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A REMARK ON GENERALIZED COMMUTATION RELATION AND SUBNORMALITY

Abstract. Tillmann [11] proved that every operator A which fulfils the canonical commutation relation $A^*A - AA^* = Id$ is an orthogonal sum of canonical creation operators. We extend this result for operators which fulfil generalized commutation relation

$$A^*A - AA^* = E^2, \quad \text{where} \quad EA = AE.$$

In addition, some inequalities which are helpful in describing analytic vectors of operators A^*A, where A fulfils the generalized commutation relation, are estabished.

Keywords: Hilbert space, generalized commutation relation, creation operator, analytic vectors.

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1. INTRODUCTION

It is well known [1] that every subnormal operator S in a Hilbert space fulfils the *Halmos–Bram condition* on a suitable dense subset of its domain M, i.e.

$$\sum_{i,j=1}^{n} \langle S^i f_j, S^j f_i \rangle \geq 0 \quad \text{for all natural} \quad n \quad \text{and} \quad f_1, \ldots, f_n \in M. \tag{1.1}$$

The canonical creation operator can be represented (in a traditional manner) as the operator S^+, defined on

$$D(S^+) := \left\{ \sum_{i \in Z_+} f_i e_i \colon \ \sum_{i \in Z_+} i \, |f_i|^2 < \infty \right\} \tag{1.2}$$

by the formula

$$S^{+}e_i := \sqrt{i+1}e_{i+1} \quad \text{for} \quad i \in Z_+,$$

where $\{e_i : i \in Z_+\}$ denotes the orthonormal basis indexed by the set Z_+ of all nonnegative integers. The canonical creation operator is in a natural way subnormal.

Tillman in [11] proved the following theorem:

Theorem 1.1. *Let S be a linear and closed densely defined operator in a Hilbert space K and suppose that $D := D(S^*S) = D(SS^*)$, and that the relation*

$$S^*S - SS^* = Id \tag{1.3}$$

holds on the set D.

Then the operator S^*S is self-adjoint and has a purely discrete spectrum with eigenenvalues $\{0, 1, 2, \ldots\}$, each of the same multiplicity. In addition, the Hilbert space K can be written as a direct sum $K = \oplus K_\alpha$, where for each α K_α is an irreducible reducing subspace of S and the restriction of S to the Hilbert space K_α is isomorphic to the canonical creation operator.

A consequence of the above theorem, is that each operator with *canonical commutation relation property* (1.3) is subnormal. The crucial point in the proof of this result is to show that the symmetric operator S^*S is self-adjoint.

Besides the Halmos–Bram condition (see (1.1)), subnormality of unbounded operators requires some additional assumptions about analyticity of vectors in the domains of their operators [5,6,7]. Actually, Szafraniec and J. Stochel [5] proved the following theorem:

Theorem 1.2. *Let a dense linear subset M be invariant for a closed operator S. The operator S is subnormal if the Halmos–Bram condition is fulfilled on M and the set M is linearly spanned by the set $\{S^i f\colon\ i \in Z_+, f \in \mathcal{A}(S)\}$, where $\mathcal{A}(S)$ denotes the set of all analytic vectors for S and f is analytic for S if and only if $f \in D(S^m)$ for all natural m and*

$$\lim_{m\to\infty} \left[\frac{\|S^m f\|}{m!}\right]^{\frac{1}{m}} < \infty.$$

In [8] it has been proved that the generalized commutation relation implies the Halmos–Bram condition, i.e.,

Theorem 1.3. *Let S be a linear operator in Hilbert space $\mathcal{H}$ and let M be a dense linear manifold of $\mathcal{H}$, and suppose that M is invariant for S. If there exists an operator E such that $M \subset D(E) \cup D(E^*)$, $EM \subset M$, $(S^*S - SS^*)f = E^2 f$, $SEf = ESf$ for each $f \in M$ and $\langle f, Eg\rangle = \langle Ef, g\rangle$ for each $f, g \in M$, then the Halmos–Bram condition holds on M.*

Theorems 1.2 and 1.3 show us, how important for a subnormality is the information about analyticity of vectors from the domain of a suitable operator.

