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# WELL-POSEDNESS AND STABILITY ANALYSIS OF HYBRID FEEDBACK SYSTEMS USING SHKALIKOV'S THEORY

**Abstract.** The modern method of analysis of the distributed parameter systems relies on the transformation of the dynamical model to an abstract differential equation on an appropriately chosen Banach or, if possible, Hilbert space. A linear dynamical model in the form of a first order abstract differential equation is considered to be well-posed if its right-hand side generates a strongly continuous semigroup. Similarly, a dynamical model in the form of a second order abstract differential equation is well-posed if its right-hand side generates a strongly continuous cosine family of operators.

Unfortunately, the presence of a feedback leads to serious complications or even excludes a direct verification of assumptions of the Hille–Phillips–Yosida and/or the Sova–Fattorini Theorems. The class of operators which are similar to a normal discrete operator on a Hilbert space describes a wide variety of linear operators. In the papers [12, 13] two groups of similarity criteria for a given hybrid closed-loop system operator are given. The criteria of the first group are based on some perturbation results, and of the second, on the application of Shkalikov's theory of the Sturm–Liouville eigenproblems with a spectral parameter in the boundary conditions. In the present paper we continue those investigations showing certain advanced applications of the Shkalikov's theory. The results are illustrated by feedback control systems examples governed by wave and beam equations with increasing degree of complexity of the boundary conditions.

**Keywords:** infinite-dimensional control systems, semigroups, spectral methods, Riesz bases.

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## 1. INTRODUCTION

The mathematical models of systems involving physical phenomena such as diffusion, wave propagation as well as information and transport delays engage the partial and/or functional differential equations and integral operators. Particular examples can be found in the mathematical description of diffusion of heat, electric charges, molecules participating in chemical reactions, genetic characters, pathogenic viruses, oscillations of overhead high-voltage transmission lines, lifting ropes, antenna masts, deformations of shafts, beams and mechanical constructions, oscillations of robot elastic arms, propagation of electromagnetic waves in transmission lines, wave-guides, oscillations of quantum generators, etc.

Such systems are called distributed parameter systems, as opposed to lumped parameter systems described by ordinary differential equations.

Feedback is an essential feature of many distributed parameter systems in automatic control, electronics (nonlinear oscillation generators), chemistry (reactors with recycles), mechanical engineering (stabilizers and dampers of mechanical construction) and must be taken into account in the analysis.

The modern method of analysis of the distributed parameter systems relies on the transformation of the dynamical model to an abstract differential equation on an appropriately chosen Banach or, if possible, Hilbert space.

The first order abstract differential equation has the form

$$\dot{u}(t) = Au(t), \qquad u(0) = u_0 \in \mathrm{H} \tag{1.1}$$

where H denotes a real Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ and $A : (D(A) \subset \mathrm{H}) \longrightarrow \mathrm{H}$ is an unbounded linear operator.

The family $\{T(t)\}_{t\geq 0} \subset \mathbf{L}(\mathrm{H})$ will be called a $\mathrm{C}_0$-*semigroup* if $T(0) = I$, $T(s+t) = T(t)T(s)$ for all $t, s \geq 0$ and $T(t)u \longrightarrow u$ as $t \to 0+$ for all $u \in \mathrm{H}$. If additionally, the mapping $t \longrightarrow T(t)u$ is an analytic function on $(0, \infty)$ for any fixed $u \in \mathrm{H}$, then we say that $A$ generates an *analytic semigroup* on H. If both $A$ and $-A$ generate $\mathrm{C}_0$-semigroups then we say that $A$ generates a $\mathrm{C}_0$-*group* on H.

The following conditions are equivalent:

(i) $A$ is a linear, closed densely defined operator and for any $u_0 \in D(A)$, $T > 0$ there exists a unique classical solution $u \in \mathrm{C}^1([0,T],\mathrm{H}) \cap \mathrm{C}([0,T], D_A)$ of problem (1.1), where $D_A$ denotes the Banach space $D(A)$ equipped with the norm $\|u\|_A = \|u\| + \|Au\|$;

(ii) $A$ generates a $\mathrm{C}_0$–semigroup $\{T(t)\}_{t\geq 0}$ on H.

