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A REMARK ON THE LINEARIZATION TECHNIQUE IN HALF-LINEAR OSCILLATION THEORY*

Abstract. We show that oscillatory properties of the half-linear second order differential equation

$$\big(r(t)\Phi(x')\big)' + c(t)\Phi(x) = 0, \quad \Phi(x) = |x|^{p-2}x,\ p > 1,$$

can be investigated via oscillatory properties of a certain associated second order *linear* differential equation. In contrast to paper [6], we do not need to distinguish between the cases $p \geq 2$ and $p \in (1, 2]$. Our results also improve the oscillation and nonoscillation criteria given in [4].

Keywords: half-linear oscillation theory, oscillation and nonoscillation criteria, Riccati technique, perturbation principle.

Mathematics Subject Classification: 34C10

1. INTRODUCTION

In this paper we deal with oscillatory properties of the half-linear second order differential equation

$$(r(t)\Phi(x'))' + c(t)\Phi(x) = 0, \quad \Phi(x) = |x|^{p-2}x,\ p > 1, \tag{1}$$

where r, c are continuous functions and $r(t) > 0$. Even if the oscillation theory of (1) is very similar to that of the second order *linear* differential equation

$$(r(t)x')' + c(t)x = 0 \tag{2}$$

*) Research supported by the grant A1163401/04 of the Grant Agency of the Czech Academy of Sciences and by the Research Project MSM0021622409 of the Ministry of Education of the Czech Government

(which is the special case $p = 2$ in (1)), the missing additivity of solution space of (1) and some consequences of this fact (we mention some of them in the next section) cause that some methods of the half-linear oscillation theory are more complicated than in the linear case. For basic methods and results of the half-linear oscillation theory we refer to [1, Chap. 3] or [7, Chap. 3].

A typical example of discrepancies between "linear" and "half-linear" is the so-called *perturbation principle* in the oscillation theory of (1) (see, e.g. [3, Sec. 5.2]). We explain this discrepancy as follows. In the classical linear oscillation theory, equation (2) is viewed as a perturbation of the one-term differential equation

$$(r(t)x')' = 0 \tag{3}$$

and (non)oscillation criteria impose conditions on the function c. Roughly speaking, (2) is oscillatory (nonoscillatory) if the function c is "sufficiently positive" ("not too positive"). A more refined criteria regard (2) not as a perturbation of one term-equation (3), but as a perturbation of a general nonoscillatory equation

$$(r(t)x')' + \tilde{c}(t)x = 0, \tag{4}$$

i.e., (2) is written in the form

$$(r(t)x')' + \tilde{c}(t)x + [c(t) - \tilde{c}(t)]x = 0. \tag{5}$$

(Non)oscillation criteria are then formulated in terms of the behaviour of the function $c(t) - \tilde{c}(t)$. A typical example of this approach when $r(t) \equiv 1$ is to view the equation $x'' + c(t)x = 0$ as a perturbation of the Euler differential equation (with the so-called critical coefficient) $x'' + \frac{1}{4t^2}x = 0$. We refer to [13] for the survey of (non)oscillation criteria for (2) up to the 1970s.

However, in view of the linear transformation formula, the idea of "smuggling" the term $\tilde{c}(t)x$ into (2) actually brings no substantially new phenomena. To see this, consider the transformation of the dependent variable $x = h(t)y$, where h is a positive differentiable function such that rh' is also differentiable. The following identity, which can be verified by a short computation (suppressing the argument t) holds true:

$$h[(rx')' + cx] = (rh^2y')' + h[(rh')' + ch]y. \tag{6}$$

Now, if h is a solution of (4) and we apply the previous formula to (5), we see that x is a solution of (2) if and only if y is a solution of the equation

$$(r(t)h^2(t)y')' + [c(t) - \tilde{c}(t)]h^2(t)y = 0. \tag{7}$$

The last equation can be again viewed as a perturbation of the *one term* equation $(rh^2y')' = 0$. Therefore, regarding (2) as a perturbation of nonoscillatory *two-term* equation (4) is principially the same as regarding (2) as a perturbation of *one-term* equation (3) (which is the classical approach of the linear oscillation theory).

Concerning half-linear equations (1), there is *no half-linear analogue* of transformation formula (6). In particular, the two-term nonoscillatory equation

$$(r(t)\Phi(x'))' + \tilde{c}(t)\Phi(x) = 0 \tag{8}$$

cannot be transformed into the one-term equation of the form

$$(r(t)\Phi(x'))' = 0, \tag{9}$$

since the solution space of (8) is generally only homogeneous, while the solution space of (9) is a two-dimensional linear space spanned over the functions $x_1(t) \equiv 1$, $x_2(t) = \int^t r^{1-q}(s)\,ds$, where q is the conjugate number of p, i.e., $\frac{1}{p} + \frac{1}{q} = 1$.

