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RATES OF CONVERGENCE FOR THE MAXIMUM LIKELIHOOD ESTIMATOR IN THE CONVOLUTION MODEL

Abstract. Rates of convergence for the maximum likelihood estimator in the convolution model, obtained recently by S. van de Geer, are reconsidered and corrected.

Keywords: maximum likelihood estimator, entropy, Hellinger distance, rate of convergence.

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1. INTRODUCTION

Consider independent, identically distributed random variables $X_1, X_2, \ldots, X_n$ in a measurable space $(\mathcal{X}, \mathcal{A})$ with distribution P. Suppose that

$$f_0 = \frac{dP}{d\mu} \in \mathcal{F},$$

where μ is a dominating, σ-finite measure, and $\mathcal{F}$ is a given class of densities with respect to μ. Throughout the whole paper, $\hat{f}_n$ will denote the maximum likelihood estimator (MLE) of f_0 and the accuracy of the estimation will be measured in the Hellinger distance defined as

$$h(\hat{f}_n, f_0) = \left(\frac{1}{2} \int \left(\sqrt{\hat{f}_n} - \sqrt{f_0} \right)^2 d\mu \right)^{\frac{1}{2}}.$$

Our interest will be focused on upper bounds for the convergence rates, when $\mathcal{F}$ is a class of convolution densities.

The paper is organized as follows. In this section, basic notations are introduced and some technical results are formulated. In Sections 2 and 3, the rates of convergence, given in [3] and [2] for two special convolution models, are reconsidered and corrected.

For a class $\mathcal{K}$ of functions on $(\mathcal{X}, \mathcal{A})$, let $conv(\mathcal{K})$ be the convex hull of $\mathcal{K}$, and $\overline{conv}(\mathcal{K})$ be its closure in the pointwise convergence topology.

For a measure Q on $(\mathcal{X}, \mathcal{A})$ and $\delta > 0$, we denote by $N(\delta, \mathcal{K}, Q)$ the δ-covering number and by $H(\delta, \mathcal{K}, Q)$ the δ-entropy of $\mathcal{K}$ with respect to the $L_2(Q)$-norm. Formally, for $\mathcal{K} \subset L_2(Q)$, the δ-covering number $N(\delta, \mathcal{K}, Q)$ is defined as the number of $L_2(Q)$-balls with radius δ, necessary to cover $\mathcal{K}$. The δ-entropy of $\mathcal{K}$ is $H(\delta, \mathcal{K}, Q) = \log N(\delta, \mathcal{K}, Q)$.

The following theorem, proved in [3], is an example of a relatively simple tool for obtaining the rate of convergence for the Hellinger distance between f_0 and $\hat{f}_n$ in case f_0 belongs to a convex class of densities. For a set of indices $\mathcal{Y}$ and some fixed $k_0(\cdot, \cdot)$, let $\mathcal{K} = \{k_0(\cdot, y) : y \in \mathcal{Y}\}$ be a class of densities on $(\mathcal{X}, \mathcal{A})$ with the envelope function $K := \sup_{k \in \mathcal{K}} k$, and let $f_0 \in \mathcal{F} = \overline{conv}(\mathcal{K})$. For $\sigma_n \downarrow 0$, let us define the class of functions

$$\tilde{\mathcal{K}}_n = \left\{ \left(\frac{k_0(\cdot, y)}{f_0} \right) \mathbf{1}\{f_0 > \sigma_n\} : y \in \mathcal{Y} \right\},$$

and moreover, let us denote by P_n, the empirical measure based on observations $X_1, \ldots, X_n$ (i.e., $P_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i}$.)

