\epsilon > 0$ such that $y_n > \epsilon$ for sufficiently large n.

From condition (4) there follow

$$-f(y_{n+1}) \leqslant -Ky_{n+1},$$

for some constant K. Thus, we obtain

$$\Delta w_n \leqslant \frac{\Delta\rho_n}{\rho_{n+1}}w_{n+1} - \frac{\rho_n\Delta y_{n+l}(a_{n+1}\Delta(b_{n+1}\Delta y_{n+1}))}{y_{n+l}y_{n+l+1}} - \frac{\rho_n}{y_{n+l}}Kp_n y_{n+l}.$$

Because for $n > n_0$ the sequence y increases, then

$$\Delta w_n \leqslant \frac{\Delta\rho_n}{\rho_{n+1}}w_{n+1} - \frac{\rho_n\Delta y_{n+l}(a_{n+1}\Delta(b_{n+1}\Delta y_{n+1}))}{(y_{n+l+1})^2} - \rho_n K p_n.$$

Because for $n > n_0$ the sequence $(b_n\Delta y_n)$ is positive and sequence $(\Delta(a_n(\Delta(b_n\Delta y_n))))$ decreases, we see that

$$b_n\Delta y_n - b_{n_0}\Delta y_{n_0} = \sum_{i=n_0}^{n-1}\Delta(b_i\Delta y_i) > (n-n_0)\Delta(b_{n+1}\Delta y_{n+1}).$$

The sequence $(b_n\Delta y_n)$ decrease for large n, hence

$$-\Delta y_{n+l} < -\frac{(n-n_0)\Delta(b_{n+1}\Delta y_{n+1})}{b_{n+l}}.$$

From the above,

$$\Delta w_n < \frac{\Delta\rho_n}{\rho_{n+1}}w_{n+1} - \frac{\rho_n(n-n_0)(a_{n+1}\Delta(b_{n+1}\Delta y_{n+1}))^2}{(y_{n+l+1})^2 a_{n+1}b_{n+1}} - K\rho_n p_n =$$
$$= \frac{\Delta\rho_n}{\rho_{n+1}}w_{n+1} - \frac{\rho_n(n-n_0)w_{n+1}^2}{\rho_{n+1}^2 a_{n+1}b_{n+1}} - K\rho_n p_n.$$

Using condition (5), we obtain

$$\Delta w_n \leqslant \frac{\Delta\rho_n}{\rho_{n+1}a_{n+1}b_{n+1}}w_{n+1} - \frac{\rho_n(n-n_0)w_{n+1}^2}{\rho_{n+1}^2 a_{n+1}b_{n+1}} - K\rho_n p_n =$$

$$= -K\rho_n p_n + \frac{(\Delta\rho_n)^2}{4\rho_n(n-n_0)a_{n+1}b_{n+1}} -$$

$$- \frac{\rho_n(n-n_0)}{\rho_{n+1}^2 a_{n+1}b_{n+1}}w_{n+1}^2 + \frac{\Delta\rho_n}{\rho_{n+1}a_{n+1}b_{n+1}}w_{n+1} - \frac{(\Delta\rho_n)^2}{4\rho_n(n-n_0)a_{n+1}b_{n+1}} =$$

$$= -K\rho_n p_n + \frac{(\Delta\rho_n)^2}{4\rho_n(n-n_0)a_{n+1}b_{n+1}} -$$

$$- \left(\frac{w_{n+1}}{\rho_{n+1}}\sqrt{\frac{\rho_n(n-n_0)}{a_{n+1}b_{n+1}}} - \frac{\Delta\rho_n}{2\sqrt{\rho_n(n-n_0)a_{n+1}b_{n+1}}}\right)^2 <$$

$$< -K\rho_n p_n + \frac{(\Delta\rho_n)^2}{4\rho_n(n-n_0)a_{n+1}b_{n+1}} = -\left(K\rho_n p_n - \frac{(\Delta\rho_n)^2}{4\rho_n(n-n_0)a_{n+1}b_{n+1}}\right).$$

Summing the above inequality from $n_1 > n_0$ to n, we get

$$w_{n+1} - w_{n_1} < -\sum_{i=n_1}^{n}\left(K\rho_i p_i - \frac{(\Delta\rho_i)^2}{4\rho_i(i-n_0)a_{i+1}b_{i+1}}\right).$$