Consequently, in contrast to the linear case, when equation (1) is rewritten in the form

$$(r(t)\Phi(x'))' + \tilde{c}(t)\Phi(x) + [c(t) - \tilde{c}(t)]\Phi(x) = 0 \tag{10}$$

and then regarded as a perturbation of (8), it requires a substantial modification of oscillation techniques comparing with the classical approach, when (1) is viewed as a perturbation of (9), see, e.g., [2, 5, 10, 11, 12].

The aim of this paper is to use the modified *Riccati technique* applied to (10), and using this approach to compare oscillatory properties of (1) with oscillatory properties of a certain associated *linear* equation of form (2). This enables us to use the deeply developed linear oscillation theory in investigating (1). In contrast to some previous papers, e.g., [6, 10, 11], we do not need to distinguish between the cases $p \geq 2$ and $p \in (1, 2]$ in (1). We also improve (non)oscillation criteria given in [4].

2. PRELIMINARIES

As we have mentioned before, the oscillation theory of half-linear equations is very similar to the linear oscillation theory. In particular, Sturmian theorems extend verbatim to (1), hence this equation can be classified as *oscillatory* or *nonoscillatory* according to whether any nontrivial solution has/does not have infinitely many zeros on any interval of the form $[T, \infty)$. On the other hand, in addition to the above mentioned missing half-linear analogue of transformation formula (6) (and, of course, the fact that the solution space of (1) is only homogeneous, but generally not additive), the most flagrant difference between linear and half-linear equations is the missing Wronskian-type identity in half-linear case. Recall that the linear Wronskian identity says that

$$r(t)[x_1'(t)x_2(t) - x_1(t)x'(t)] \equiv \text{const}$$

for any pair of linearly independent solutions x_1, x_2 of (2). For some other differences between linear and half-linear equations, we refer to [3, Sec. 3].

The results of our paper are based on the so-called *Riccati technique*, which consists in the fact that if x is a solution of (1) such that $x(t) \neq 0$ in some interval I, then the function $w = r\Phi(x'/x)$ solves in I the Riccati-type differential equation

$$w' + c(t) + (p-1)r^{1-q}(t)|w|^q = 0, \quad q = \frac{p}{p-1}. \tag{11}$$

More precisely, in view of the Sturmian comparison theorem, we will use the following refinement of the Riccati equivalence, which can be found, e.g., in [3, Theorem 5.3].

Lemma 1. *Equation* (1) *is nonoscillatory if and only if there exists a differentiable function w such that*

$$R[w](t) := w'(t) + c(t) + (p-1)r^{1-q}(t)|w(t)|^q \leq 0 \tag{12}$$

for large t.

We will also need the following integral modification of the Riccati equivalence, which is usually referred to as the half-linear version of the Hartman–Wintner theorem; the proof of this statement can be found e.g. in [3, Theorem 5.6].

Lemma 2. *Suppose that* (1) *is nonoscillatory and*

$$\int^{\infty} r^{1-q}(t)\, dt = \infty. \tag{13}$$

Then the following statements are equivalent:

(i) *There exists a finite limit*

$$\lim_{t\to\infty} \frac{1}{\int^t r^{1-q}(s)\, ds} \int^t r^{1-q}(s) \left(\int^s c(\tau)\, d\tau \right) ds;$$

(ii) *The integral*

$$\int^{\infty} r^{1-q}(t)|w(t)|^q\, dt$$

is convergent for every solution w of (11).

In particular, if (13) *holds and the integral $\int^{\infty} c(t)\, dt$ is convergent, then every solution of* (11) *satisfies the Riccati integral equation*

$$w(t) = \int_t^{\infty} c(s)\, ds + (p-1) \int_t^{\infty} r^{1-q}(s)|w(s)|^q\, ds. \tag{14}$$

We finish this section with a technical result which we will need in the proof of our main results. It concerns the function

$$P(u,v) := \frac{|u|^p}{p} - uv + \frac{|v|^q}{q} \tag{15}$$

and its proof can be found, e.g., in [8].

