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THE DELSARTE-DARBOUX TYPE BINARY TRANSFORMATIONS, THEIR DIFFERENTIAL-GEOMETRIC AND OPERATOR STRUCTURE WITH APPLICATIONS. PART 1

Abstract. The structure properties of multidimensional Delsarte–Darboux transmutation operators in parametric functional spaces are studied by means of differential-geometric and topological tools. It is shown that kernels of the corresponding integral operator expressions depend on the topological structure of related homological cycles in the coordinate space. As a natural realization of the construction presented we build pairs of Lax type commutative differential operator expressions related via a Delsarte–Darboux transformation and having a lot of applications in soliton theory.

Keywords: Delsarte transmutation operators, parametric functional spaces, Darboux transformations, inverse spectral transform problem, soliton equations, Zakharov–Shabat equations.

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1. INTRODUCTION: CONJUGATED DIFFERENTIAL OPERATORS AND THEIR PROPERTIES

Let us consider the following functional space $\mathcal{H} := L_2(\mathrm{T}^2_{(t,y)}; H)$, where $H = L_2(\mathbb{R}_x; \mathbb{C}^N)$, $N \in \mathbb{Z}_+$, in which the next matrix-differential expressions

$$
\begin{aligned}
\frac{\partial}{\partial y} - L &:= \frac{\partial}{\partial y} - \sum_{i=0}^{n(L)} a_i(x; y, t) \frac{\partial^i}{\partial x^i} := \mathrm{L}, \\
\frac{\partial}{\partial t} - M &:= \frac{\partial}{\partial t} - \sum_{j=0}^{n(M)} b_j(x; y, t) \frac{\partial^j}{\partial x^j} := \mathrm{M},
\end{aligned} \tag{1.1}
$$

are defined. We here denoted $\mathrm{T}^2 := \mathrm{T}^2_{(t,y)} = [0,Y]\times[0,T] \subset \mathbb{R}^2_+$, matrices $a_i, b_j \in C^1\left(\mathrm{T}^2_{(t,y)}; S(\mathbb{R}; End\mathbb{C}^N)\right)$, $i = \overline{1, n(L)}$, $j = \overline{1, n(M)}$, where $S(\mathbb{R}; End\mathbb{C}^N)$ is the space of matrix-valued Schwartz class coefficient functions, and $n(M)$, $n(L) \in \mathbb{Z}_+$ are fixed orders.

Let $\mathcal{H}^* := L_2\left(\mathrm{T}^2_{(t,y)}; H^*\right)$ be the corresponding conjugated to $\mathcal{H}$ space. On the space $\mathcal{H}^*\times\mathcal{H}$ define a usual semi-linear scalar form according to the rule: for any pair $(\varphi,\psi)\in\mathcal{H}^*\times\mathcal{H}$

$$(\varphi,\psi) := \int\limits_{\mathrm{T}^2_{(t,y)}} dt\,dy \int\limits_{\mathbb{R}} dx\ \langle\varphi,\psi\rangle = \int\limits_{\mathrm{T}^2_{(t,y)}} dt\,dy \int\limits_{\mathbb{R}} dx\,(\bar{\varphi}^{\intercal}\psi), \tag{1.2}$$

where $\langle\cdot,\cdot\rangle$ is the standard semi-linear scalar form on $\mathbb{C}^N$, the bar "$-$" means the usual complex conjugation and "$\intercal$" means the standard matrix transposition. Concerning the scalar form (1.2) let us study a problem of existence the corresponding to (1.1) conjugated matrix differential operators in the space $\mathcal{H}^*$.

Differential expressions $L, M : H \to H$ are defined as closeable normal operators in a domain $Dom(L,M)\subset H$, being dense in H. Then, by definition, the conjugated operators L^*, $M^* : H^* \to H^*$ exist and the following equalities

$$(\mathrm{L}^*\varphi,\psi) = (\varphi,\mathrm{L}\psi), \quad (\mathrm{M}^*\varphi,\psi) = (\varphi,\mathrm{M}\psi) \tag{1.3}$$

hold obviously for all $\varphi,\psi\in W_1^1(\mathrm{T}^2_{(t,y)};\ Dom(L,M))$. Then one can consider the corresponding to (1.3) frelationships being analogs of the classical Lagrange identities for the operator L

$$\langle\mathrm{L}^*\varphi,\psi\rangle - \langle\varphi,\mathrm{L}\psi\rangle = -\frac{\partial}{\partial x}Z_{(L)}[\varphi,\psi] + \frac{\partial}{\partial t}(\bar{\varphi}^{\intercal}\psi) \tag{1.4}$$

and for the operator M

$$\langle\mathrm{M}^*\varphi,\psi\rangle - \langle\varphi,\mathrm{M}\psi\rangle = -\frac{\partial}{\partial x}\,Z_{(M)}[\varphi,\psi] - \frac{\partial}{\partial y}(\bar{\varphi}^{\intercal}\psi), \tag{1.5}$$

where $Z_{(L)}[\varphi,\psi]$ and $Z_{(M)}[\varphi,\psi]$ are some semi-linear forms on $\mathcal{H}^*\times\mathcal{H}$. From (1.4) and (1.5) one sees that the conjugated operators $L^* : H^*\to H^*$ and $M^* : H^* \to H^*$ are defined if there exists a scalar function $\Omega\in C^1(\mathbb{R}^2\times\mathrm{T}^2;\mathbb{C})$ satisfying the expressions

$$\bar{\varphi}^{\intercal}\psi := \partial\Omega/\partial x, \qquad Z_{(L)}[\varphi,\psi] = \partial\Omega/\partial t, \quad Z_{(M)}[\varphi,\psi] = \partial\Omega/\partial y, \tag{1.6}$$

together with conditions

$$\partial\Omega/\partial y, \partial\Omega/\partial t \in W_1^1\left(\mathrm{T}^2_{(t,y)}; H\right). \tag{1.7}$$

By integrating (1.4) and (1.5) with respect to the measures $dt\wedge dx$ and $dy\wedge dx$, respectively, we find that the function $\Omega\in C^1(\mathbb{R}^1\times\mathrm{T}^2;\mathbb{C})$, called a Delsarte transmutation generator, exists if the following condition is satisfied: the differential form

$$Z^{(1)}[\varphi,\psi] := Z_{(L)}[\varphi,\psi]dy + Z_{(M)}[\varphi,\psi]dt) + \bar{\varphi}^{\intercal}\psi dx = d\Omega[\varphi,\psi], \tag{1.8}$$

is exact, that is one can write down the following relationship

$$\Omega[\varphi,\psi] = \Omega_0[\varphi,\psi] + \int\limits_{S(P;P_0)} Z^{(1)}[\varphi,\psi], \tag{1.9}$$