If the above conditions are satisfied then the function $u(t) = T(t)u_0$, where $u_0 \in \mathrm{H}$, is called a *weak solution* of (1.1).

The second order abstract differential equation has the form

$$\ddot{u}(t) = Au(t), \qquad u(0) = u_0 \in \mathrm{H}, \qquad \dot{u}(0) = u_1 \in \mathrm{H} \tag{1.2}$$

where H denotes a real Hilbert space with scalar product $\langle \cdot, \cdot \rangle$, and $A\colon (D(A) \subset \mathrm{H}) \longrightarrow \mathrm{H}$, is generally, an unbounded linear operator.

The family $\{C(t)\}_{t\in\mathbb{R}} \subset \mathbf{L}(\mathrm{H})$, such that $C(0) = I$, $C(s+t)+C(t-s) = 2C(t)C(s)$ for all $t, s \in \mathbb{R}$ and the function $\mathbb{R} \ni t \longmapsto C(t)u$ is continuous for any fixed $u \in \mathrm{H}$ is called a *strongly continuous cosine family of operators* on H.

The following conditions are equivalent:

(i) $A$ is a linear, closed densely defined operator and for any $u_0 \in D(A)$, $T > 0$ there exists a unique classical solution $u \in \mathrm{C}^2([0,T],\mathrm{H}) \cap \mathrm{C}([0,T],D_A)$ of the problem (1.2).

(ii) $A$ generates a strongly continuous cosine family $\{C(t)\}_{t\in\mathbb{R}}$ on H.

If the above conditions are satisfied then the function

$$u(t) = \int_0^t C(s)u_1 ds + C(t)u_0,$$

where $u_0$, $u_1 \in \mathrm{H}$, is called a *weak solution* of (1.2).

The concept of semigroup is a formal extension of the definition of the exponential scalar function $\mathbb{C} \ni \lambda \longmapsto e^{t\lambda}$ $(t \geq 0)$, to an argument being an unbounded linear operator $A$, while the strongly continuous cosine family of operators is a similar extension of the scalar entire function $\mathbb{C} \in \lambda \longmapsto \cosh(t\sqrt{\lambda})$ $(t \in \mathbb{R})$. This justifies the notation $T(t) = e^{tA}$, $t \geq 0$ and $C(t) = \cosh tA^{1/2}$, $t \in \mathbb{R}$. The fundamental results of the semigroup theory as the Hille–Phillips–Yosida theorem – see [28, Corollary 3.8, p. 12] and the Sova–Fattorini [8, Theorem 5.1, p. 37] theorem determine those classes of linear unbounded operators on a general Banach space for which such extensions are possible. To verify the assumptions of the above theorems one should estimate the norm $\|(\lambda I - A)^{-n}\|$ of the $n$-th power of the resolvent of $A$ on appropriate subsets of $\mathbb{C}$ (observe that for the semigroup generator $A$,

$$(\lambda I - A)^{-1}u_0 = \int_0^\infty e^{-t\lambda}T(t)u_0\, dt$$

is the Laplace transform of a weak solution of (1.1). This is a difficult task especially for an operator $A$ describing a feedback system with boundary control and/or boundary observation.

Let $A : (D(A) \subset \mathrm{H}) \longrightarrow \mathrm{H}$ be a closed, densely defined linear operator on a Hilbert space H.

$$D(A^*) := \{v \in \mathrm{H} :\ \exists(!)\ h_v \in \mathrm{H} :\ \langle Au, v\rangle = \langle u, h_v\rangle \qquad \forall u \in D(A)\}$$

is the domain of the *adjoint operator* $A^* :\ (D(A^*) \subset \mathrm{H}) \longrightarrow \mathrm{H}$ with respect to $A$, defined as $A^*v := h_v$, $v \in D(A^*)$. $A$ is called *normal* if

$$D(A) = D(A^*), \qquad AA^* = A^*A.$$

It follows from the spectral theorem for normal operators [37, Theorem 7.32, p. 215] that:

(i) The resolvent of $A$ satisfies an estimate

$$\left\|(\lambda I - A)^{-n}\right\| \leq [\mathrm{dist}(\lambda, \sigma(A))]^{-n}, \tag{1.3}$$

where $\lambda \in \mathbb{C} \setminus \sigma(A)$, $n \in \mathbb{N}$ and $\sigma(A)$ denotes the spectrum of $A$.