Theorem 1. *Assume that for some non-decreasing sequence* $\rho_n \geq 1$

$$\int\limits_{f_0 > \sigma_n} \frac{K^2}{f_0} d\mu \leq \rho_n^2, \quad n = 1, 2, \ldots,$$

and

$$\lim_{C \to \infty} \limsup_{n \to \infty} P \left(\sup_{0 < \delta < \delta_0} \left(\frac{\delta}{\rho_n} \right)^w N(\delta, \tilde{\mathcal{K}}_n, P_n) > C \right) = 0,$$

for some $0 < w < \infty$ *and* $\delta_0 > 0$. *Then, for*

$$\tau_n^2 \geq \int\limits_{f_0 \leq \sigma_n} f_0 d\mu, \qquad n = 1, 2, \ldots,$$

$$\tau_n \geq n^{-(2+w)/(4+4w)} \rho_n^{w/(2+2w)}, \qquad n = 1, 2, \ldots,$$

there is

$$h(\hat{f}_n, f_0) = O_P(\tau_n).$$

The following lemma will be used in entropy calculations. Although it is prooved in [1], we present another proof, along the lines suggested in [3], because the technique applied will be useful in the next section.

Lemma 1. *Let*

$$\mathcal{G} = \{g : [0, \infty) \to [0, 1], g \quad \textit{non-increasing}\}.$$

Then there exists a constant C such that for each probabilistic measure Q on $[0, \infty)$,

$$H(\delta, \mathcal{G}, Q) \leq C\delta^{-1}, \ \textit{for all } \delta > 0.$$

Proof. It is easy to see that

$$\mathcal{G} \subset \overline{conv}(\mathcal{K}), \tag{1}$$

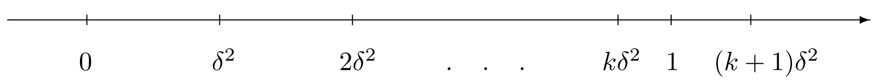
where $\mathcal{K} = \{\mathbf{1}_{[0,y)} : y \in [0, \infty)\}$. It is a consequence of the fact that $conv(\mathcal{K})$ consists of functions $\sum_{i=1}^{n} w_i \mathbf{1}_{[y_{i-1},y_i)}$, where $0 = y_0 < \ldots < y_n < \infty$, $1 \geq w_1 > \ldots > w_n > 0$ and $n \in \mathbb{N}$, and that any function $g \in \mathcal{G}$ can be approximated by a sequence of functions from $conv(\mathcal{K})$.

Inclusion (1) implies that $H(\delta, \mathcal{G}, Q) \leq H(\delta, \overline{conv}(\mathcal{K}), Q)$. Therefore, by the Ball and Pajor Theorem (see, e.g., [4]), it suffices to show that there exists a constant C_1 such that for each probabilistic measure Q

$$N(\delta, \mathcal{K}, Q) \leq C_1 \delta^{-2}.$$

Note that $\mathcal{G}$ is a subset of the ball of radius 1 centered at zero. Hence, for $\delta \geq 1$ the entropy equals 0 and the statement of the lemma holds. Therefore, it is enough to consider $\delta \in (0, 1)$.

If Q has no atoms, i.e., $Q[0, x)$ is a continuous function of x, the δ-covering may be constructed as follows. Take $0 < \delta < 1$ and divide the interval $(0, 1)$ as in the following figure,



where $k\delta^2$ is the maximal multiplicity of δ^2, which is less than 1.

Therefore,

$$k = \begin{cases} \left\lfloor \dfrac{1}{\delta^2} \right\rfloor & \text{for} \quad \left\lfloor \dfrac{1}{\delta^2} \right\rfloor \neq \dfrac{1}{\delta^2}, \\ \left\lfloor \dfrac{1}{\delta^2} \right\rfloor - 1 & \text{for} \quad \left\lfloor \dfrac{1}{\delta^2} \right\rfloor = \dfrac{1}{\delta^2}, \end{cases}$$

where $\lfloor \cdot \rfloor$ is the floor function. Then we select a set of $k + 2$ points and a set of k functions in the following way

$$\begin{aligned} &x_0 = 0, \\ &x_1 : \ Q[0, x_1) = \delta^2, \quad f_1(x) := \mathbf{1}_{[0,x_1)}(x), \\ &\vdots \\ &x_k : \ Q[0, x_k) = k\delta^2, \quad f_k(x) := \mathbf{1}_{[0,x_k)}(x), \\ &x_{k+1} = \infty. \end{aligned}$$