where $S(P;P_0) \subset \mathbb{R}\times\mathrm{T}^2$ is some smooth curve connecting a running point $P(x;y,t) \in \mathbb{R}\times\mathrm{T}^2$ with a fixed point $P(x_0;y_0,t_0) \in \mathbb{R}\times\mathrm{T}^2$, a function $\Omega_0[\varphi,\psi]$ being a semilinear form on $\mathcal{H}^* \times \mathcal{H}$ constant with respect to variables $(x;y,t) \in \mathbb{R}\times\mathrm{T}^2$. It is clear that conditions (1.7) for mapping (1.9) are certain restrictions concerning $(x;t,y)$–parametric dependence of functions $(\varphi,\psi) \in \mathcal{H}_0^* \times \mathcal{H}_0$. Let $\mathcal{H}_0^* \times \mathcal{H}_0$ be a closed subspace of pairs of functions $(\varphi,\psi) \in \mathcal{H}_-^* \times \mathcal{H}_-$ where $\mathcal{H}_-^* \times \mathcal{H}_-$ is the corresponding Hilbert–Schmidt rigged [4, 3] space $L_1(\mathrm{T}^2_{(t,y)}; H_-^*) \times L_1(\mathrm{T}^2_{(t,y)}; H_-)$. Consider expression (1.8) for $\varphi \in \mathcal{H}_0^* \subset \mathcal{H}_-^*$ and $\psi \in \mathcal{H}_0 \subset \mathcal{H}_-$ satisfying conditions (1.7). It is enough to assume that $\mathrm{L}\psi = 0$ and $\mathrm{M}\psi = 0$ for $\psi \in \mathcal{H}_0$ and $\mathrm{L}^*\varphi = 0$ and $\mathrm{M}^*\varphi = 0$ for all $\varphi \in \mathcal{H}_0^*$, where

$$\begin{aligned}
\mathcal{H}_0 := \{&\psi(\lambda;\xi) \in \mathcal{H}_- : \mathrm{L}\psi(\lambda;\xi) = 0,\ \mathrm{M}^*(\lambda;\xi) = 0,\\
&\psi(\lambda;\xi)|_{\substack{t=0^+\\ y=0^+}} = \psi_\lambda \in H_-^*,\ L\psi_\lambda = \lambda\psi_\lambda,\\
&\psi(\xi)|_{x=x_0} = 0, (\lambda;\xi) \in \Sigma := \sigma(L,M) \cap \bar{\sigma}(L^*,M^*) \times \Sigma_\sigma\},\\
\mathcal{H}_0^* := \{&\varphi(\lambda;\xi) \in \mathcal{H}_-^* : \mathrm{L}^*\varphi(\lambda;\xi) = 0,\ \mathrm{M}^*\varphi(\lambda;\xi) = 0,\\
&\varphi(\lambda;\xi)|_{\substack{t=0^+\\ y=0^+}} = \varphi_\lambda \in H_-^*,\ M\varphi_\lambda = \bar{\lambda},\\
&\varphi_\lambda, \varphi(\xi)|_{x=x_0} = 0, (\lambda;\xi) \in \Sigma := \sigma(L,M) \cap \bar{\sigma}(L^*,M^*) \times \Sigma_\sigma\}
\end{aligned} \tag{1.10}$$

for some fixed point $x_0 \in \mathbb{R}$ and a "specrtral" set $\Sigma = \sigma(L,M)\cap\bar{\sigma}(L^*,M^*)\times\Sigma_\sigma \subset \mathbb{C}^p$, $\sigma(L,M) \subset \mathbb{C}$ is a combined spectrum of operators L and M with $\sigma(L^*,M^*) \subset \mathbb{C}$ being the spectrum for the adjoint pair L^* and M^* in H. Thereby we have stated the following proposition.

Proposition 1.1. *For a pair of functions $(\varphi,\psi) \in \mathcal{H}_0^* \times \mathcal{H}_0$ there exists a semilinear form $\Omega : \mathcal{H}_0^* \times \mathcal{H}_0 \to \mathbb{C}$, such that relationships* (1.8) *and* (1.9) *hold.*

The proposition above makes it now possible to study local prperties of operators L and M in $\mathcal{H}$ withrespect to their dependence on parameter variables $(y,t) \in \mathrm{T}^2$. To study this dependence let us proceed to its corresonding analysis in the next Chapter.

2. STRUCTURE OF DELSARTE–DARBOUX TRANSFORMATIONS

Consider now another pair of operators $\tilde{\mathrm{L}}$ and $\tilde{\mathrm{M}} : \mathcal{H} \to \mathcal{H}$ for which there exist the corresponding conjugated operators $\tilde{L}^*$ and $\tilde{M}^* : H^* \to H^*$. Making use of Proposition 1, we can easily find another Delsrate transmutation generator $\tilde{\Omega} \in C^1(\mathbb{R}\times\mathrm{T}^2;\mathbb{C})$ being a semilinear form on suitable pairs of functions $(\tilde{\varphi},\tilde{\psi}) \in \tilde{\mathcal{H}}_0^* \times$

$\tilde{\mathcal{H}}_0$ for which there exist the corresponding semilinear forms $\tilde{Z}_{(\tilde{L})}[\tilde{\varphi},\tilde{\psi}]$, $\tilde{Z}_{(\tilde{M})}[\tilde{\varphi},\tilde{\psi}]$ satisfying the conditions like (1.4), (1.5), (1.6) and (1.7):

$$\bar{\tilde{\varphi}}^{\intercal}\tilde{\psi} := \partial\tilde{\Omega}/\partial x, \quad \tilde{Z}_{(\tilde{L})}[\tilde{\varphi},\tilde{\psi}] = \partial\tilde{\Omega}/\partial t, \tilde{Z}_{(\tilde{M})}[\tilde{\varphi},\tilde{\psi}] = \partial\tilde{\Omega}/\partial y, \tag{2.1}$$

together with the conditions

$$\partial\tilde{\Omega}/\partial y, \partial\tilde{\Omega}/\partial t \in W_1^1(\mathrm{T}^2_{(t,y)}; H). \tag{2.2}$$

By means of analyzing conditions (2.1) we can, similarly as before, state that a function $\tilde{\Omega} \in C^1(\mathbb{R}^1 \times \mathrm{T}^2; \mathbb{C})$ exists if the differential form

$$\tilde{Z}^{(1)}[\tilde{\varphi},\tilde{\psi}] := \tilde{Z}_{(\tilde{L})}[\tilde{\varphi},\tilde{\psi}]dy + \tilde{Z}_{(\tilde{M})}[\tilde{\varphi},\tilde{\psi}]dt) + \bar{\tilde{\varphi}}^{\intercal}\tilde{\psi}dx = d\tilde{\Omega}[\tilde{\varphi},\tilde{\psi}], \tag{2.3}$$

is exact, that is one can write down the corresponding to (2.3) relationship for $\tilde{\Omega}[\tilde{\varphi},\tilde{\psi}] := \tilde{\Omega}(\lambda;\xi|\mu;\eta)$ as follows:

$$\tilde{\Omega}(\lambda;\xi|\mu;\eta) = \tilde{\Omega}_0(\lambda;\xi|\mu;\eta) + \int\limits_{\tilde{S}(P;\tilde{P}_0)} \tilde{Z}^{(1)}[\tilde{\varphi}(\lambda;\xi),\tilde{\psi}(\mu;\eta)] \tag{2.4}$$