In this paper, we try to use Tillman's method to obtain more information about operators which fulfil generalized commutation relation. First, we slightly generalize Theorem 1.1, and next, we prove some estimates for the norm $\|S^*Sf\|$ (Proposition 3.3, 3.5 and Corollary 3.6), which could be helpful in the proving of the self-adjointnes of the operator S^*S.

2. AN IMPROVED VERSION OF TILLMAN'S THEOREM

Denote by N and R the set of all natural numbers and real numbers, respectively, and by H a complex Hilbert space. Given a dense linear subspace D of H, by $L^{\#}(D)$ we denote the set of all linear operators A from D in D such that the domain $D(A^*)$ of the adjoint A^* of A contains D and $A^*(D) \subseteq D$. Then $L^{\#}(D)$ becomes *-algebra with the involution $A^{\#} := A^*|_D$.

Definition 2.1. *An operator $S \in L^{\#}(D)$ is said to satisfy the generalized semicommutation relation if it commutes with its selfcommutator $[S^{\#}; S] := S^{\#}S - SS^{\#}$.*

Definition 2.2. *We say that $S \in L^{\#}(D)$ satisfies the generalized commutation relation if there exists $E \in L^{\#}(D)$ such that $E^{\#} = E$, $ES = SE$ and $[S^{\#}; S] = E^2$ (see Theorem 1.3).*

Roughly speaking, the canonical commutation relation (see (1.3)) can be regarded as a special case of the generalized one with $E = Id$, where Id stands for the identity operator.

In [8] it was shown that the generalized commutation relation implies Halmos–Bram condition, which is a necessary condition for an operator to be subnormal [5–7]. Tillmann [11] proves that if a closed densly defined operator S in H satisfies the commutation relation, i.e., $S^*S - SS^* \subseteq Id$ and $D(S^*S) = D(SS^*)$, then S is an orthogonal sum of creation operators. Consequently, S is subnormal [5]. It is worthwhile to notice that in the case of S^+ (see (1.2)), the space $D[(S^+)^*S^+]$ is not invariant under S^+. Indeed, since

$$D[(S^+)^*S^+] = \left\{ \sum_{i\in\mathbf{N}} f_i e_i \in H : \sum_{i\in\mathbf{N}} i^2 |f_i|^2 < \infty \right\}$$

then

$$g := \sum_{i\in\mathbf{N}} \frac{1}{i(i+1)} e_i \quad \in \quad D[(S^+)^*S^+] \quad \text{and} \quad S^+g \notin D[(S^+)^*S^+].$$

We can slightly generalize Tillmann's theorem as follows. Below, $\sigma(C)$ and $\sigma_p(C)$ stand for the spectrum and point spectrum, respectivaty, of a densly defined operator C in H.

Theorem 2.1 ([10]). *Let S be a closed, densly defined linear operator in H. Assume that the domains of the operators S^*S and SS^* coincide and that the operator*

$C := S^*S - SS^*$ is bounded. Moreover, assume that $\sigma(C) = \sigma_p(C)$ and $\overline{C}Sf = S\overline{C}f$ for each $f \in D[S] := D(S^*S)$. Then there are complex numbers λ_i,μ_j such that

$$S = [\oplus\lambda_i S^+] \oplus [\oplus\mu_j (S^+)^*] \oplus N$$

where N is a normal operator with the domain $D(N) = D[S] \cap \ker(\overline{C})$. If $C \geq 0$, then $S = [\oplus\lambda_i S^+] \oplus N$ and the operator S is subnormal.

