Jarosław Mikołajski, Ewa Schmeidel

COMPARISON OF PROPERTIES OF SOLUTIONS OF DIFFERENTIAL EQUATIONS AND RECURRENCE EQUATIONS WITH THE SAME CHARACTERISTIC EQUATION (ON EXAMPLE OF THIRD ORDER LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS)

Abstract. Third order linear homogeneous differential and recurrence equations with constant coefficients are considered. We take the both equations with the same characteristic equation. We show that these equations (differential and recurrence) can have solutions with different properties concerning oscillation and boundedness. Especially the numbers of suitable types of solutions taken out from fundamental sets are presented. We give conditions under which the asymptotic properties considered are the same for the both equations.

Keywords: differential equation, recurrence, linear, third order, oscillatory solution, bounded solution.

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1. INTRODUCTION

Concurrently with the development of modern computational methods, discrete equations, so-called recurrences, have acquired a great importance. They often replace differential equations in mathematical modelling. For example, the logistic equation describing population increase is written in the form of a differential equation $y' = y(b - ay)$ (Verhulst–Pearl equation) or in the form of a recurrence equation $u_{n+1} = u_n(b - au_n)$ (Pielou equation) (see [4] and [13]).

It appears that application of a discrete mathematical model instead of the continuous one can lead to qualitatively different solutions, particularly with respect to oscillation or boundedness. Different quantities of solutions can be seen even in the case of linear equations with constant coefficients. Oscillation of such third order

equations is investigated in [11] and boundedness in [10]. In this paper we present the numbers of suitable types of solutions taken out from fundamental sets. We also give conditions under which the asymptotic properties considered are the same for two equations: differential and recurrence ones.

We consider a third order linear homogeneous equation with constant coefficients written in the continuous and discrete form, namely the differential equation

$$\frac{d^3 f}{dt^3} + a\frac{d^2 f}{dt^2} + b\frac{df}{dt} + cf = 0 \tag{1}$$

and the recurrence equation

$$u_{n+3} + au_{n+2} + bu_{n+1} + cu_n = 0, \tag{2}$$

where $f : R \to R$, $u : N \to R$, R denotes the set of real numbers, N the set of positive integers and a, b, c are real constants.

A function $f : R \to R$ (a sequence $u : N \to R$) is called trivial if there exists a $t_0 \in R$ (an $n_0 \in N$) such that $f(t) = 0$ for every $t > t_0$ ($u_n = 0$ for every $n > n_0$). Otherwise, a function f (a sequence u) is said to be nontrivial. Every nontrivial real function f (sequence u) satisfying equation (1) ((2)) is called a solution of this equation.

A solution of equation (1) ((2)) is called nonoscillatory if there exists a $t_0 \in R$ (an $n_0 \in N$) such that it is a positive function (sequence) on the set $[t_0, \infty)$ ($\{n \in N : n \geqslant n_0\}$) or a negative function (sequence) on this set. Otherwise, a solution is said to be oscillatory.

A solution of equation (1) ((2)) is called bounded if it is a bounded function (sequence) on the set $[0, \infty)$ (N). Otherwise a solution is said to be unbounded.

Equation (1) or (2) is called oscillatory (nonoscillatory, bounded, unbounded) if each of its solutions is oscillatory (nonoscillatory, bounded, unbounded).

Fundamentals of differential equations theory can be found in many monographs, e.g. Stiepanow [19]. The background of recurrences theory is given by Agarwal [1], Elaydi [4], Kelley and Peterson [6] and, in Polish, by Koźniewska [8]. Oscillation of solutions of third order difference equations was considered for example by Kobza [7], Migda, Schmeidel and Drozdowicz [9], Popenda and Schmeidel [14], Saker [15], Smith [17], Thandapani and Mahalingam [20] and boundedness by Andruch-Sobiło and Migda [2], Došla and Kobza [3], Graef and Thandapani [5], Smith [16], Smith and Taylor [18].