for all pairs $(\lambda;\xi)$ and $(\mu;\eta) \in \Sigma$, where by definition,

$$\begin{aligned}
\tilde{\mathcal{H}}_0 := \Big\{ & \tilde{\psi}(\lambda;\xi) \in \mathcal{H}_-^* :\ \tilde{\mathrm{L}}\tilde{\psi}(\lambda;\xi) = 0,\ \tilde{\mathrm{M}}\tilde{\psi}(\lambda;\xi) = 0,\\
& \tilde{\psi}(\lambda;\xi)|_{\substack{t=0^+\\ y=0^+}} = \tilde{\psi}_\lambda \in H_-,\ \tilde{L}\tilde{\psi}_\lambda = \lambda\psi_\lambda,\ \tilde{\psi}(\lambda;\xi)|_{x=\tilde{x}_0} = 0,\\
& (y,t) \in \mathrm{T}^2,\ (\lambda;\xi) \in \Sigma = \sigma(L,M) \cap \bar{\sigma}(\tilde{L}^*,\tilde{M}^*) \times \Sigma\sigma \Big\}\\
\tilde{\mathcal{H}}_0^* := \Big\{ & \tilde{\varphi}(\lambda;\eta) \in \mathcal{H}_-^* :\ \tilde{\mathrm{L}}^*\tilde{\varphi}(\lambda;\eta) = 0,\ \tilde{\mathrm{M}}^*\tilde{\varphi}(\lambda;\eta) = 0,\\
& \tilde{L}^*\tilde{\varphi}_\lambda = \bar{\lambda}\varphi_\lambda\tilde{\varphi}(\lambda;\eta)|_{\substack{t=0^+\\ y=0^+}} = \tilde{\varphi}_\lambda \in H_-^*,\ \tilde{\varphi}(\lambda;\eta)|_{x=\tilde{x}_0} = 0,\\
& (y,t) \in \mathrm{T}^2,\ (\lambda;\eta) \in \Sigma = \sigma(\tilde{L},\tilde{M}) \cap \bar{\sigma}(\tilde{L}^*,\tilde{M}^*) \times \Sigma_\sigma \Big\}
\end{aligned} \tag{2.5}$$

for some fixed point $\tilde{x}_0 \in \mathbb{R}$ and the "spectral" parameter set $\Sigma := \sigma(\tilde{L},\tilde{M}) \times \Sigma_\sigma \subset \mathbb{C}^p$. Assume now that there exists a bounded isomorphic mapping $\mathbf{\Omega} : \mathcal{H}_0 \rightleftarrows \tilde{\mathcal{H}}_0$, parametrized by pairs $(\varphi(\lambda;\xi), \tilde{\varphi}(\mu;\eta) \in \mathcal{H}_0^* \times \tilde{\mathcal{H}}_0^*$ and $(\psi(\lambda;\xi), \tilde{\psi}(\mu;\eta)) \in \mathcal{H}_0 \times \tilde{\mathcal{H}}_0$, $(\lambda;\xi)$ and $(\mu;\eta) \in \Sigma$, which we will define as

$$\mathbf{\Omega} : \psi(\lambda;\xi) \to \tilde{\psi}(\lambda;\xi) := \psi(\lambda;\xi) \cdot \Omega^{-1}\Omega_0, \tag{2.6}$$

where one supposes that the expression $\Omega^{-1} \colon L_2^{(\rho)}(\Sigma;\mathbb{C}) \to L_2^{(\rho)}(\Sigma;\mathbb{C})$ denotes the inverse operator for the corresponding operator $\Omega : L_2^{(\rho)}(\Sigma;\mathbb{C}) \to L_2^{(\rho)}(\Sigma;\mathbb{C})$ with the kernel $\Omega(\lambda;\xi|\mu,\eta) \in L_2^{(\rho)}(\Sigma;\mathbb{C}) \otimes L_2^{(\rho)}(\Sigma;\mathbb{C})$, and ρ being some finite Borel measure on the Borel subsets of the "spectral" parameter set Σ. The corresponding bounded isomorphic mapping between subspaces $\mathcal{H}_0^*$ and $\tilde{\mathcal{H}}_0^*$, i.e. $\hat{\Omega}^{\circledast}\colon\ \mathcal{H}_0^* \rightrightarrows \tilde{\mathcal{H}}_0^*$ such that for any pair $(\varphi(\lambda;\xi), \tilde{\varphi}(\mu;\eta)) \in \mathcal{H}_0^* \times \tilde{\mathcal{H}}_0^*$

$$\hat{\mathbf{\Omega}}^{\circledast}\colon\ \varphi(\lambda;\xi) \to \tilde{\varphi}(\lambda;\xi) = \varphi(\lambda;\xi)\cdot \Omega^{\circledast,-1}\Omega_0^{\circledast}, \tag{2.7}$$

where, by definition, the kernel $\Omega^{\circledast}(\lambda;\xi) := \bar{\Omega}^{\intercal}(\lambda;\xi) \in L_2^{(\rho)}(\Sigma;\mathbb{C})\otimes L_2^{(\rho)}(\Sigma;\mathbb{C})$ for all $(\lambda;\xi)\in\Sigma$. It is easy now to see that the following proposition holds.

Proposition 2.1. *The constructed above pair of mappings* $(\mathbf{\Omega}^{\circledast}, \mathbf{\Omega})$ *is consistent, i.e. there exists such a kerenel* $\tilde{\Omega}(\lambda;\xi) \in L_2^{(\rho)}(\Sigma;\mathbb{C})\otimes L_2^{(\rho)}(\Sigma;\mathbb{C})$ *that analogs of* (2.2) *and* (2.3) *hold.*

Proof. Indeed, by using expressions (1.8), (2.3), (2.6) and (2.7), we easily obtain:

$$d\tilde{\Omega} = \Omega_0\Omega^{-1}d\Omega\Omega^{-1}\Omega_0 = -d\left(\Omega_0\Omega^{-1}\Omega_0\right),$$

whence $\tilde{\Omega} = -\Omega_0\Omega^{-1}\Omega_0$ and, moreover, the condition $\tilde{\Omega}_0 = -\Omega_0$ holds. □

Since the functional subspaces $\mathcal{H}_0$ and $\tilde{\mathcal{H}}_0$ are consistent, the expressions for $\tilde{\mathrm{L}} := \mathbf{\Omega}\mathrm{L}\mathbf{\Omega}^{-1}$ and $\tilde{\mathrm{M}} := \mathbf{\Omega}\mathrm{M}\mathbf{\Omega}^{-1}$ prove to be differential too. The corresponding extensions onto the entire $\mathcal{H}$ mappings $\mathbf{\Omega}$ and $\mathbf{\Omega}^{\circledast} : \mathcal{H}\to\mathcal{H}$ defined by (2.6) and (2.7) are often called Delsarte–Darboux type transformations and were for the first time used by Darboux [8] and in general form were studied by B. M. Levitan and I. S. Sargsian [7], V. A. Marchenko [6], J. Delsarte and J. Lions [1, 2].