Lemma 3. *The function $P(u,v)$ defined in (15) satisfies the following inequalities*

$$\begin{aligned} P(u,v) &\geq \frac{1}{2}|u|^{2-p}(v-\Phi(u))^2 \quad \textit{for} \quad p \leq 2, \\ P(u,v) &\leq \frac{1}{2}|u|^{2-p}(v-\Phi(u))^2 \quad \textit{for} \quad p \geq 2,\ u \neq 0. \end{aligned}$$

Futhermore, let $T > 0$ be arbitrary. Then there exists a constant $K = K(T) > 0$ such that

$$\begin{aligned} P(u,v) &\geq K|u|^{2-p}(v-\Phi(u))^2 \quad \textit{for} \quad p \geq 2 \\ P(u,v) &\leq K|u|^{2-p}(v-\Phi(u))^2 \quad \textit{for} \quad p \leq 2, \end{aligned}$$

and every $u, v \in \mathbb{R}$ satisfying $\left|\frac{v}{\Phi(u)}\right| \leq T$.

3. OSCILLATION AND NONOSCILLATION CRITERIA

Our first main result reads as follows.

Theorem 1. *Let $\int^\infty r^{1-q}(t)\,dt = \infty$, $\int^\infty c(t)\,dt$ be convergent, and $\int_t^\infty c(s)\,ds \geq 0$ for large t. Further suppose that equation (8) is nonoscillatory and possesses a positive solution h such there exists a finite limit*

$$\lim_{t\to\infty} r(t)h(t)\Phi(h'(t)) =: L > 0 \tag{16}$$

and

$$\int^\infty \frac{dt}{r(t)h^2(t)(h'(t))^{p-2}} = \infty. \tag{17}$$

Finally suppose that

$$\int^\infty [c(t)-\tilde{c}(t)]h^p(t)\,dt \quad \textit{converges.} \tag{18}$$

Denote

$$R(t) = r(t)h^2(t)(h'(t))^{p-2}, \quad C(t) = [c(t)-\tilde{c}(t)]h^p(t). \tag{19}$$

If there exists $\varepsilon > 0$ such that the linear equation

$$(R(t)y')' + \left(\frac{q}{2} - \varepsilon\right)C(t) = 0 \tag{20}$$

is oscillatory, then also (1) is oscillatory.

Proof. Suppose, by contradiction, that (1) is nonoscillatory, i.e., by Lemma 1, there exists a differentiable function w such that (12) holds for large t. Denote $w_h = r\Phi(h'/h)$ and put $v = h^p[w - w_h]$. Then (suppressing the argument t)

$$\begin{aligned} v' &= ph^{p-1}h'[w - w_h] + h^p[-c - (p-1)r^{1-q}|w|^q + \tilde{c} + (p-1)r^{1-q}|w_h|^q] = \\ &= pr^{1-q}h^p\left\{\Phi^{-1}(w_h)w - |w_h|^q - \frac{1}{q}|w|^q + \frac{1}{q}|w_h|^q\right\} - C = \\ &= -C - pr^{1-q}h^p P(\Phi^{-1}(w_h), w), \end{aligned}$$

where the function P is given by (15). Note that the last equation is called the *modified Riccati equation* in [5], since when $\tilde{c} \equiv 0$ and $h \equiv 1$, it reduces to (11). By integrating, we get

$$v(s)|_t^T = \int_T^t C(s)\,ds + p\int_T^t r^{1-q}(s)h^p(s)P(\Phi^{-1}(w_h), w)\,ds. \tag{21}$$

Since $\int^\infty r^{1-q}(t)dt = \infty$ and $0 \le \int_t^\infty c(s)ds < \infty$, by Lemma 2 w also solves integral Riccati equation (14) and, therefore, $w(t) \ge 0$ for large t. Hence

$$h^p(w_h - w)|_T^t \le h^p w_h(t) + h^p(w(T) - w_h(T))$$

and letting $t \to \infty$ in (21) we obtain (with L given by (16))

$$L + h^p(w(T) - w_h(T)) \ge \int_T^\infty (c(s) - \tilde{c}(s))h^p(s)\,ds + p\int_T^\infty r^{1-q}(s)h^p(s)P(\Phi^{-1}(w_h), w)\,ds.$$

Since $P(u, v) \ge 0$ and (18) holds, this means that

$$\int^\infty r^{1-q}(t)h^p(t)P(\Phi^{-1}(w_h(t)), w(t))\,dt < \infty. \tag{22}$$

Now, since (16), (18), and (22) hold, from (21) it follows that there exists a finite limit

$$\lim_{t\to\infty} h^p(t)(w(t) - w_h(t)) =: \beta$$

and also the limit

$$\lim_{t\to\infty} \frac{w(t)}{w_h(t)} = \lim_{t\to\infty} \frac{h^p(t)w(t)}{h^p(t)w_h(t)} = \frac{L+\beta}{L}. \tag{23}$$