Obviously, for $n = 0, \ldots, k$, there is $Q[x_n, x_{n+1}) \leq \delta^2$. Take any $y \in [0, \infty)$. Then, for some $n \in \{0, \ldots, k\}$, there is $y \in [x_n, x_{n+1})$, and

$$\left\|\mathbf{1}_{[0,y)} - f_n\right\|^2_{L_2(Q)} = \int \mathbf{1}^2_{[x_n,y)} dQ = Q[x_n, y) \leq Q[x_n, x_{n+1}) \leq \delta^2.$$

In other words, the $L_2(Q)$-balls of radius δ, centered at $f_1, \ldots, f_k$ cover the class $\mathcal{K}$, therefore

$$N(\delta, \mathcal{K}, Q) \leq k \leq \delta^{-2}.$$

Now let us consider the general case, when Q is any probabilistic measure. For an arbitrarily chosen δ, we construct, as previously, the sequence of centers, but if for some n there exists no such x that $Q[0, x) = n\delta^2$, then instead of x_n, we take x such, that $Q[0, x) < n\delta^2 < Q[0, x]$. For the chosen points $x_1, \ldots, x_l$, there is $l \leq k$ and $Q(x_n, x_{n+1}) \leq \delta^2$. Let us take $y \in [0, \infty)$. If for some $n \in \{1, \ldots, l\}$ $y = x_n$, then $\left\|\mathbf{1}_{[0,y)} - \mathbf{1}_{[0,x_n)}\right\|^2_{L_2(Q)} = 0$. Otherwise, if $y \in (x_n, x_{n+1})$ for some n, then

$$\left\|\mathbf{1}_{[0,y)} - \mathbf{1}_{[0,x_{n+1})}\right\|^2_{L_2(Q)} = \int \mathbf{1}_{[y,x_{n+1})} dQ \leq Q(x_n, x_{n+1}) \leq \delta^2,$$

and, since $l \leq k$, there is $k \leq \delta^2$. □

2. CONVOLUTION MODEL WITH A MONOTONIC KERNEL

Let Y and Z be independent random variables on $[0, 1]$. Suppose that Z has a given density k_0 with respect to the Lebesgue measure. The distribution θ of Y is unknown. We observe independent copies $X_1, \ldots, X_n$ of $X = Z + Y$. Therefore,

$$f_0 \in \mathcal{F} = \left\{ \int_0^1 k_0(\cdot - y) d\theta(y) \colon \theta \in \Theta \right\},$$

where Θ is the class of all probabilistic measures on $[0, 1]$. If we put $\mathcal{K} = \{k_0(\cdot - y) \colon y \in [0, 1]\}$, then $\mathcal{F} = \overline{conv}(\mathcal{K})$ (see [3]). In this section, the special case of a monotonic kernel $k_0(x) = 2x\mathbf{1}\{0 \leq x \leq 1\}$ will be handled. As in [3], in order to simplify the analysis of the shape of f_0, we assume that θ is the uniform distribution (a more general case, when θ has a density bounded away from zero and infinity gives similar results).

We want to apply Theorem 1, so we need to calculate the covering number of $\tilde{\mathcal{K}}_n = \{(k/f_0)\mathbf{1}\{f_0 > \sigma_n\} \colon k \in \mathcal{K}\}$. With θ being the uniform distribution, one obtains

$$f_0(x) = \begin{cases} x^2 & \text{for } \ 0 \leq x \leq 1; \\ x(2 - x) & \text{for } \ 1 \leq x \leq 2; \\ 0 & \text{otherwise} \end{cases}$$

and it is convenient to obtain the covering numbers in two steps.