Consider now the compatible pair of invertible Delsarte mappings $(\mathbf{\Omega}, \mathbf{\Omega}^{\circledast})$ from the closed functional subspace $\mathcal{H}_0\times\mathcal{H}_0^* \subset \mathcal{H}_-\times\mathcal{H}_-^*$ to closed subspaces $\tilde{\mathcal{H}}_0\times\tilde{\mathcal{H}}_0^* \subset \mathcal{H}_-\times\mathcal{H}_-^*$ reduced naturally upon $\mathcal{H}\times\mathcal{H}^*$. It means that the following diagram

$$\begin{array}{ccccc} & \mathcal{H} & \mathbf{\Omega}\to & \mathcal{H} & \\ \mathrm{M},\mathrm{L} & \downarrow & & \downarrow & \tilde{\mathrm{L}},\tilde{\mathrm{M}} \\ & \mathcal{H} & \mathbf{\Omega}\to & \mathcal{H} & \end{array}$$

is commutative, and consequently the relationships $\mathbf{\Omega}\cdot\mathrm{L} = \tilde{\mathrm{L}}\cdot\mathbf{\Omega}$ and $\mathbf{\Omega}\cdot\mathrm{M} = \tilde{\mathrm{M}}\cdot\mathbf{\Omega}$ hold. These relationships connect evolution operators L and M in the entire space $\mathcal{H}$ with the corresponding evolution operators $\tilde{L}$ and $\tilde{M}$.

In order to define the exact form of mappings $\mathbf{\Omega} : \mathcal{H}\to\mathcal{H}$ and $\mathbf{\Omega}^{\circledast} : \mathcal{H}^*\to\mathcal{H}^*$, we will make use of mappings (2.6) and (2.7) on fixed elements $(\varphi(\lambda;\xi), \psi(\mu;\eta)) \in \mathcal{H}_0^*\times\mathcal{H}_0$, where $(\lambda;\xi)$, $(\mu;\eta)\in\Sigma$. Namely, from (2.4) we obtain expressions

$$\tilde{\psi}(\lambda;\xi) = \mathbf{\Omega}\big(\psi(\lambda;\xi)\big) := \\ \int_\Sigma d\rho(\mu;\eta)\int_\Sigma d\rho(\nu;\gamma)\psi(\mu;\eta)\times\Omega_x^{-1}(\mu;\eta|\nu;\gamma)\,\Omega_{x_0}(\nu;\gamma|\lambda;\xi),$$

$$\tilde{\varphi}(\lambda;\xi) = \mathbf{\Omega}^{\circledast}\big(\varphi(\lambda;\xi)\big) := \\ \int_\Sigma d\rho(\mu;\eta)\int_\Sigma d\rho(\nu;\gamma)\varphi(\mu;\eta)\times\Omega_x^{\circledast,-1}(\mu;\eta|\nu;\gamma)\,\Omega_{x_0}^{\circledast}(\nu;\gamma|\lambda;\xi),$$

which make it possible to define the extended operators $\mathbf{\Omega}:\mathcal{H}_+\to\mathcal{H}_+$ and $\mathbf{\Omega}^{\circledast}:\mathcal{H}_+^*\to\mathcal{H}_+^*$ as

$$\mathbf{\Omega}:=\mathbf{1}-\int\limits_{\Sigma}d\rho(\mu;\eta)\int\limits_{\Sigma}d\rho(\nu;\gamma)\tilde{\psi}(\mu;\eta)\Omega_{x_0}^{-1}(\mu;\eta|\nu;\gamma)\times\int\limits_{P_0}^{P}Z^{(m-1)}[\varphi(\nu;\gamma),(\cdot)],$$

$$\mathbf{\Omega}^{\circledast}:=\mathbf{1}-\int\limits_{\Sigma}d\rho(\mu;\eta)\int\limits_{\Sigma}d\rho(\nu;\gamma)\tilde{\varphi}(\mu;\eta)\Omega_{x_0}^{\circledast,-1}(\mu;\eta|\nu;\gamma)\times\int\limits_{P_0}^{P}\bar{Z}^{(m-1),\intercal}[(\cdot),\psi(\nu;\gamma)],\tag{2.8}$$

with ρ being as before some finite Borel measure on the Borel subsets of the "spectral" parameter set Σ.

Now based on expressions (2.8), one can easily find the "dressed" operators $\tilde{\mathrm{L}},\tilde{\mathrm{M}}:\mathcal{H}\to\mathcal{H}$, and thereby their coefficient matrix functions subject to the corresponding coefficients of operators $\mathrm{L},\mathrm{M}:\mathcal{H}\to\mathcal{H}$, which are also called the Darboux–Backlund transformations [8].

Note also here that the compatibility condition for the dressed differential operators $\tilde{\mathrm{L}},\tilde{\mathrm{M}}$ is equivalent to some system of nonlinear evolution equations in partial derivatives and often this pair is called [9, 6, 8] a Lax type or a Zakharov–Shabat pair.

Consider now the structure of "dressed" operators

$$\tilde{\mathrm{L}}=\mathbf{\Omega}\mathrm{L}\hat{\mathbf{\Omega}}^{-1},\quad \tilde{\mathrm{M}}=\mathbf{\Omega}\mathrm{M}\mathbf{\Omega}^{-1}\tag{2.9}$$

as elements of orbits of some Volterra group G_- [15, 16]. As one can see from (2.10), these operators lie in orbits of elements $\mathrm{L},\mathrm{M}\in\mathcal{G}_-^*$, respectively, with respect to the natural co-adjoint group action of the group of pseudo-differential operators G_-, whose Lie co-algebra $\mathcal{G}_-^*$ consists of Volterra type integral operators of the form $l:=\sum_{i=0}^{n(l)}a_i\partial^{-1}\bar{b}_i^{\intercal}$, where $n(l)\in\mathbb{Z}_+$ is some finite number, i.e.,

$$\mathcal{G}_-^*=\left\{l=\sum_{i=0}^{n(l)}a_i\partial^{-1}\bar{b}_i^{\intercal}:\ a_i,b_i\in C^1\left(\mathrm{T}^2_{(t,y)};S(\mathbb{R};End\mathbb{C}^N)\right),i=\overline{1,n},\ n\in\mathbb{Z}_+\right\}.\tag{2.10}$$

Let us show that these orbits leave the space $\mathcal{G}_-^*$ invariant, i.e. the "dressed" operators $\tilde{\mathrm{L}}$ and $\tilde{\mathrm{M}}:\mathcal{H}\to\mathcal{H}$ under transformation (2.10) remain to be differential and preserve their orders. To do it let us consider an arbitrary pseudo-differential operator $\mathrm{P}:\mathcal{H}\to\mathcal{H}$ and note that the following identity

$$\mathrm{Tr}\left(\mathrm{P}f\partial^{-1}\bar{h}^{\intercal}\right):=\left(\mathrm{P}f,\partial^{-1}\bar{h}^{\intercal}\right)_{\mathcal{G}}=(h,\mathrm{P}_+f)_H\tag{2.11}$$

holds, where, by definition, $(\cdot,\cdot)_H$ denotes the inner product in the Hilbert space H,

$$\mathrm{Tr}(\cdot):=\int\limits_{\mathbb{R}}dx\,\operatorname*{res}_{\partial}\mathrm{Sp}(\cdot),$$

and the operation $(\cdot)_+$ means the projection upon the differential part of a given pseudo-differential expression. Based on relationship(2.11), it is easy to prove the following [13] lemma.