Proof. It follows from [12] that for $f \in D(S)$ there exists a sequence $\{f_n\} \subseteq D(S)$ such that

$$f_n \longrightarrow f \quad \text{and} \quad Sf_n \longrightarrow Sf.$$

Then $\overline{C}f_n \longrightarrow \overline{C}f$ and $S\overline{C}f_n = \overline{C}Sf_n \longrightarrow \overline{C}Sf$. Since the operator S is closed, we get $\overline{C}f \in D(S)$ and $S\overline{C}f = \overline{C}Sf$. Thus, $\overline{C}S \subset S\overline{C}$ and, consequently, $F(\delta)S \subset SF(\delta)$ for every Borel subset δ of $\mathbf{R}$, where F is the spectral measure of the selfadjoint operator $\overline{C}$. Since $\ker(\lambda Id - \overline{C}) = F(\{\lambda\})$ for $\lambda \in \mathbf{R}$, we conclude that $\ker(\lambda Id - \overline{C})$ reduces S for every $\lambda \in \mathbf{R}$. Denote by S_λ the restriction of S to $\ker(\lambda Id - \overline{C})$, $\lambda \in \mathbf{R}$. Then, by the equality $D(S^*S) = D(SS^*)$, we get

$$D(S_\lambda^* S_\lambda) = D\left(S^*S|_{\ker(\lambda Id - \overline{C})}\right) = D[S] \cap \ker(\lambda Id - \overline{C}) = D(S_\lambda S_\lambda^*),$$
$$(S_\lambda^* S_\lambda - S_\lambda S_\lambda^*)f = \lambda f, \quad f \in D[S_\lambda], \quad \lambda \in \mathbf{R}.$$

This and Tillman's Theorem imply that S_λ is unitarily equivalent to the orthogonal sum of creation (resp. annihilation) operators, provided that $\lambda \in \sigma_p(C)$ and $\lambda > 0$ (resp, $\lambda < 0$). Since $S_0^*S_0 = S_0S_0^*$, then operator S_0 is normal in $(D[S] \cap \ker(\overline{C}))^-$ and $D(S_0) = D[S] \cap \ker(\overline{C})$. The conclusion follows from the fact that

$$H = \bigoplus_{\lambda \in \sigma(\overline{C})} \ker(\lambda Id - \overline{C}). \qquad \square$$

Let M_0 be a dense linear subspace of H, $A \in L^\#(M_0)$ such that $A = A^\#$ and

$$M := \{(f_1, \ldots, f_m, 0, 0, \ldots) \in K \colon \ f_n \in M_0 \text{ and } m \in \mathbf{N}\}$$

where $K = \bigoplus_{n \in \mathbf{N}} K_n$ and $K_n := H$ for each $n \in \mathbf{N}$. Then M is a dense linear subspuce of K. We define the operator $\mathcal{A} \in L^\#(M)$ by

$$\mathcal{A}f := (0, Af_1, \sqrt{2}Af_2, \sqrt{3}Af_3, \ldots), \quad f := (f_1, f_2, \ldots) \in M.$$

Then the adjoint $\mathcal{A}^\#$ of $\mathcal{A}$ acts accordingly to the following formula:

$$\mathcal{A}^\# f = (A^\# f_2, \sqrt{2}A^\# f_3, \sqrt{3}A^\# f_4, \ldots), \quad f := (f_1, f_2, \ldots) \in M.$$

The selfcommutator of the operator $\mathcal{A}$ acts on M as follows:

$$(\mathcal{A}^\#\mathcal{A} - \mathcal{A}\mathcal{A}^\#)(0, \ldots, 0, f_n, 0, \ldots) =$$
$$= (0, \ldots, 0, \ n\, A^\# A f_n - (n-1)AA^\# f_n, 0, \ldots) = (0, \ldots, 0, A^2 f_n, 0, .) \ \text{ if } \ n \geq 2$$

and

$$(\mathcal{A}^{\#}\mathcal{A} - \mathcal{A}\mathcal{A}^{\#})(f_1, 0, 0, \ldots) = (A^{\#}Af_1, 0, 0 \ldots) = (A^2 f_1, 0, 0, \ldots).$$