Therefore, letting $t \to \infty$ in (21) and then replacing T by t, we get the equation

$$\begin{aligned} &h^p(t)(w(t) - w_h(t)) - \beta = \\ &= \int_t^\infty (c(s) - \tilde{c}(s))h^p(s)\,ds + p\int_t^\infty r^{1-q}(s)h^p(s)P(\Phi^{-1}(w_h), w)\,ds. \end{aligned} \tag{24}$$

Since (23) holds, according to Lemma 3, there exists a positive constant K such that

$$K|\Phi^{-1}(w_h)|^{2-p}(w-w_h)^2 \le P(\Phi^{-1}(w_h),w)$$

for large t, and hence

$$Kr^{1-q}h^p w_h^{q-2}(w-w_h)^2 \le r^{1-q}h^p P(\Phi^{-1}(w_h),w).$$

Now, using the fact that $w_h^{q-2} = r^{q-2}(h')^{2-p}h^{p-2}$, we get the inequality

$$\frac{K}{r(t)h^2(t)(h'(t))^{p-2}}[(w(t)-w_h(t))h^p(t)]^2 \le r^{1-q}(t)h^p(t)P(\Phi^{-1}(w_h(t)),w(t)). \quad (25)$$

Denote $G(t) = r^{-1}(t)h^{-2}(t)(h'(t))^{2-p}$, then after integrating over $[T,\infty)$ the last inequality reads

$$K\int_T^\infty G(t)[(w(t)-w_h(t))h^p(t)]^2\,dt \le \int_T^\infty r^{1-q}(t)h^p(t)P(\Phi^{-1}(w_h(t)),w(t))\,dt.$$

By (17) it holds $\int^t G(s)\,ds \to \infty$ as $t\to\infty$. This implies that $\beta = \lim_{t\to\infty} h^p(t)(w(t)-w_h(t)) = 0$. Indeed, if $\beta \ne 0$, then

$$\int^\infty G(t)[(w(t)-w_h(t))h^p(t)]^2\,dt = \infty,$$

which, in view of (25), implies that $\int^\infty r^{1-q}h^pP(\Phi^{-1}(w_h),w)\,dt = \infty$, and this contradicts (22). Now, denote

$$Q(s) = \frac{|s|^q}{q} - s + \frac{1}{p} = P(1,s).$$

Using the fact that $w/w_h \to 1$ as $t\to\infty$, by the second degree Taylor formula, for $\varepsilon > 0$ as in (20) there exists $T \in \mathbb{R}$ such that

$$Q(w/w_h) \ge \left(\frac{q-1}{2} - \frac{\varepsilon}{p}\right)\left(\frac{w}{w_h} - 1\right)^2 \quad (26)$$

for $t > T$. This estimate implies that

$$0 = v' + C + pr^{1-q}h^pP(\Phi^{-1}(w_h),w) = v' + C + pr^{1-q}h^p|w_h|^qQ(w/w_h) \ge$$
$$\ge v' + C + \left(\frac{q}{2} - \varepsilon\right) r^{1-q}h^p|w_h|^q\left(\frac{w}{w_h} - 1\right)^2 = v' + C + \left(\frac{q}{2} - \varepsilon\right) r^{1-q}h^{-p}|w_h|^{q-2}v^2 =$$
$$= v' + C + \left(\frac{q}{2} - \varepsilon\right)\frac{v^2}{R}$$

which means that the linear second order equation

$$\left(\frac{2}{q-2\varepsilon}R(t)y'\right)' + C(t)y = 0, \tag{27}$$

is nonoscillatory by Lemma 1, but (27) is the same equation as (20) and we have reached a contradiction with the assumption that this equation is oscillatory. □

The next statement is a nonoscillatory counterpart of Theorem 1.

Theorem 2. *With the notation of the previous theorem, suppose that its assumptions are satisfied, except for the requirements concerning the integral* $\int_t^\infty c(s)\,ds$ *and assumption* (13)*. If there exists* $\varepsilon > 0$ *such that the second order linear equation*

$$(R(t)y')' + \left(\frac{q}{2}+\varepsilon\right)C(t)y = 0 \tag{28}$$

is nonoscillatory, then also (1) *is nonoscillatory.*

Proof. Nonoscillation of (28), which is the same equation as (27) with $+\varepsilon$ instead of $-\varepsilon$, implies the existence of a differentiable function v for which

$$v' + C(t) + \left(\frac{q}{2}+\varepsilon\right)\frac{v^2}{R(t)} = 0 \tag{29}$$

for large t and by Lemma 2 this function also verifies the Riccati integral equation

$$v(t) = \int_t^\infty C(s)\,ds + \left(\frac{q}{2}+\varepsilon\right)\int_t^\infty \frac{v^2(s)}{R(s)}\,ds,$$