Lemma 2.2. *A pseudo-differential operator* $\mathrm{P}\colon \mathcal{H} \to \mathcal{H}$ *is pure differential if and only if the following equality*

$$\big(h, (\mathrm{P}\partial^i)_+ f\big)_H = \big(h, \mathrm{P}_+\partial^i f\big)_H \tag{2.12}$$

holds with respect to the inner product $(\cdot,\cdot)_H$ *in* H *for all* $i \in \mathbb{Z}_+$ *and any dense in* $H^* \times H$ *set of pairs* $(h, f) \in \mathcal{H}_0^* \times \mathcal{H}_0$*. That is the condition* (2.12) *is equivalent to the equality* $\mathrm{P}_+ = \mathrm{P}$*.*

Making use of this Lemma in the case when $\mathrm{P} := \mathrm{L}\colon \mathcal{H} \to \mathcal{H}$ and taking into consideration condition (2.12), we obtain:

$$\begin{aligned}
&\left(h, (\tilde{\mathrm{L}}\partial^i)_+ f\right) = \left(h, \left(\boldsymbol{\Omega}\left(\frac{\partial}{\partial t} - L\right)\boldsymbol{\Omega}^{-1}\cdot\partial^i\right)_+ f\right) = \\
&= \left(h, \frac{\partial}{\partial t}\partial^i f\right) - \left(h, \left[\left(\boldsymbol{\Omega}_t\boldsymbol{\Omega}^{-1} + \boldsymbol{\Omega}L\boldsymbol{\Omega}^{-1}\right)\partial^i\right]_+ f\right) = \\
&= \left(h, \frac{\partial}{\partial t}\partial^i f\right) - \mathrm{Tr}\left\{\left(\boldsymbol{\Omega}_t\boldsymbol{\Omega}^{-1}\partial^i + \boldsymbol{\Omega}\ell\boldsymbol{\Omega}^{-1}\partial^i\right) f\partial^{-1}\bar{h}^{\intercal}\right\} = \\
&= \left(h, \frac{\partial}{\partial t}\partial^i f\right) - \mathrm{Tr}\left\{\left(1 - \tilde{\psi}\Omega_0^{-1}\partial^{-1}\bar{\varphi}^{\intercal}\right)_t\left(1 + \psi\Omega_0^{-1}\partial^{-1}\bar{\tilde{\varphi}}^{\intercal}\right)\partial^i + \right. \\
&\qquad\qquad \left. + \left(1 - \tilde{\psi}\Omega_0^{-1}\partial^{-1}\bar{\varphi}^{\intercal}\right) L\left(1 + \psi\Omega_0\partial^{-1}\bar{\varphi}^{\intercal}\right)\partial^i f\partial^{-1}\bar{h}^{\intercal})\right\} \equiv \\
&\equiv \mathrm{Tr}\left(\tilde{L}(\partial^i f)\partial^{-1}\bar{h}^{\intercal}\right) = \left(h, \tilde{L}_+\partial^i f\right).
\end{aligned} \tag{2.13}$$

When deriving (2.13) we made use of the equalities $\mathrm{L}\psi = 0$, $\mathrm{L}^*\varphi = 0$ for any pair $(\varphi, \psi) \in \mathcal{H}_0^* \times \mathcal{H}_0$ and the obvious condition $\mathrm{L}_+ = \mathrm{L}$. Thereby, by virtue of Lemma 2.2, the operator $\tilde{\mathrm{L}}\colon \tilde{\mathcal{H}} \to \tilde{\mathcal{H}}$ continues to be differential and, moreover, the order $ord\,\tilde{L} = ord\,L$, owing to definition (2.9). Similarly, the same proposition also holds for the "dressed" operator $\tilde{\mathrm{M}}\colon \mathcal{H} \to \mathcal{H}$, i.e. $\tilde{\mathrm{M}}_+ = \tilde{\mathrm{M}}$ and $ord\,\tilde{M} = ord\,M$. As a conclusion from the results obtained above we derive the following proposition.

Proposition 2.3. *The pair of "dressed" differential operators* $\tilde{\mathrm{L}}, \tilde{\mathrm{M}}\colon \mathcal{H} \to \mathcal{H}$ *of the form* (2.9)*, obtained as a result of the Delsarte–Darboux type transformation from a compatible Zakharov–Shabat commuting pair of differential operators* $\mathrm{L}, \mathrm{M}\colon \mathcal{H} \to \mathcal{H}$ *of the form* (1.1) *remains a compatible pair of commuting differential operators in* $\mathcal{H}$ *preserving their differential orders. The corresponding coefficient matrix functions of the Delsarte–Darboux transformed differential operators* $\tilde{\mathrm{L}}, \tilde{\mathrm{M}}\colon \mathcal{H} \to \mathcal{H}$ *define a so-called Backlund–Darboux transformation for the coefficient matrix functions of the initially chosen compatible pair* $\mathrm{L}, \mathrm{M}\colon \mathcal{H} \to \mathcal{H}$ *of differential operators.*

From the practical point of view of Proposition 2.3, it is clear that the Delsarte–Darboux transformations are especially useful in the construction of a wide class of so-called soliton [8, 11, 9, 10, 22] and algebraic solutions to the corresponding system of nonlinear evolution differential equations, which is equivalent to the compatibility condition for the obtained pair of “dressed” operators (1.1). A great deal of papers (see, for example [8, 10])have been devoted to such calculations, where particular solutions of solitons and other types were built for different evolution differential equations of mathematical physics.

3. GENERAL STRUCTURE OF DELSARTE–DARBOUX TRANSFORMATIONS: A DIFFERENTIAL-GEOMETRIC ASPECTS

A preliminary analysis of the Delsarte–Darboux type transformation operators constructed above for differential operator expressions in the case of a single variable $x \in \mathbb{R}$ shows that its form is rather restrictive concerning a class of possible transformations for operator differential expressions depending on two or more variables and admitting Lax type representations [15, 16, 9, 14, 6, 17]. Therefore, it is important to consider a nontrivial multi-dimensional generalization of the scheme proposed above for constructing these Delsarte–Darboux type transformations. Below we will outline such an approach to this problem based on the preliminary results obtained in [22, 12, 21, 20].

As before we consider a parametric functional space $\mathcal{H} := L_1(\mathrm{T}_t; H)$, $\mathrm{T}_t := [0, T] \in \mathbb{R}_+$, where we take $H := L_2(\mathbb{R}^2; \mathbb{C}^N)$, in which there acts a (2+1)-dimensional differential closeable normal operator $\mathrm{L} : \mathcal{H} \to \mathcal{H}$ of the form

$$\begin{gathered} \mathrm{L} = \partial/\partial t - L(t; x, y|\partial), \\ L(t; x, y|\partial) := \sum_{0 \le i+j \le n(L)} u_{ij} \frac{\partial^{i+j}}{\partial x^i \partial y^j} \end{gathered} \tag{3.1}$$

with coefficients $u_{ij} \in C^1(\mathrm{T}; \mathcal{S}(\mathbb{R}^1; End\mathbb{C}^N))$, $i, j = \overline{1, n(L)}$. Applying the same scheme as above, we find that for expression (3.1) the standard identity

$$\langle L^* \varphi, \psi \rangle - \langle \varphi, L\psi \rangle = \frac{\partial}{\partial t}(\bar{\varphi}^\intercal \psi) + \frac{\partial}{\partial x} Z^{(x)}[\varphi, \psi] + \frac{\partial}{\partial y} Z^{(y)}[\varphi, \psi] \tag{3.2}$$