Putting $f(t) = \exp(rt)$ in equation (1) or $u_n = r^n$ in equation (2), the same characteristic equation

$$r^3 + ar^2 + br + c = 0 \tag{3}$$

is obtained in the continuous as well as in the discrete case. Solutions of equation (3) are described in [12].

Let us denote

$$q = c - \frac{1}{3}ab + \frac{2}{27}a^3, \tag{4}$$

$$\Delta = c^2 + \frac{4}{27}b^3 - \frac{2}{3}abc - \frac{1}{27}a^2b^2 + \frac{4}{27}a^3c. \tag{5}$$

We will use the following theorem, which can be found in [11].

Theorem 1.

1° *If $\Delta > 0$, then (3) has one real root r_1 and two complex conjugate roots r_2, r_3 with non-vanishing imaginary parts:*

$$r_1 = \sqrt[3]{\frac{-q-\sqrt{\Delta}}{2}} + \sqrt[3]{\frac{-q+\sqrt{\Delta}}{2}} - \frac{a}{3},$$

$$r_2 = -\frac{1}{2}\left(\sqrt[3]{\frac{-q-\sqrt{\Delta}}{2}} + \sqrt[3]{\frac{-q+\sqrt{\Delta}}{2}}\right) - \frac{a}{3} + i\frac{\sqrt{3}}{2}\left(\sqrt[3]{\frac{-q-\sqrt{\Delta}}{2}} - \sqrt[3]{\frac{-q+\sqrt{\Delta}}{2}}\right), \tag{6}$$

$$r_3 = -\frac{1}{2}\left(\sqrt[3]{\frac{-q-\sqrt{\Delta}}{2}} + \sqrt[3]{\frac{-q+\sqrt{\Delta}}{2}}\right) - \frac{a}{3} - i\frac{\sqrt{3}}{2}\left(\sqrt[3]{\frac{-q-\sqrt{\Delta}}{2}} - \sqrt[3]{\frac{-q+\sqrt{\Delta}}{2}}\right).$$

2° *If $\Delta = 0$, then (3) has three real roots given by (7), but one is multiple:*

$$r_1 = -2\sqrt[3]{\frac{q}{2}} - \frac{a}{3}, \quad r_2 = r_3 = \sqrt[3]{\frac{q}{2}} - \frac{a}{3}. \tag{7}$$

3° *If $\Delta < 0$, then (3) has three different real roots:*

$$r_{k+1} = \sqrt[6]{16(q^2-\Delta)}\cos\frac{\arccos\frac{-q}{\sqrt{q^2-\Delta}} + 2k\pi}{3} - \frac{a}{3}, \quad k = 0, 1, 2. \tag{8}$$

Remark 1. *The following equality*

$$q^2 - \Delta = \frac{4}{27}\left(\frac{1}{27}a^6 - \frac{1}{3}a^4b + a^2b^2 - b^3\right)$$

holds.

Remark 2. *If $\Delta = 0$ and $q \neq 0$, then the roots of (3) can also be written in form (8).*

Remark 3. *The case $\Delta = 0$ and $q = 0$ holds only for $r_1 = r_2 = r_3$.*

2. PARTITION OF SOLUTIONS IN FUNDAMENTAL SETS

Assume $c \neq 0$, which is required for (2) to be a third order equation. Then roots of equation (3) are not equal to zero. In the case $\Delta > 0$, the roots r_2 and r_3 can be written in the form

$$r_2 = |r_2|(\cos\varphi + i\sin\varphi),$$
$$r_3 = |r_2|(\cos\varphi - i\sin\varphi),$$