As a consequence, we obtain

$$C := \mathcal{A}^{\#}\mathcal{A} - \mathcal{A}\mathcal{A}^{\#} = A^2 \oplus A^2 \oplus \ldots = (A \oplus A \oplus \ldots)^2 \in L^{\#}(M)$$

and

$$\sigma(C) = \sigma(A^2).$$

It is not difficult to see that the operator $\mathcal{A}$ fulfils the generalized semicommutation relation on M. If we assume that continuous spectrum of A^2 coincides with $\sigma(A^2)$, then we obtain that C has a continuous spectrum. So, the assumption $\delta(C) = \delta_p(C)$ in Theorem 2.1 is essentailal.

3. ANALYCITY OF VECTORS FROM $D(S^{\#}S)$

In [9] it is shown that generalized creation operators acting on a Bergmann space of an infinite order are subnormal. These operators fulfil the generalized commmutation relation on some invariant space M, in which all vectors are analytic. If we look at the proof of Tillmann's Theorem [11], we note that the crucial point in this proof is to show that the symmetric operator S^*S is selfadjoint (in the notation used in Theorem 1.1). On the other hand, we know that a symmetric operator with the set of analytic (resp. quasi-analytic, C^∞) vectors in its domain rich enough is essentially selfadjoint [2,3,4]. These two above approaches to commutation relations have common parts. They require some kind of regularity of vectors from the domains of adequate operators. In the sequel, we will try to show how useful can the generalized commutation property be in describing analytic (resp. quasi-analytic) properties of vectors from the domain of operators $S^{\#}S$. First we recall the definition of analytic and quasi-analytic vectors of some symmetric operator $A \in L^{\#}(D)$. We say that $f \in D$ is an analytic (resp., quasi-analytic) vector of A if

$$\sum_{n=0}^{\infty} \frac{\|A^n f\|}{n!} t^n < \infty \quad \text{for some} \quad t > 0$$
$$\left(\text{resp.,} \sum_{n=1}^{\infty} \|A^n f\|^{-1/n} = +\infty\right).$$

From these definitions it follows that if we want to know something about the "analyticity" of a vector f, we must first know how the norms $\|A^n f\|$ behave. Next, we will consider the operator $A := S^{\#}S$, where S is an operator which fulfils the generalized commutation relation.

The following property is shown to hold [8]:

Proposition 3.1. *The following formula is true:*

$$(S^{\#})^i S^j f = \sum_{m=0}^{\infty} m! \binom{j}{m} \binom{i}{m} S^{j-m} (S^{\#})^{i-m} E^{2m} f, \quad f \in D$$

where $S^{-l} := (S^{\#})^{-l} := 0$ *if* $l > 0$, $\binom{i}{j} := 0$ *for* $j > i$ *and* $S^0 := (S^{\#})^0 := Id$.

Using the above property we obtain:

Proposition 3.2. *There exist integers* $\alpha(n,k)$, $n,k = 0,1,2,\dots$, $n \geq k$, *such that:*

(a) $(S^{\#}S)^n f = \sum\limits_{k=0}^{n} \alpha(n,k)(S^{\#})^k S^k E^{2(n-k)} f$, $f \in D$,

(b) $\alpha(n,0) = 0$, $\quad n = 1,2,\dots$
$\alpha(n,n) = 1$, $\quad n = 0,1,2,\dots$
$\alpha(n+1,k) = \alpha(n,k-1) - k\alpha(n,k)$, $\quad n = 2,3,4$ *and* $1 \leq k \leq n$,

(c) $\alpha(n+1,1) = (-1)^n$, $\quad n = 1,2,\dots$
$\alpha(n+1,2) = (-1)^n(1-2^n)$, $\quad n = 1,2,\dots$
$\alpha(n,n-1) = -\frac{n(n-1)}{2}$, $\quad n = 1,2,\dots$
$\alpha(n,n-k) = (-1)^k \sum\limits_{i_k=1}^{n-k} \cdot \dots \cdot \sum\limits_{i_2=1}^{i_3} \sum\limits_{i_1=1}^{i_2} i_1 \dots i_k$, $\quad 1 \leq k < n$,

(d) $\alpha(n,n-k) > 0$ *for even integers* k, $1 \leq k < n$,
$\alpha(n,n-k) < 0$ *for odd integers* k, $1 \leq k < n$.