holds for all pairs $(\varphi, \psi) \in D(\mathrm{L}^*) \times D(\mathrm{L}) \subset \mathcal{H}^* \times \mathcal{H}$, where $Z^{(x)}[\varphi, \psi]$ and $Z^{(y)}[\varphi, \psi]$ are some easily computable semi-linear forms on $\mathcal{H}^* \times \mathcal{H}$. From (3.2), with respect to the oriented measure $dt \wedge dx \wedge dy$, one easily gets that

$$\begin{aligned} &(\langle \mathrm{L}^* \varphi, \psi \rangle - \langle \varphi, \mathrm{L}\psi \rangle)\, dt \wedge dx \wedge dy = \\ &= d\left(\bar{\varphi}^\intercal \psi \wedge dx \wedge dy + Z^{(x)}[\varphi, \psi] dy \wedge dt - Z^{(y)}[\varphi, \psi]\, dx \wedge dt\right) := dZ^{(2)}[\varphi, \psi], \end{aligned} \tag{3.3}$$

where, by definition,

$$Z^{(2)}[\varphi, \psi] = \bar{\varphi}^\intercal \psi\, dx \wedge dy + Z^{(x)}[\varphi, \psi]\, dy \wedge dt + Z^{(y)}[\varphi, \psi]\, dt \wedge dx \tag{3.4}$$

is a semilinear on $\mathcal{H}_-^* \times \mathcal{H}_-$ differential 2-form on $\mathbb{R}^2 \times \mathrm{T}$. Therefore, for all $t \in \mathrm{T}$ and any $(\varphi(\lambda;\xi), \psi(\mu;\eta)) \in \mathcal{H}_0^* \times \mathcal{H}_0 \subset \mathcal{H}_-^* \times \mathcal{H}_-$, $(\lambda;\xi), (\mu;\eta) \in \Sigma$, from a f closed subspace of the correspondingly Hilbert–Schmidt rigged [4, 3] parametric functional spaces $\mathcal{H}_-^* \times \mathcal{H}_-$ with $\Sigma \subset \mathbb{C}^p$ being some "spectral" parameter set, the expression on the right-hand side of relationship (3.3) can be made to become identically zero if the conditions

$$\mathrm{L}^*\varphi = 0, \quad \mathrm{L}\psi = 0 \tag{3.5}$$

hold on $\mathcal{H}_0^* \times \mathcal{H}_0$. Thereby, similarly to (1.10), one can define the following closed dense subspaces $\mathcal{H}_0^* \subset \mathcal{H}_-^*$ and $\mathcal{H}_0 \subset \mathcal{H}_-$ as

$$\begin{aligned}
\mathcal{H}_0 := \Big\{ & \psi(\lambda;\xi) \in \mathcal{H}_- : \mathrm{L}\psi(\lambda;\xi) = 0,\ \mathrm{M}^*(\lambda;\xi) = 0, \\
& \psi(\lambda;\xi)|_{t=0^+} = \psi_\lambda \in H_-^*,\ L\psi_\lambda = \lambda\psi_\lambda,\ \psi(\xi)|_\Gamma = 0,\ t \in \mathrm{T}, \\
& (\lambda;\xi) \in \Sigma = \sigma(L,M) \cap \bar{\sigma}(L^*,M^*) \times \Sigma_\sigma \Big\}, \\
\mathcal{H}_0^* := \Big\{ & \varphi(\lambda;\xi) \in \mathcal{H}_-^* : \mathrm{L}^*\varphi(\lambda;\xi) = 0,\ \mathrm{M}^*\varphi(\lambda;\xi) = 0, \\
& \varphi(\lambda;\xi)|_{t=0^+} = \varphi_\lambda \in H_-^*,\ M\varphi_\lambda = \bar{\lambda},\ \varphi_\lambda,\ \varphi(\xi)|_\Gamma = 0,\ t \in \mathrm{T}, \\
& (\lambda;\xi) \in \Sigma = \sigma(L,M) \cap \bar{\sigma}(L^*,M^*) \times \Sigma_\sigma \Big\},
\end{aligned} \tag{3.6}$$

where we may possibly have imposed some boundary conditions on the spaces $\mathcal{H}_-^*$ and $\mathcal{H}_-$ at $\Gamma \subset \mathbb{R}^2$, where Γ is some one-dimensional smooth curve in $\mathbb{R}^2$. Now differential 2-form (3.6) becomes closed, i.e. $dZ^{(2)}[\varphi,\psi] = 0$, which due to the Poincare lemma [18, 19] brings about the following equality

$$Z^{(2)}[\varphi(\lambda;\xi), \psi(\mu;\eta)] = d\Omega^{(1)}[\varphi(\lambda;\xi), \psi(\mu;\eta)] \tag{3.7}$$

for some differential 1-form $\Omega^{(1)}[\varphi(\lambda;\xi), \psi(\mu;\eta)]$ on the space $\mathbb{R}^2 \times \mathrm{T}$ and all pairs $(\varphi(\xi), \psi(\eta)) \in \mathcal{H}_0^* \times \mathcal{H}_0$, $\xi, \eta \in \Sigma$. Thus, the following proposition similar to one in [4] holds.

Proposition 3.1. *If the differential 2-forms $Z^{(2)}[\varphi(\lambda;\xi), \psi(\mu;\eta)]$ are closed for all pairs $(\varphi(\xi), \psi(\eta)) \in \mathcal{H}_0^* \times \mathcal{H}_0$, $\xi, \eta \in \Sigma$, then the pair of conjugated differential operators (L, L^*) is adjoint with respect to the scalar form on $H^* \times H$.*

Applying now the Stokes theorem [18, 19] to a closed 2-form (3.7) on $\mathbb{R}^2 \times \mathrm{T}$, we obtain

$$\int\limits_{S^{(2)}\left(\sigma^{(1)},\sigma_0^{(1)}\right)} Z^{(2)}[\varphi(\lambda;\xi), \psi(\mu;\eta)] =$$

$$= \int\limits_{S^{(2)}\left(\sigma^{(1)},\sigma_0^{(1)}\right)} d\Omega^{(1)}[\varphi(\lambda;\xi), \psi(\mu;\eta)] = \int\limits_{\partial S^{(2)}\left(\sigma^{(1)},\sigma_0^{(1)}\right)} \Omega^{(1)}[\varphi(\lambda;\xi), \psi(\mu;\eta)] =$$

$$= \int\limits_{\sigma^{(1)}} \Omega^{(1)}[\varphi(\lambda;\xi), \psi(\mu;\eta)] - \int\limits_{\sigma_0^{(1)}} \Omega^{(1)}[\varphi(\lambda;\xi), \psi(\mu;\eta)] := \Omega(\lambda;\xi|\mu;\eta) - \Omega_0(\lambda;\xi|\mu;\eta) \quad (3.8)$$

for some piecewise imbedded smooth compact two-dimensional surface $S^{(2)}(\sigma^{(1)}, \sigma_0^{(1)}) \subset \mathbb{R}^2 \times \mathrm{T}$ with the boundary $\partial S^{(2)}(\sigma^{(1)}, \sigma_0^{(1)}) = \sigma^{(1)} - \sigma_0^{(1)}$, where $\sigma^{(1)}, \sigma_0^{(1)} \subset \mathbb{R}^2 \times \mathrm{T}$ are some closed homological one-dimensional cycles without self-intersections parametrized by a running point $P(x, y; , t) \in \mathbb{R}^2 \times \mathrm{T}$ and a fixed point $P(x_0, y_0; t_0) \in \mathbb{R}^2 \times \mathrm{T}$, respectively.