<div align="center">Table 1</div>

Case	Equation (1)	Equation (2)	Remarks
$\Delta > 0$	$e^{r_1 t}$, $e^{\operatorname{Re} r_2 t} \cos(\operatorname{Im} r_2 t)$, $e^{\operatorname{Re} r_2 t} \sin(\operatorname{Im} r_2 t)$	r_1^n, $\lvert r_2^n \rvert \cos n\varphi$, $\lvert r_2^n \rvert \sin n\varphi$	r_1, r_2 defined by (6), $r_2 = \lvert r_2 \rvert (\cos\varphi + i\sin\varphi)$
$\Delta = 0$, $\quad r_1 \neq r_2$	$e^{r_1 t}$, $e^{r_2 t}$, $t e^{r_2 t}$	r_1^n, r_2^n, $n r_2^n$	r_1, r_2 defined by (7)
$\Delta = 0$, $\quad r_1 = r_2$	$e^{r_1 t}$, $t e^{r_1 t}$, $t^2 e^{r_1 t}$	r_1^n, $n r_1^n$, $n^2 r_1^n$	r_1, r_2 defined by (7)
$\Delta < 0$	$e^{r_1 t}$, $e^{r_2 t}$, $e^{r_3 t}$	r_1^n, r_2^n, r_3^n	r_1, r_2, r_3 defined by (8)

where φ is an argument of r_2. We will consider fundamental sets of solutions indicated in Table 1.

Let $\Delta < 0$. Then there exist $k, l, m \in \{1, 2, 3\}$ such that $r_k < r_l < r_m$. Set $R_1 = r_k$, $R_2 = r_l$, $R_3 = r_m$.

Let $\Delta = 0$. Set $R_1 = \min\{r_1, r_2\}$, $R_2 = r_2$, $R_3 = \max\{r_1, r_2\}$. Thus $R_1 \leqslant R_2 \leqslant R_3$.

The numbers of oscillatory solutions taken out from the fundamental sets (Tab. 1) are presented in Table 2 and bounded solutions in Table 3 (se the attached interleaf).

<div align="center">Table 2</div>

Case	Equation (1)	Equation (2)
$\Delta > 0$, $r_1 < 0$	2	3
$\Delta > 0$, $r_1 > 0$	2	2
$\Delta \leqslant 0$, $R_1, R_2, R_3 < 0$	0	3
$\Delta \leqslant 0$, $R_1, R_2 < 0$, $R_3 > 0$	0	2
$\Delta \leqslant 0$, $R_1 < 0$, $R_2, R_3 > 0$	0	1
$\Delta \leqslant 0$, $R_1, R_2, R_3 > 0$	0	0

3. MAIN RESULTS

Remark 4. *Equation* (1) *is never oscillatory. Thus this equation can be nonoscillatory or it can have oscillatory and nonoscillatory solutions. However, equation* (2) *is oscillatory or nonoscillatory or it can have both oscillatory and nonoscillatory solutions.*

Let us

$$d = \sqrt[3]{\frac{-q - \sqrt{\Delta}}{2}} + \sqrt[3]{\frac{-q + \sqrt{\Delta}}{2}} \quad \text{for} \quad \Delta \geqslant 0,$$

$$D = \sqrt{d^2 + \frac{1}{3}ad + \frac{1}{9}a^2 - \frac{3}{2}\sqrt[3]{2(q^2 - \Delta)}},$$

$$p_k = \cos \frac{\arccos \frac{-q}{\sqrt{q^2 - \Delta}} + 2k\pi}{3} \quad \text{for} \quad \Delta < 0 \quad (k = 0, 1, 2).$$

Theorem 2. *Equations* (1) *and* (2) *are nonoscillatory if and only if one of the following cases holds:*

1° $\Delta \leqslant 0$, $\Delta^2 + q^2 > 0$, $p_k > \frac{a}{3\sqrt[6]{16(q^2 - \Delta)}}$ *for* $k = 0, 1, 2$,

2° $\Delta = q = 0$, $a < 0$.