Proof. We prove properties (a) and (b) by induction with respect "n". For $n = 0,1$, a proof is obvious. Let $n \geq 1$. Then from the inductive step there follows:

$$\begin{aligned}
&(S^{\#}S)^{n+1} f = S^{\#}S(S^{\#}S)^n f = \\
&\quad = S^{\#}S \sum_{k=0}^{n} \alpha(n,k)(S^{\#})^k S^k E^{2(n-k)} f = \sum_{k=0}^{n} \alpha(n,k) S^{\#}[S(S^{\#})^k] S^k E^{2(n-k)} f \stackrel{(*)}{=} \\
&\quad \stackrel{(*)}{=} \sum_{k=0}^{n} \alpha(n,k)(S^{\#})[(S^{\#})^k S - k \cdot (S^{\#})^{k-1} E^2] S^k E^{2(n-k)} f = \\
&\quad = \sum_{k=0}^{n} \alpha(n,k)(S^{\#})^{k+1} S^{k+1} E^{2(n-k)} f - \sum_{k=0}^{n} \alpha(n,k) \cdot k \cdot (S^{\#})^k S^k E^{2(n+1-k)} f = \\
&\quad = \sum_{l=1}^{n+1} \alpha(n,l-1)(S^{\#})^l S^l E^{2(n+1-l)} f - \sum_{k=1}^{n} k \cdot \alpha(n,k)(S^{\#})^k S^k E^{2(n+1-k)} f = \\
&\quad = \sum_{k=0}^{n+1} \alpha(n+1,k)(S^{\#})^k S^k E^{2(n+1-k)} f,
\end{aligned}$$

where

$$\begin{aligned}
&\alpha(n+1,0) := 0,\\
&\alpha(n+1,n+1) := \alpha(n,n),\\
&\alpha(n+1,k) := \alpha(n,k-1) - k\cdot\alpha(n,k) \quad \text{for} \quad 1\le k\le n
\end{aligned}$$

and equality (*) is a consequence of Proposition 3.1 for $i = k$, $j = 1$.

In case (c), we also proceed by induction:

$$\alpha(2,1) = \alpha(1,0) - 1\alpha(1,1) = -1$$

and for $n \ge 1$, from the inductive step, we obtain:

$$\alpha(n+2,1) = \alpha(n+1,0) - 1\cdot\alpha(n+1,1) = 0 - 1\cdot(-1)^n = (-1)^{n+1}.$$

Similarly, $\alpha(2,2) = 1 = (-1)^1(1-2^1)$ and

$$\begin{aligned}
&\alpha(n+2,2) = \alpha(n+1,1) - 2\cdot\alpha(n+1,2) =\\
&\qquad = (-1)^n - 2\cdot(-1)^n(1-2^n) = (-1)^n[1-2+2^{n+1}] = (-1)^{n+1}(1-2^{n+1})
\end{aligned}$$

for $n \ge 1$. By the last equality, we will proceed analogously but by induction with respect to "k".

$$\begin{aligned}
&\alpha(n,n-1) = \alpha(n-1,n-2) - (n-1)\alpha(n-1,n-1) =\\
&\qquad = \alpha(n-1,n-2) - (n-1) = \ldots = \alpha(2,1) - \sum_{i=2}^{n-1} i = -\sum_{i=1}^{n-1} i = -\frac{(n-1)\cdot n}{2}
\end{aligned}$$