Define

$$\tilde{\mathcal{K}}_n^{(1)} = \left\{ \tilde{k}\mathbf{1}_{[0,1]}\colon\ \tilde{k} \in \tilde{\mathcal{K}}_n \right\}, \qquad \tilde{\mathcal{K}}_n^{(2)} = \left\{ \tilde{k}\mathbf{1}_{[1,2]}\colon\ \tilde{k} \in \tilde{\mathcal{K}}_n \right\}.$$

Lemma 2 (see [3]**).** *There exists a constant A_1 such that*

$$N(\delta, \tilde{\mathcal{K}}_n^{(1)}, P_n) \le A_1 \delta^{-1}, \qquad \text{for all } \ \delta \in (0,1) \ a.s.,$$

for each n sufficiently large.

In order to calculate the δ-covering number for the class $\tilde{\mathcal{K}}_n^{(2)}$, let us deal with the class $\mathcal{K}$ first. It is asserted in [3] that there exists a constant C such that for any probabilistic measure on $[0,2]$ there is $N(\delta, \mathcal{K}, Q) \le C\delta^{-1}$. The suggested line of the proof is, however, incorrect (it is asserted that such an inequality holds true for the δ-covering number in the supremum norm. However, it cannot be true, because for any $k_1 \neq k_2 \in \mathcal{K}$, there is $\|k_1 - k_2\|_\infty = 2$ and, hence, for $\delta < 1$, there follows $N_\infty(\delta, \mathcal{K}) = \infty$).

The following lemma gives a corrected upper bound for the covering number.

Lemma 3. *There exists a constant A_0 such that for any probabilistic measure Q on $[0,2]$,*

$$N(\delta, \mathcal{K}, Q) \le A_0 \delta^{-2}, \qquad \text{for all } \ \delta \in (0,1). \tag{2}$$

Proof. Take $\delta \in (0,1)$ and define $\tilde{k}_0(x) := 2x\mathbf{1}\{0 \le x < 1\}$. Let y_i, $i = 1, \dots, N$, be points chosen in such a way that $Q(1 + y_{i-1}, 1 + y_i) \le \delta^2$, $i = 2, \dots, N$ (the proof of Lemma 1 implies that $N < 1/\delta^2$). Moreover, let $y_{N+k} := k\delta$, for $k = 1, \dots, \lfloor 1/\delta \rfloor$, $y_0 := 0$, and $y_{N+\lfloor 1/\delta \rfloor + 1} := 1$. For simplicity, we assume that the points y_i are arranged increasingly. Obviously, for $i = 1, \dots, N + \lfloor 1/\delta \rfloor + 1$,

$$y_i - y_{i-1} \le \delta \quad \text{and} \quad Q(1 + y_{i-1}, 1 + y_i) \le \delta^2. \tag{3}$$

As the centers of the balls for the δ-covering of the class $\mathcal{K}$, we take $\tilde{k}_0(\cdot - y_i)$ and $k_0(\cdot - y_i)$, for $i = 0, \dots, N + \lfloor 1/\delta \rfloor + 1$. Since

$$2\left(N + \left\lfloor \frac{1}{\delta} \right\rfloor + 2\right) \le \frac{8}{\delta^2} \quad \text{for } \delta \in (0,1), \tag{4}$$

it suffices to show that the balls cover $\mathcal{K}$. Take $y \in [0,1]$ such that $y \neq y_i$ for all i (otherwise, $k_0(\cdot - y)$ is one of the chosen centers). Since $y \in (y_{i-1}, y_i)$ for some $i \in \{1, \dots, N + \lfloor 1/\delta \rfloor + 1\}$, there is

$$\begin{aligned}
&\int\limits_{[0,2]} \left[k_0(x-y) - \tilde{k}_0(x-y_i)\right]^2 dQ(x) \le \\
&\qquad \le \int\limits_{[y,1+y]} 4(y_i - y_{i-1})^2 dQ(x) + \int\limits_{(1+y,1+y_i)} 4 dQ(x) \le \\
&\qquad\qquad \le 4(y_i - y_{i-1})^2 + 4Q(1 + y_{i-1}, 1 + y_i) \le 8\delta^2,
\end{aligned}$$

for this i, because of (3). In view of inequality (4), it follows that $N(\sqrt{8}\delta, \mathcal{K}, Q) \le 8\delta^{-2}$ for $\delta \in (0,1)$. Hence, $N(\delta, \mathcal{K}, Q) \le 64\delta^{-2}$ for $\delta \in (0,1)$. $\square$