Making use of surface integral (3.8) and assuming that the closed cycle $\sigma_0^{(1)} \subset \mathbb{R}^2 \times \mathrm{T}$ is fixed, one can define the following mappings for the corresponding Delsarte–Darboux transformations on pairs of functions $(\varphi, \psi) \in \mathcal{H}_0^* \times \mathcal{H}_0$:

$$\begin{aligned} \tilde{\psi}(\lambda;\xi) = \mathbf{\Omega}(\psi(\lambda;\xi)) := \int\limits_{\Sigma} d\rho(\mu;\eta) \int\limits_{\Sigma} d\rho(\nu;\gamma)\psi(\mu;\eta) \times \\ \times\, \Omega^{-1}(\mu;\eta|\nu;\gamma)\Omega_0(\nu;\gamma|\lambda;\xi), \end{aligned} \quad (3.9)$$

$$\begin{aligned} \tilde{\varphi}(\lambda;\xi) = \mathbf{\Omega}^{\circledast}(\varphi(\lambda;\xi)) := \int\limits_{\Sigma} d\rho(\mu;\eta) \int\limits_{\Sigma} d\rho(\nu;\gamma)\varphi(\mu;\eta) \times \\ \times\, \Omega^{\circledast,-1}(\mu;\eta|\nu;\gamma)\Omega_0^{\circledast}(\nu;\gamma|\lambda;\xi), \end{aligned} \quad (3.10)$$

where the Delsarte transmutation generator kernel expressions $\Omega(\lambda;\xi|\mu;\eta)$ and $\Omega_0^{\circledast}(\lambda;\xi|\mu;\eta) \in L_2^{(\rho)}(\Sigma;\mathbf{C}) \otimes L_2^{(\rho)}(\Sigma;\mathbf{C})$, $(\lambda;\xi), (\mu;\eta) \in \Sigma$, are as before considered to be non-degenerate kernels from $L_2^{(\rho)}(\Sigma;\mathbf{C}) \otimes L_2^{(\rho)}(\Sigma;\mathbf{C})$. The following proposition concerning the pair of spaces $\tilde{\mathcal{H}}_0 \ni \tilde{\psi}$ and $\tilde{\mathcal{H}}_0^* \ni \tilde{\varphi}$ holds.

Proposition 3.2. *The pair of closed functional subspaces $\tilde{\mathcal{H}}_0^* \subset \mathcal{H}_-$ and $\tilde{\mathcal{H}}_0 \subset \mathcal{H}_-$ defined by expressions* (3.10) *can be equivalently characterized as follows:*

$$\begin{aligned} \tilde{\mathcal{H}}_0 := \Big\{ & \tilde{\psi}(\lambda;\xi) \in \mathcal{H}_-^* : \ \tilde{\mathrm{L}}\tilde{\psi}(\lambda;\xi) = 0, \ \tilde{\mathrm{M}}\tilde{\psi}(\lambda;\xi) = 0, \\ & \tilde{\psi}(\lambda;\xi)|_{t_0} = \tilde{\psi}_\lambda \in H_-, \ \tilde{L}\tilde{\psi}_\lambda = \lambda\psi_\lambda, \ \tilde{\psi}(\lambda;\xi)|_{\tilde{\Gamma}} = 0, \\ & (\lambda;\xi) \in \Sigma = \sigma(\tilde{L}, \tilde{M}) \cap \bar{\sigma}(\tilde{L}^*, \ \tilde{M}^*) \times \Sigma_\sigma \Big\}, \\ \tilde{\mathcal{H}}_0^* := \Big\{ & \tilde{\varphi}(\lambda;\eta) \in \mathcal{H}_-^* : \ \tilde{\mathrm{L}}^*\tilde{\varphi}(\lambda;\eta) = 0, \ \tilde{\mathrm{M}}^*\tilde{\varphi}(\lambda;\eta) = 0, \\ & \tilde{L}^*\tilde{\varphi}_\lambda = \bar{\lambda}\varphi_\lambda, \ \tilde{\varphi}(\lambda;\eta)|_{t_0} = \tilde{\varphi}_\lambda \in H_-^*, \ \tilde{\varphi}(\lambda;\eta)|_{\tilde{\Gamma}} = 0, \\ & (\lambda;\eta) \in \Sigma = \sigma(\tilde{L}, \tilde{M}) \cap \bar{\sigma}(\tilde{L}^*, \tilde{M}^*) \times \Sigma_\sigma \Big\} \end{aligned} \quad (3.11)$$

for some piecewise smooth curve $\tilde{\Gamma} \subset \mathbb{R}^2$.

Now based on this Proposition, mappings (3.11) can be naturally extended on the entire space $\mathcal{H}_-^* \times \mathcal{H}_-$ by means of the just used before classical method of variation of constants [21, 19, 12, 20] and easily give rise to the exact forms of the pair of Delsarte–Darboux mapping $(\boldsymbol{\Omega}, \boldsymbol{\Omega}^{\circledast})$ upon the entire space $\mathcal{H}^* \times \mathcal{H}$:

$$\begin{aligned} \boldsymbol{\Omega} &:= \mathbf{1} - \int_{\Sigma} d\rho(\mu;\eta) \int_{\Sigma} d\rho(\nu;\gamma) \tilde{\psi}(\mu;\eta) \Omega_{\sigma_0}^{-1}(\mu;\eta|\nu;\gamma) \times \\ &\times \int_{S^{(2)}\left(\sigma^{(1)},\sigma_0^{(1)}\right)} Z^{(m-1)}[\varphi(\nu;\gamma),(\cdot)], \\ \boldsymbol{\Omega}^{\circledast} &:= \mathbf{1} - \int_{\Sigma} d\rho(\mu;\eta) \int_{\Sigma} d\rho(\nu;\gamma) \tilde{\varphi}(\mu;\eta) \Omega_{\sigma_0}^{\circledast,-1}(\mu;\eta|\nu;\gamma) \times \\ &\times \int_{S^{(2)}\left(\sigma^{(1)},\sigma_0^{(1)}\right)} \bar{Z}^{(m-1),\intercal}[(\cdot),\psi(\nu;\gamma)], \end{aligned} \tag{3.12}$$

defined for some imbedded into piecewise smooth two-dimensional surface $S^{(2)}(\sigma^{(1)}, \sigma_0^{(1)}) \subset \mathbb{R}^2 \times \mathrm{T}$, spanned between two closed homological cycles $\sigma^{(1)}$ and $\sigma_0^{(1)} \subset \mathbb{R}^2 \times \mathrm{T}$ as its boundary, that is $\partial S^{(2)}(\sigma^{(1)}, \sigma_0^{(1)}) := \sigma^{(1)} - \sigma_0^{(1)}$. It is seen from (3.12) that found above Delsarte transmutation operators $\boldsymbol{\Omega} : \mathcal{H} \to \mathcal{H}$ and $\boldsymbol{\Omega}^{\circledast} : \mathcal{H}^* \to \mathcal{H}^*$ are bounded of Volterra type integral operators, strongly depending on a measure ρ on the "spectral" parameter space Σ and some piecewise smooth two-dimensional surface $S^{(2)}(\sigma^{(1)}, \sigma_0^{(1)})$ parametrized by a running point $P(x, y; t) \in \mathbb{R}^2 \times \mathrm{T}$ and a fixed point $P(x_0, y_0; t_0) \in \mathbb{R}^2 \times \mathrm{T}$.