(ii) For any Borel, function $f$ bounded on $\sigma(A)$, the formula

$$\langle f(A)u, v\rangle = \int\limits_{\sigma(A)} f(\lambda) d\,\langle E(\lambda)u, v\rangle \qquad \forall u, v \in \mathrm{H} \tag{1.4}$$

determines an operator $f(A) \in \mathbf{L}(\mathrm{H})$. Here $E(\lambda)$ is the unique (by the spectral theorem) spectral resolution of identity. If, additionally, $A$ has a compact resolvent ($A$ is a *discrete operator*) then (1.4) takes an equivalent form

$$f(A)u = \sum_{i=1}^{\infty} f(\lambda_i)\,\langle u, e_i\rangle\, e_i,$$

where $\{e_i\}_{i=1}^{\infty}$ is the orthonormal system of eigenvectors of $A$, corresponding to the eigenvalues of $A$ denoted by $\{\lambda_i\}_{i=1}^{\infty}$, $Ae_i = \lambda_i e_i$.

The result (i) requires an explanation. If $\lambda \in \mathbb{C}\setminus\sigma(A)$ then applying the result from [37, Theorem 7.34(b), p. 217] we get

$$\left\|(\lambda I - A)^{-1}\right\| = [\mathrm{dist}(\lambda, \sigma(A))]^{-1}.$$

Moreover, from [37, Corollary, p. 126] we know that the resolvent $(\lambda I - A)^{-1}$ is also normal. Hence

$$\left\|(\lambda I - A)^{-n}\right\| = \left\|(\lambda I - A)^{-1}\right\|^{n} = [\mathrm{dist}(\lambda, \sigma(A))]^{-n}$$

– see [37, Theorem 5.44, p. 127] or [14, Problem 162]. The results (ii) are known as the *functional calculus* for normal operators.

A closed, densely defined linear operator $A : (D(A) \subset \mathrm{H} \longrightarrow \mathrm{H}$ is *similar* to a normal operator $N$, if there exists an isomorphism $S \in \mathbf{L}(\mathrm{H})$ such that $S^{-1}AS = N$. The similarity relation does not change the spectrum of operators.

Putting: $f(\lambda) = e^{t\lambda}$ (for semigroup $t \geq 0$ and $t \in \mathbb{R}$ for group), $f(\lambda) = (\mu - \lambda)^{-1}$ (for an analytic semigroup, $\mu \in S_{b,\theta}$) and $f(\lambda) = \cosh(t\sqrt{\lambda})$ (for a strongly cosine family of operators, $t \in \mathbb{R}$), in (ii) we obtain, respectively statements (a), (b), (c) and (d) of the following theorem.

**Theorem 1.1.** *If $A$ is similar to a normal operator then:*

(a) *$A$ generates a $\mathrm{C}_0$-semigroup iff* $\sup\limits_{\lambda\in\sigma(A)} \mathrm{Re}\,\lambda < \infty$,

(b) *$A$ generates a $\mathrm{C}_0$-group iff* $-\infty < \inf\limits_{\lambda\in\sigma(A)} \mathrm{Re}\,\lambda \leq \sup\limits_{\lambda\in\sigma(A)} \mathrm{Re}\,\lambda < \infty$,

(c) *$A$ generates an analytic semigroup iff there exist $b \in \mathbb{R}$ and $\theta \in \left(\frac{\pi}{2}, \pi\right)$ such that*

$$S_{b,\theta} = \{\lambda \in \mathbb{C} : \ |\arg(\lambda - b)| \leq \theta, \quad \lambda \neq b\} \subset \mathbb{C}\setminus\sigma(A),$$

(d) *A generates a strongly cosine family of operators iff there exists* $\omega \in \mathbb{R}$ *such that*

$$\sigma(A) \subset \left\{ \lambda \in \mathbb{C} : \ \operatorname{Re} \lambda \leq \omega^2 - \frac{1}{\omega^2} \operatorname{Im}^2 \lambda \right\}.$$

Frequently, in the analysis of finite-dimensional dynamics it is enough to consider the state matrices of simple structure (matrices with linear divisors exclusively or, equivalently, with a diagonal Jordan form). The class of such matrices is identical with the class of matrices which are similar to normal ones.