Note that (2) holds true for all finite (not necessarily probabilistic) measures and apply Lemma 3 with $dQ = ((1/f_0^2)\mathbf{1}\{f_0 > \sigma_n\}\mathbf{1}_{[1,2]}dP_n)/A^2\rho_n^2$, to obtain

$$N(\delta, \tilde{\mathcal{K}}_n^{(2)}, P_n) \leq A^2 A_0 \left(\frac{\rho_n}{\delta}\right)^2, \qquad \text{for all} \quad \delta \in (0,1), \tag{5}$$

on the set

$$\left\{ \int\limits_{f_0 > \sigma_n} \frac{1}{f_0^2}\mathbf{1}_{[1,2]}dP_n \leq A^2\rho_n^2 \right\}.$$

So, for

$$\int\limits_{f_0 > \sigma_n} \frac{1}{f_0}\mathbf{1}_{[1,2]}dx \leq \rho_n^2, \tag{6}$$

there is

$$\limsup_{n\to\infty} P\left(\sup_{0<\delta<1} \left(\frac{\delta}{\rho_n}\right)^2 N(\delta, \tilde{\mathcal{K}}_n^{(2)}, P_n) > A_0 A^2 \right) \leq$$
$$\leq \limsup_{n\to\infty} P\left(\int \frac{1}{f_0^2\rho_n^2}\mathbf{1}\{f_0 > \sigma_n\}\mathbf{1}_{[1,2]}dP_n > A^2 \right) \longrightarrow 0, \quad \text{as} \quad A \to \infty.$$

Because of Lemma 2, if (6) holds, we can write

$$\lim_{A\to\infty} \limsup_{n\to\infty} P\left(\sup_{0<\delta<1} \left(\frac{\delta}{\rho_n}\right)^2 N(\delta, \tilde{\mathcal{K}}_n^{(i)}, P_n) > A \right) = 0,$$

for $i = 1, 2$.

Some effort is needed to see that the above remains true for the whole class $\mathcal{K}$. To this end, it will be shown that

$$N(\delta, \tilde{\mathcal{K}}_n, Q) \leq N(\delta, \tilde{\mathcal{K}}_n^{(1)}, Q) + N(\delta, \tilde{\mathcal{K}}_n^{(2)}, Q), \qquad \text{for all} \quad \delta \in (0,1). \tag{7}$$

Notice that the functions from $\tilde{\mathcal{K}}_n$ are continuous at $x = 1$ and can be obtained as 'junctions' of the functions from $\tilde{\mathcal{K}}_n^{(1)}$ and $\tilde{\mathcal{K}}_n^{(2)}$. It is not hard to verify that the balls covering the classes $\tilde{\mathcal{K}}_n^{(1)}$ and $\tilde{\mathcal{K}}_n^{(2)}$ can be represented in $\mathbb{R}^2$ as sets, bounded by two functions from the corresponding class.

Therefore, if we construct the centers of the balls for the covering of $\tilde{\mathcal{K}}_n$ as 'junctions' of the centers of the balls from the coverings of $\tilde{\mathcal{K}}_n^{(1)}$ and $\tilde{\mathcal{K}}_n^{(2)}$, it is sufficient to choose those pairs of centers only for which the representations of the corresponding balls do touch each other at $x = 1$. The number of such pairs is less then the sum of the numbers of balls covering the sets $\tilde{\mathcal{K}}_n^{(1)}$ and $\tilde{\mathcal{K}}_n^{(2)}$, so that (7) holds true (see Fig. 1).