Hence

$$-w_{n_1} < -\sum_{i=n_0}^{n}\left(K\rho_i p_i - \frac{(\Delta\rho_i)^2}{4\rho_i(i-n_0)-a_{i+1}b_{i+1}}\right),$$

which yields

$$\sum_{i=n_1}^{n}\left(K\rho_i p_i - \frac{(\Delta\rho_i)^2}{4\rho_i(i-n_0)a_{i+1}b_{i+1}}\right) < C$$

for all large n. The above inequality contradicts (7).
Next we consider case (9). Since y is a positive and decreasing sequence, it follows that

$$\lim_{n\to\infty} y_n = c \geqslant 0.$$

Set $c > 0$. This implies that there exists $n_2 \in N$ such that $y_n \geqslant c$ for $n \geqslant n_2$. Therefore, from equation (1) and condition (4), we get

$$\Delta(a_n\Delta(b_n\Delta y_n)) + Kcp_n \leqslant 0, \qquad \text{for} \qquad n \geqslant n_2.$$

Hence

$$\Delta(a_n\Delta(b_n\Delta y_n)) \leqslant -Kcp_n.$$

Choose n_3 so large that inequality given by (9) holds, and $n_4 = max\{n_2, n_3\}$. Summing the above inequality from n_4 to $n-1$, we obtain

$$a_n\Delta(b_n\Delta y_n) \leqslant a_{n_4}\Delta(b_{n_4}\Delta y_{n_4}) - Kc\sum_{i=n_4}^{n-1} p_i. \tag{10}$$

Two cases are possible:

1° $\sum\limits_{i=1}^{\infty} p_i = \infty$,

or

2° $\sum\limits_{i=1}^{\infty} p_i < \infty$.

In case 1°, the left hand side of inequality (10) is positive for $n > n_4$, but the left hand side of this inequality approaches minus infinity. This contradiction give us $c = 0$.

Consider case 2°. Set $\max\limits_{[\frac{c}{2},2c]} f(x) = m$. Since $x \in [\frac{c}{2}, 2c]$, $c > 0$ and (3) there is $m > 0$. From equation (1) and continuity of function f, we get

$$0 < a_n \Delta(b_n \Delta y_n) = \sum_{i=n}^{\infty} p_i f(y_{i+l}) \leqslant m \sum_{i=n}^{\infty} p_i,$$

for sufficiently large n. Hence

$$\lim_{n\to\infty} a_n \Delta(b_n \Delta y_n) = 0.$$

Letting $n \to \infty$ in (10), from the above we obtain

$$a_k \Delta(b_k \Delta y_k) \geqslant Kc \sum_{i=k}^{\infty} p_i.$$

Rewrite it as follows

$$a_n \Delta(b_n \Delta y_n) \geqslant Kc \sum_{i=n}^{\infty} p_i.$$

Dividing by a_n and summing the above inequality from n_1 to $n-1$ we obtain

$$b_n \Delta y_n - b_{n_4} \Delta y_{n_4} \geqslant Kc \sum_{j=n_4}^{n-1} \frac{1}{a_j} \sum_{i=n}^{\infty} p_i.$$

Since $b_{n_4} \Delta y_{n_4} > 0$ we get

$$b_n \Delta y_n > Kc \sum_{j=n_4}^{n-1} \frac{1}{a_j} \sum_{i=n}^{\infty} p_i.$$

Dividing by b_n and summing again we derive

$$y_n - y_{n_4} > Kc \sum_{k=n_4}^{n-1} \frac{1}{b_j} \sum_{j=n_4}^{n-1} \frac{1}{a_j} \sum_{i=n}^{\infty} p_i.$$

Since $y_{n_4} > 0$ we obtain

$$y_n > Kc \sum_{k=n_4}^{n-1} \frac{1}{b_k} \sum_{j=n_4}^{n-1} \frac{1}{a_j} \sum_{i=n}^{\infty} p_i.$$

Since (9) and (6) hold, this is not possible. This contradiction give us $c = 0$. The proof is complete. □

Example 1. Consider the difference equation

$$\Delta^3 y_n + 8y_n = 0. \tag{11}$$

All assumption of Theorem 3 hold (with $\rho_n \equiv 1$). It is easy to check that $y_n = (-1)^n$ is an oscillatory solution of equation (11).