Making now use of the bounded Delsarte–Darboux integral transformation operators (3.12) of Volterra type, one can now as before to construct the corresponding Delsarte–Darboux transformed differential operator $\tilde{\mathrm{L}} : \mathcal{H} \to \mathcal{H}$ as follows:

$$\tilde{\mathrm{L}} = \mathrm{L} + [\boldsymbol{\Omega}, \mathrm{L}]\boldsymbol{\Omega}^{-1}. \tag{3.13}$$

Since expression (3.13) contains the inverse integral operator $\boldsymbol{\Omega}^{-1} : \mathcal{H} \to \mathcal{H}$, with use of the symmetry properties between closed subspaces $\mathcal{H}_0^* \times \mathcal{H}_0$ and $\tilde{\mathcal{H}}_0^* \times \tilde{\mathcal{H}}_0$, from (3.12) there follows

$$\begin{aligned} \boldsymbol{\Omega}^{-1} &:= \mathbf{1} - \int_{\Sigma} d\rho(\mu;\eta) \int_{\Sigma} d\rho(\nu;\gamma) \psi(\mu;\eta) \tilde{\Omega}_0^{-1}(\mu;\eta|\nu;\gamma) \times \\ &\times \int_{S^{(2)}\left(\sigma^{(1)},\sigma_0^{(1)}\right)} \tilde{Z}[\tilde{\varphi}(\nu;\gamma),(\cdot)], \\ \boldsymbol{\Omega}^{\circledast,-1} &:= \mathbf{1} - \int_{\Sigma} d\rho(\mu;\eta) \int_{\Sigma} d\rho(\nu;\gamma) \varphi(\mu;\eta) \tilde{\Omega}_0^{\circledast,-1}(\mu;\eta|\nu;\gamma) \times \\ &\times \int_{S^{(2)}\left(\sigma^{(1)},\sigma_0^{(1)}\right)} \tilde{\bar{Z}}^{(m-1),\intercal}[(\cdot),\tilde{\psi}(\nu;\gamma)], \end{aligned} \tag{3.14}$$

for $(\tilde{\varphi}(\lambda;\xi),\tilde{\psi}(\mu;\eta)) \in \tilde{\mathcal{H}}_0^* \times \tilde{\mathcal{H}}_0$, $(\lambda;\xi),(\mu;\eta) \in \Sigma$, satisfying conditions (3.11). As a result of direct calculations in (3.13), based on expressions (3.14), one can find the corresponding Delsarte–Darboux transformed coefficient functions of the transformed operator $\tilde{\mathrm{L}} : \mathcal{H} \to \mathcal{H}$ parametrized by piecewise smooth closed one-dimensional homological cycles $\sigma^{(1)}$ and $\sigma_0^{(1)} \subset \mathbb{R}^2 \times \mathrm{T}$. We do not present these expressions here in the general case of operator (3.1), as they are too cumbersome for writing them down. We are going to present in detail an application of the constructions developed in the article in Part 2.

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REFERENCES

[1] Delsarte J., *Sur certaines transformations fonctionelles relative aux equations linearines aux derives partielles du second ordre*, C. R. Acad. Sci. Paris 206 (1938), 178–182.

[2] Delsarte J., Lions J., *Transmutations d'operateurs differentiles dans le domain complexe*, Comment. Math. Helv. 52 (1957), 113–128.

[3] Berezin F. A., Shubin M. A., *The Schrodinger equation*, Moscow, Nauka Publ., 1983 (in Russian).

[4] Berezansky Yu. M., *Eigenfunctions expansions with respect to self-adjoint differential operators*, Kiev, Nauk.Dumka Publ., 1965 (in Russian).

[5] Berezin F. A., Shubin M. A., *Schrodinger equation*, Moscow, the Moscow University Publisher, 1983 (in Russian).

[6] Marchenko V. A., *Spectral theory of Sturm-Liouville operators*, Kiev, Nauk. Dumka Publ., 1972 (in Russian).

[7] Levitan B. M., Sargsian I. S., *Sturm-Liouville and Dirac operators*, Moscow, Nauka Publ., 1988 (in Russian).

[8] Matveev V. B., Salle M. I., *Drboux-Backlund transformations and applications*, NY: Springer, 1993.

[9] Novikov S. P. (Ed.), *Soliton theory*, Moscow: Nauka, 1980, 319.

[10] Ablowitz M. J., Segur H., *Solitons and the invesre scattering transform*, SIAM, Philadelphia, USA, 1981.

[11] Nimmo J. C. C., *Darboux transformations from reductions of the KP-hierarchy*, Preprint of the Department of Mathemtics at the University of Glasgow. November, 8, 2002, 11 p.

[12] Golenia J., Prykarpatsky Y. A., Samoilenko A. M., Prykarpatsky A. K., *The general differential-geometric structure of multidimensional Delsarte transmutation operators in parametric functional spaces and their applications in soliton theory. Part 2*, Opuscula Mathematica, 24 (2004) 1, 74–83.

[13] Samoilenko A. M., Prykarpatsky Ya. A., *Algebraic-analytic aspects of completely integrable dynamical systems and their perturbations.* Proceedings of Institute of Mathematics of NAS of Ukraina. – Kyiv: Institute of Mathematics (41) 2002, 236.

[14] Prykarpatsky, A. K., Mykytiuk, I. V., *Algebraic integrability of dynamical systems on manifolds*, Kluwer Academic Publisher, the Netherlands, 1998.

[15] Nizhnik L. P., *Integrability of nonlinear multidimensional equations with inverse scattering transform approach*, Doklady of AN USSR. 254 (1980) 2 332–335.

[16] Nizhnik L. P., *Inverse scattering problems for hyperbolic equations*, Kyiv, Nauk. dumka, 1991, 232 p.

[17] Manakov S. V., *Inverse scattering transform approach and two-dimensional evolution equations*, Uspekhi mat. nauk. 31 (1976) 5, 245–246.

[18] Godbillon K., *Differential geometry and analitical mechanics*, Moscow, Mir, 1973, 188 p.

[19] Cartan A., *Differential calculus. Differntial forms*, Moscow, Mir, 1971, 392 p.

[20] Hentosh O. E., Prytula M. M., Prykarpatsky A. K., *Differential -geometric and algebraic foundations of studying integrable nonlinear dynamical systems on functional manifolds*, Lviv University Publisher, Ukraine, 2005 (in Ukrainian).

[21] Prykarpatsky A. K., Samoilenko A. M., Prykarpatsky Y. A., *The multidimensional Delsarte transmutation operators, their differential-geometric structure and applications. Part 1.*, Opuscula Mathematica, 23 (2003), 71–80

[22] Prykarpatsky Y. A., Samoilenko A. M., Samolyenko V. G., *The structure of Darboux-type binary transformations and their applications in soliton theory*, Ukr. Mat. Zhurnal, 55 (2003) 12, 1704–1719 (in Ukrainian).

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