From that, for the whole class $\tilde{\mathcal{K}}_n$, we obtain

$$\lim_{A\to\infty} \limsup_{n\to\infty} P\left(\sup_{0<\delta<1} \left(\frac{\delta}{\rho_n}\right)^2 N(\delta, \tilde{\mathcal{K}}_n, P_n) > A \right) = 0.$$

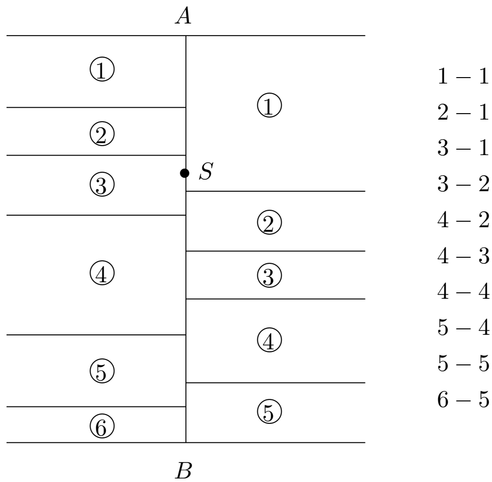


Fig. 1. Schematic representation of the balls covering $\tilde{\mathcal{K}}_n^{(1)}$ (to the left of AB, which corresponds to $x = 1$) and $\tilde{\mathcal{K}}_n^{(2)}$ (to the right of AB). In order to construct a center of the ball for the covering of $\tilde{\mathcal{K}}_n$, two centers are joined to form a (not necessarily continuous) function on $[0, 2]$: one from the covering of $\tilde{\mathcal{K}}_n^{(1)}$ and one from the covering of $\tilde{\mathcal{K}}_n^{(2)}$. For example, a (continuous) function from $\tilde{\mathcal{K}}_n$ that crosses the $x = 1$ line at the point S would belong to the ball centered at the junction $3-1$. On the right, the list of junctions sufficent to form a covering of $\tilde{\mathcal{K}}_n$ in this particular configuration. Obviously, the covering number of $\tilde{\mathcal{K}}_n$ is not greater than $N(\delta, \tilde{\mathcal{K}}_n^{(1)}, Q) + N(\delta, \tilde{\mathcal{K}}_n^{(2)}, Q) - 1$

The envelope function of the class $\mathcal{K}$ takes the form

$$K(x) = 2x\mathbf{1}_{[0,1)}(x) + 2\mathbf{1}_{[1,2]}(x).$$

Hence, using the specific form of f_0,

$$\int_{f_0 > \sigma_n} \frac{K^2}{f_0} dx = 4 - 4\sqrt{\sigma_n} + 2\log\left|\frac{\sqrt{1-\sigma_n}+1}{\sqrt{1-\sigma_n}-1}\right| \asymp \log\frac{1}{\sigma_n}$$

and

$$\int_{f_0 \le \sigma_n} f_0 dx = \frac{1}{3}\sigma_n^{3/2} + \frac{2}{3} - \sqrt{1-\sigma_n} + \frac{1}{3}(1-\sigma_n)^{3/2} = \frac{1}{3}\sigma_n^{3/2} + o\left(\sigma_n^{3/2}\right).$$

Because

$$\int_{f_0 > \sigma_n} \frac{1}{f_0}\mathbf{1}_{[1,2]} dx = \frac{1}{2}\log\left|\frac{\sqrt{1-\sigma_n}+1}{\sqrt{1-\sigma_n}-1}\right| \asymp \log\frac{1}{\sigma_n},$$

in order to satisfy condition (6) and the assumptions of Theorem 1, we need to hold

$$\rho_n^2 \ge A\log\frac{1}{\sigma_n}, \quad \tau_n^2 \ge B\sigma_n^{3/2}, \quad \text{and} \quad \tau_n \ge Cn^{-1/3}\rho_n^{1/3},$$

with suitably chosen constants. So, with the optimal $\sigma_n \asymp n^{-4/9}$, we arrive at the rate

$$h(\hat{f}_n, f_0) = O_P\left(n^{-1/3}(\log n)^{1/6}\right).$$

Note that the rate asserted in [3] was $O_P(n^{-3/8}(\log n)^{1/8})$, but that result does not seem to be correct, because of the faulty proof of Lemma 3 in [3].