Example 2. Consider the difference equation

$$\Delta^3 y_n + \frac{1}{4} y_{n+1} = 0. \tag{12}$$

All assumption of Theorem 3 hold (with $\rho_n \equiv 1$). It is easy to check that $y_n = \frac{1}{2^n}$ is a solution of equation (12) which tends to zero as n tends to infinity.

Example 3. Consider the difference equation

$$\Delta \left(\frac{2}{n-1} \Delta(2n\Delta y_n) \right) + 2^{n-1} y_{n+1}^2 = 0. \tag{13}$$

All assumption of Theorem 3 hold (with $\rho_n \equiv 1$). It is easy to check that $y_n = \frac{1}{2^n}$ is an solution of equation (13) which tends to zero as n tends to infinity.

Example 4. Consider the difference equation

$$\Delta^2((n+1)\Delta y_n)) + \frac{2}{(n+1)(n+2)} y_n = 0. \tag{14}$$

All assumptions of Theorem 3 hold (with $\rho_n = n$). It is easy to check that $y_n = \frac{1}{n}$ is a solution of equation (14) which tends to zero as n tends to infinity.

Example 5. Consider the difference equation

$$\Delta^2((n+1)\Delta y_n)) + \frac{2}{n(n+2)} y_{n+1} = 0. \tag{15}$$

All assumption of Theorem 3 hold (with $\rho_n = n$). It is easy to check that $y_n = \frac{1}{n}$ is a solution of equation (15) which tends to zero as n tends to infinity.

Example 6. Consider the difference equation

$$\Delta^2((n+1)\Delta y_n)) + \frac{2}{n(n+1)} y_{n+2} = 0. \tag{16}$$

All assumption of Theorem 3 hold (where $\rho_n = n$). It is easy to check that $y_n = \frac{1}{n}$ is a solution of equation (16) which tends to zero as n tends to infinity.

REFERENCES

[1] Agarwal R. P., *Difference Equations and Inequalities.* Theory, Methods and Applications, Marcel Dekker, Inc., New York, 2000 (2nd ed.).

[2] Andruch-Sobiło A., Migda M., *Bounded solutions of third order nonlinear difference equations,* Rocky Mountain Journal of Mathematics 36 (2006) 1, 23–24.

[3] Došla Z., Kobza A., *Global asymptotic properties of third order difference equations,* Computers and Mathematics with Applications 48, 1–2 (2004), 191–200.

[4] Elaydi S. N., *An Introduction to Difference Equations*, Springer, 1996.

[5] Graef J. R., Thandapani E., *Oscillatory and asymptotic behavior of solutions of third order delay difference equations*, Funkcialaj Ekvacioj 42 (1999) 3, 355–369.

[6] Kelley W. G., Peterson A. C., *Difference Equations*, an Introduction with Applications, Academic Press, 1991.

[7] Kobza A., *Property A for third order difference equations*, Studies of the University of Zilina, Mathematical Series 17 (2003), 109–114.

[8] Migda M., *On the discrete version of generalized Kiguradzes Lemma*, Facsiculi Mathematici 35 (2005), 77–84.

[9] Migda M., Schmeidel E., Drozdowicz A., *Nonoscillation results for some third order nonlinear difference equation*, Folia Facultatis Scientiarium Naturalium Universitatis Masarykianae Brunensis Mathematica 13 (2003), 185–192.

[10] Popenda J., Schmeidel J., *Nonoscillatory solutions of third order difference equations*, Portugaliae Mathematica 49 (1992), 233–239.

[11] Saker S. H., *Oscillation of third-order difference equations*, Portugaliae Mathematica 61 (2004) 3, 249–257.

[12] Smith B., *Oscillatory and asymptotic behavior in certain third order difference equations*, Rocky Mountain Journal of Mathematics 17 (1987) 3, 597–606.

[13] Smith B., *Oscillation and no oscillation theorems for third order quasi-adjoint difference equations*, Portugaliae Mathematica 45 (1988) 3, 229–243.

[14] Smith B., Taylor W. E., *Oscillatory and asymptotic behavior of certain fourth order difference equations*, Rocky Mountain Journal of Mathematics 16 (1986) 2, 403–406.

[15] Thandapani E., Mahalingam K., *Oscillatory properties of third order neutral delay difference equatons,* Demonstratio Mathematica 35 (2002), 325–337.

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Received: August 29, 2005.