and

$$\begin{aligned}
&\alpha(n,n-(k+1)) =\\
&\quad = \alpha(n-1,n-1-(k+1)) - [n-(k+1)]\alpha(n-1,n-(k+1)) =\\
&\quad = \ldots = \alpha(k+2,1) - \sum_{s=2}^{n-(k+1)} s\cdot\alpha(s+k,s) =\\
&\quad = (-1)^{k+1} - \sum_{s=2}^{n-(k+1)} s\cdot(-1)^k \sum_{i_k=1}^{s}\cdots\sum_{i_2=1}^{i_3}\sum_{i_1=1}^{i_2} i_1\cdot\ldots\cdot i_k =\\
&\quad = (-1)^{k+1} + (-1)^{k+1}\sum_{i_{k+1}=s=2}^{n-(k+1)}\sum_{i_k=1}^{i_{k+1}}\cdots\sum_{i_1=1}^{i_2} i_1\cdot\ldots\cdot i_k\cdot i_{k+1} =\\
&\quad = (-1)^{k+1}\sum_{i_{k+1}=1}^{n-(k+1)}\cdots\sum_{i_1=1}^{i_2} i_1\cdot\ldots\cdot i_{k+1}.
\end{aligned}$$

Property (d) follows immediately from property (c). □

The above proposition reveals some possible methods for calculating the numbers $\alpha(n,k)$. We are now in a position to compute the numbers $\|(S^{\#}S)f\|$.

Proposition 3.3. *The following formula is true:*

$$\|(S^{\#}S)^n f\|^2 = \sum_{k=1}^{2n} \alpha(2n,k) \ \left\|S^k E^{2n-k} f\right\|^2 \ \textit{for } f \in D.$$

Proof. From Proposition 3.2, we conclude that

$$\begin{aligned}\|(S^{\#}S)^n f\|^2 &= \left\langle (S^{\#}S)^n f, (S^{\#}S)^n f\right\rangle = \left\langle (S^{\#}S)^{2n} f,\ f\right\rangle = \\ &= \left\langle \sum_{k=0}^{2n} \alpha(2n,k)\ (S^{\#})^k (S)^k E^{2(2n-k)} f,\ f \right\rangle = \\ &= \sum_{k=0}^{2n} \alpha(2n,k) \left\langle S^k E^{2n-k} f,\ S^k E^{2n-k} f\right\rangle = \\ &= \sum_{k=0}^{2n} \alpha(2n,k)\ \left\|S^k E^{2n-k} f\right\|^2 .\end{aligned}$$

□

It would be convenient to have some estimates of the norms $\left\|S^k E^{2n-k} f\right\|$. We know that the operator S fulfils the Halmos–Bram condition. It helps us to prove the following inequality:

Proposition 3.4. *The following inequality is true:*

$$\left\|S^k E^{2n-k} f\right\|^{\frac{1}{k}} \le \left\|E^{2n-k} f\right\|^{\frac{2n-k}{2n\cdot k}} \left\|E^{2n-k} S^{2n} f\right\|^{\frac{1}{2n}} \ \textit{for } f \in D.$$

Proof. In [6] it is shown that there exists a finite non-negative measure μ such that

$$\|S^n g\|^2 = \int_0^{+\infty} t^n \mu(dt) \text{ for } g \in D,\ n = 0, 1, 2, \dots .$$

Using Hölder Inequality with $p = n+1$ and $q = \frac{n+1}{n}$, we obtain:

$$\begin{aligned}\|S^n g\|^2 = \int_0^{\infty} t^n \mu(dt) \le \left[\int_0^{\infty} \mu(dt)\right]^{\frac{1}{n+1}} \cdot \left[\int_0^{\infty} t^{n+1} \mu(dt)\right]^{\frac{n}{n+1}} = \\ = \|g\|^{\frac{2}{n+1}} \cdot \left\|S^{n+1} g\right\|^{\frac{2n}{n+1}} .\end{aligned}$$

Therefore,

$$\|S^n g\|^{\frac{1}{n}} \le \|g\|^{\frac{1}{n(n+2)}} \cdot \left\|S^{n+1} g\right\|^{\frac{1}{n+1}} .$$