Proof. In Case (1°), by Theorem 1 and Remark 2, from (8) there follows

$$r_k = \sqrt[6]{16(q^2 - \Delta)}\, p_{k-1} - \frac{a}{3} > 0$$

and, obviously, $R_k > 0$ for $k = 1, 2, 3$. In Case (ii), by Remark 3 and (7) we obtain $r_1 = r_2 = r_3 > 0$. By Table 2, the proof is completed. □

Theorem 3. *Equations* (1) *and* (2) *are bounded if and only if one of the following cases holds:*

1° $\Delta > 0$, $\max(-\frac{2}{3}a, \frac{1}{3}a - 1) \leqslant d < \frac{1}{3}a$, $D \leqslant 1$,

2° $\Delta \leqslant 0$, $\Delta^2 + q^2 > 0$, $\frac{a-3}{3\sqrt[6]{16(q^2 - \Delta)}} \leqslant p_k < \frac{a}{3\sqrt[6]{16(q^2 - \Delta)}}$ *for* $k = 0, 1, 2$ *(the equality* $p_k = \frac{a-3}{3\sqrt[6]{16(q^2 - \Delta)}}$ *is valid for at most one* k*),*

3° $\Delta = q = 0$, $0 < a < 3$.

Proof. In Case (1°), by Theorem 1, from (6) there follows $r_1 = d - \frac{1}{3}a < 0$ and $r_1 \geqslant -1$. Moreover,

$$\operatorname{Re} r_2 = -\frac{1}{2}d - \frac{1}{3}a \leqslant -\frac{1}{2}(-\frac{2}{3}a) - \frac{1}{3}a = 0$$

and

$$|r_2| = \sqrt{\frac{1}{4}d^2 + \frac{1}{3}ad + \frac{1}{9}a^2 + \frac{3}{4}\left(d^2 - 4\sqrt[3]{\frac{q^2 - \Delta}{4}}\right)} =$$

$$= \sqrt{d^2 + \frac{1}{3}ad + \frac{1}{9}a^2 - 3\sqrt[3]{\frac{q^2 - \Delta}{4}}} = D \leqslant 1.$$

In Case (2°), by Theorem 1 and Remark 2, from (8) we obtain $-1 \leqslant r_k < 0$ for $k = 1, 2, 3$ and $r_k = -1$ is valid for at most one k. The same inequalities hold for R_k ($k = 1, 2, 3$). In Case (3°), by Remark 3 and (7), there is $-1 < r_1 = r_2 = r_3 < 0$. By Table 3, the proof is completed. □

Theorem 4. *Equations* (1) *and* (2) *are unbounded if and only if one of the following cases holds:*

1° $\Delta > 0$, $\frac{1}{3}a + 1 < d < -\frac{2}{3}a$, $D > 1$,

2° $\Delta \leqslant 0$, $\Delta^2 + q^2 > 0$, $p_k > \frac{a+3}{3\sqrt[6]{16(q^2 - \Delta)}}$ *for* $k = 0, 1, 2$,

3° $\Delta = q = 0$, $a < -3$.

Proof. Analogously as in proof of Theorem 3, we obtain:
in Case (1°): $r_1 > 1$, $\operatorname{Re} r_2 > 0$, $|r_2| > 1$;
in Case (2°): R_1, R_2, $R_3 > 1$ for $k = 1, 2, 3$;
in Case (3°): $r_1 = r_2 = r_3 > 1$.
By Table 3, this completes the proof. □

Remark 5. *Case* (1°) *of Theorem 3 can only hold for* $a > 0$ *and Case* (1°) *of Theorem 4 only for* $a < -1$.

Indeed,

if $-\frac{2}{3}a < \frac{1}{3}a$, then $a > 0$;

if $\frac{1}{3}a + 1 < -\frac{2}{3}a$, then $a < -1$.

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Jarosław Mikołajski
jmikolaj@math.put.poznan.pl

Poznań University of Technology
Institute of Mathematics, Faculty of Electrical Engineering
ul. Piotrowo 3a, 60-965 Poznań, Poland

Ewa Schmeidel
ewa.schmeidel@put.poznan.pl

Poznań University of Technology
Institute of Mathematics, Faculty of Electrical Engineering
ul. Piotrowo 3a, 60-965 Poznań, Poland