3. CONVOLUTION MODEL WITH A STRICTLY CONVEX KERNEL

Let us now consider the convolution model with a strictly convex kernel

$$k_0(x) = [3 - 12x(1-x)]\,\mathbf{1}_{[0,1]}(x),$$

which was studied in [2]. Again, the rate $O_P(n^{-3/8}(\log n)^{1/8})$, asserted in [2], does not seem to be correct, because the δ-covering number for the class $\mathcal{K}$ cannot be of the order δ^{-1} (k_0 is discontinuous at 0 and 1).

For $y_1 < y_2$, one has

$$\int (k_0(\cdot - y_2) - k_0(\cdot - y_1))^2\, dQ =$$

$$= \int\limits_{[y_2, 1+y_1]} (k_0(\cdot - y_2) - k_0(\cdot - y_1))^2\, dQ + \int\limits_{[y_1, y_2)} k_0^2(\cdot - y_1) dQ + \int\limits_{(1+y_1, 1+y_2]} k_0^2(\cdot - y_2) dQ \leq$$

$$\leq [36(y_2 - y_1)]^2 + 9Q[y_1, y_2) + 9Q(1+y_1, 1+y_2].$$

Hence, reasoning as in the proof of Lemma 3, one can easily see that, for some constant A and for any probabilistic measure Q on $[0,2]$,

$$N(\delta, \mathcal{K}, Q) \leq A\delta^{-2} \quad \text{for all } \delta \in (0,1). \tag{8}$$

Let us assume that θ has a density g_0 with respect to the Lebesgue measure, and that, for some constant $c_1 > 0$,

$$\frac{1}{c_1} \leq g_0(y) \leq c_1, \quad \text{for all } y \in [0,1]. \tag{9}$$

Then,

$$\int\limits_{f_0 > \sigma_n} \frac{K^2(x)}{f_0(x)} dx = 9 \int\limits_{f_0 > \sigma_n} \frac{1}{f_0(x)} dx \asymp c_2 \log\left(\frac{1}{\sigma_n}\right), \tag{10}$$

and

$$\int\limits_{f_0 \leq \sigma_n} f_0(x) dx \geq c_3 \sigma_n^2, \tag{11}$$

for some suitable, strictly positive constants c_2 and c_3 depending on c_1.

Using (8) with $dQ = dP_n(1/f_0^2)\mathbf{1}\{f_0 > \sigma_n\}/(C\rho_n^2)$, one obtains

$$N(\delta, \tilde{\mathcal{K}}_n, P_n) \le AC\left(\frac{\rho_n}{\delta}\right)^2 \qquad \text{for all} \quad \delta \in (0,1),$$

on the set

$$\left\{ \int\limits_{f_0>\sigma_n} \frac{1}{f_0^2} dP_n \le C\rho_n^2 \right\}.$$

So, for

$$\int\limits_{f_0>\sigma_n} \frac{1}{f_0} dx \le \rho_n^2, \tag{12}$$

there is

$$\lim_{C\to\infty} \limsup_{n\to\infty} P\left(\sup_{0<\delta<1} \left(\frac{\delta}{\rho_n}\right)^2 N(\delta, \tilde{\mathcal{K}}_n, P_n) > C \right) = 0.$$

In view of (10), (11) and (12), the following inequalities must hold, if we want to apply Theorem 1

$$\rho_n^2 \ge c_4 \log\frac{1}{\sigma_n}, \quad \tau_n^2 \ge c_3\sigma_n^2, \quad \text{and} \quad \tau_n \ge n^{-1/3}\rho_n^{1/3}.$$

Hence, again, we arrive at the rate

$$h(\hat{f}_n, f_0) = O_P\left(n^{-1/3}(\log n)^{1/6}\right),$$

this time with the optimal $\sigma_n \asymp n^{-1/3}$.

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