in particular, $\lim_{t\to\infty} v(t) = 0$. Put $w = h^{-p}v + w_h$. Then the last limit relation means that

$$\lim_{t\to\infty} h^p(t)[w(t) - w_h(t)] = 0,$$

and hence by (16), using the same argument as in the proof of Theorem 1

$$\lim_{t\to\infty}\frac{w(t)}{w_h(t)} = 1. \tag{30}$$

Using (30), again similarly as in the proof of Theorem 1, for $\varepsilon > 0$ as in (28), there exists $T \in \mathbb{R}$ such that

$$pr^{1-q}(t)h^p(t)P(\Phi^{-1}(w_h(t), w(t)) \le \left(\frac{q}{2}+\varepsilon\right)\frac{[h^p(t)(w(t)-w_h(t))]^2}{r(t)h^2(t)(h'(t))^{p-2}} \tag{31}$$

for $t \ge T$.

Substituting for the function P in the final part of the computation, for $t \ge T$, we obtain

$$w' = -ph'h^{-p-1}v + h^{-p}\left[-C - \left(\frac{q}{2}+\varepsilon\right)\frac{v^2}{R}\right] - \tilde{c} - (p-1)r^{1-q}|w_h|^q =$$

$$= -\frac{ph'}{h}(w-w_h) - c + \tilde{c} - h^{-p}\left(\frac{q}{2}+\varepsilon\right)\frac{[h^p(w-w_h)]^2}{rh^2(h')^{p-2}} - \tilde{c} - (p-1)|w_h|^q \leq$$
$$\leq -pr^{1-q}\Phi^{-1}(w_h)w + p|w_h|^q - c - pr^{1-q}P(\Phi^{-1}(w_h, w) - (p-1)r^{1-q}|w_h|^q =$$
$$= -c - (p-1)|w|^q.$$

Therefore, (1) is nonoscillatory by Lemma 1. □

Remark 1. *In* [4] *we have proved that under the assumptions of Theorems 1, 2, equation* (1) *is oscillatory, provided that*

$$\liminf_{t\to\infty} \int^t \frac{ds}{r(s)h^2(s)(h'(s))^{p-2}} \int_t^\infty [c(s)-\tilde{c}(s)]h^p(s)\,ds > \frac{1}{2q}, \tag{32}$$

while it is nonoscillatory, provided that

$$\limsup_{t\to\infty} \int^t \frac{ds}{r(s)h^2(s)(h'(s))^{p-2}} \int_t^\infty [c(s)-\tilde{c}(s)]h^p(s)\,ds < \frac{1}{2q} \tag{33}$$

and

$$\liminf_{t\to\infty} \int^t \frac{ds}{r(s)h^2(s)(h'(s))^{p-2}} \int_t^\infty [c(s)-\tilde{c}(s)]h^p(s)\,ds > -\frac{3}{2q}. \tag{34}$$

These statements can be obtained as corollaries of Theorems 1, 2. To show this, let us recall that equation (2) *is oscillatory, provided that*

$$\liminf_{t\to\infty} \int^t r^{-1}(s)\,ds \int_t^\infty c(s)\,ds > \frac{1}{4} \tag{35}$$

while it is nonoscillatory if

$$\limsup_{t\to\infty} \int^t r^{-1}(s)\,ds \int_t^\infty c(s)\,ds < \frac{1}{4}, \tag{36}$$

and

$$\liminf_{t\to\infty} \int^t r^{-1}(s)\,ds \int_t^\infty c(s)\,ds > -\frac{3}{4}. \tag{37}$$

These conditions, applied to (20) *and* (28)*, give* (32) *and* (33), (34). *Indeed, concerning, e.g., the oscillation part of this remark, if* (32) *holds, this means that there exists* $\varepsilon > 0$ *(sufficiently small) such that*

$$\liminf_{t\to\infty} \int^t \frac{ds}{r(s)h^2(s)(h'(s))^{p-2}} \int_t^\infty [c(s)-\tilde{c}(s)]h^p(s)\,ds > \frac{1}{2q-4\varepsilon} \tag{38}$$

stille holds, and this implies that (*with notation* (19))

$$\liminf_{t\to\infty} \int^{t} R^{-1}(s)\,ds \left(\frac{q}{2}-\varepsilon\right) \int_{t}^{\infty} C(s)\,ds > \frac{1}{4}.$$

Hence (20) *is oscillatory by* (35)*. The proof that the assumption of nonoscillation of* (28) *is weaker than* (33)*,* (34) *is analogical.*

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Received: September 21, 2005.