In the sequel, let $a_n := \|S^n g\|^{\frac{1}{n}}$. Then $a_n \le \|g\|^{\frac{1}{n(n+1)}} \cdot a_{n+1}$ and, as a consequence,

$$a_n \le \|g\|^{\frac{1}{k(k+1)}} a_{k+1} \le \|g\|^{\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}} \cdot a_{k+2} \le \dots$$
$$\dots \le \|g\|^{\frac{1}{k(k+1)}+\dots+\frac{1}{(m-1)\cdot m}} \cdot a_m = \|g\|^{\frac{1}{k}-\frac{1}{m}} \cdot a_m \text{ for } k < m.$$

If we now substitute $g = E^{2n-k}f$ and $m = 2n$, then we obtain

$$\left\|S^k E^{2n-k} f\right\|^{\frac{1}{k}} \le \left\|E^{2n-k} f\right\|^{\frac{1}{k}-\frac{1}{2n}} \cdot \left\|S^{2n} E^{2n-k} f\right\|^{\frac{1}{2n}} =$$
$$= \left\|E^{2n-k} f\right\|^{\frac{2n-k}{2n\cdot k}} \cdot \left\|E^{2n-k} S^{2n} f\right\|^{\frac{1}{2n}}. \qquad \square$$

Suppose that E is a bounded operator and $e := \|E\|$. Then the inequality in Proposition 3.4 yields the following one:

$$\left\|S^k E^{2n-k} f\right\|^{\frac{1}{k}} \le e^{\frac{2n-k}{2n\cdot k}} \cdot \|f\|^{\frac{2n-k}{2n\cdot k}} \cdot e^{\frac{2n-k}{2n}} \cdot \left\|S^{2n} f\right\|^{\frac{1}{2n}} =$$
$$= e^{\frac{(2n-k)(k+1)}{2n\cdot k}} \cdot \|f\|^{\frac{1}{k}-\frac{1}{2n}} \cdot \left\|S^{2n} f\right\|^{\frac{1}{2n}} = e^{\frac{(2n-k)(k+1)}{2n\cdot k}} \|f\|^{\frac{1}{k}} \left\|S^{2n}\left(\frac{f}{\|f\|}\right)\right\|^{\frac{1}{2n}}.$$

Thus we obtain the following:

Proposition 3.5. *If the operator E is bounded, then*

$$\left\|(S^\# S)^n f\right\|^2 \le \|f\|^2 \cdot \sum_{k=1}^{2n} \cdot |\alpha(2n,k)| \cdot s(n,f)^k \cdot \|E\|^{\frac{(2n-k)(k+1)}{n}}$$

where $s(n,f) := \left\|S^{2n}\left(\frac{f}{\|f\|}\right)\right\|^{\frac{1}{n}}$ *and* $f \in D$.

Let now $E = \lambda \cdot Id$, where λ is a complex number. Then Proposition 3.3 and Proposition 3.5 yield the following properties:

Corollary 3.1.

$$\left\|(S^\# S)^n f\right\|^2 = \sum_{k=1}^{2n} \alpha(2n,k) \cdot \lambda^{2n-k} \cdot \left\|S^k f\right\|^2, \quad \textit{for} \;\; f \in D$$

and

$$\left\|(S^\# S)^n f\right\|^2 \le \|f\| \cdot \sum_{k=1}^{2n} |\alpha(2n,k)| \cdot s(n,f)^k \cdot \lambda^{\frac{(2n-k)(k+1)}{n}}$$

where $s(n,f) := \left\|S^{2n}\left(\frac{f}{\|f\|}\right)\right\|^{\frac{1}{n}}$ *and* $f \in D$.

The last proposition and corollary give us hope that if we know more about the numbers $\alpha(m,k)$, then we will be able the say something about the "analyticity" of vectors $f \in D$ for the operator $S^\# S$. So if we want to use the same methods as in Tillman's paper [11] to describe operators which fulfil the generalized commutation relation, we must first describe the numbers $\alpha(m,k)$ more precisely.

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