Nagy pointed out (see [8]) that if $A$ generates a uniformly bounded $\mathrm{C}_0$-group (i.e. there exists $M \geq 0$ such that $\|T(t)\| \leq M$ for all $t \in \mathbb{R}$) then $jA$ ($j = \sqrt{-1}$) is similar to a self-adjoint operator and a similarity isomorphism $S$ can be found in the class of self-adjoint, positive definite operators. This is a partial inverse of the claim (b).

Recall that a system $\{f_i\}_{i \in I}$ is a *Riesz basis* in a Hilbert space H if there exists a linear, bounded operator $S$ mapping H onto itself and an orthonormal basis $\{e_i\}_{i \in I}$ of H such that $f_i = Se_i$ for all $i \in I$. An operator $A : (D(A) \subset \mathrm{H}) \longrightarrow \mathrm{H}$ with a compact resolvent is similar to a normal operator iff $A$ possesses a system of eigenvectors forming a Riesz basis of H. This follows immediately from the fact that an operator with a compact resolvent is normal iff it possesses a system of eigenvectors forming an orthonormal basis of H. For the proof of necessity see [18, pp. 260–263 and pp. 276–277], [32, pp. 250–255] or, less explicitly [37, Theorem 7.2, p. 167]. Sufficiency can be deduced from [37, Theorem 7.2, p. 167].

**Remark 1.1.** *There are operators which are not similar to normal ones but still satisfy an estimate analogous to* (1.3). *This is the case for hyponormal operators. Recall that a densely defined operator* $A : \ (D(A) \subset \mathrm{H}) \longrightarrow \mathrm{H}$ *is called* hyponormal *if*

$$D(A) \subset D(A^*), \qquad \|Au\| \geq \|A^*u\| \qquad \forall u \in D(A).$$

*As an example of a hyponormal operator one may take the generator of a right--shift semigroup on* $L^2(0, \infty)$. *In* [17] *this observation is employed to show that the statements of Theorem 1.1 remain true for hyponormal operators. Let us recall, however, that for operators with a compact resolvent the notions of normality and hyponormality are equivalent.*

A very important feature of the spectral approach to the problem of well-posedness of systems (1.1) and (1.2) is the possibility of collecting essential information by the examination of the spectral properties of $A$, which makes considerations simpler than with other analytical tools. This enables one to investigate a wide class of infinite-dimensional systems by elementary methods available also for engineers. As an example we shall consider the stability problem of the system (1.1).

The most commonly used concepts of asymptotic stability of the system (1.1) are:

— *weak asymptotic stability*, ($T(t) \xrightarrow{w} 0$ as $t \to \infty$),

— *strong asymptotic stability*, ($T(t) \xrightarrow{s} 0$ as $t \to \infty$),

— *uniform asymptotic stability* ($T(t) \longrightarrow 0$ as $t \to \infty$).

The latter is equivalent to, *exponential stability* (**EXS**), which holds if there exist $M \geq 1$ and $\alpha > 0$ such that $\|T(t)\| \leq Me^{-\alpha t}$ for $t \geq 0$.

**EXS** implies strong asymptotic stability while strong asymptotic stability implies weak asymptotic stability. For *eventually compact* semigroups (i.e. there exists $t_0 > 0$ such that $T(t)$ is a compact operator on H for all $t \geq t_0$) all the above concepts are equivalent. In particular, this is the case if dim H $< \infty$. For the semigroup whose infinitesimal generator has a compact resolvent the strong and weak asymptotic stability are equivalent.

To derive practically checkable criteria of **EXS**, it is of great importance to characterize the notion of **EXS** in terms of the spectrum of semigroup generator. Prüss [29], Huang [15] and Weiss [38] have proved that the following conditions are equivalent:

(i) **EXS**,

(ii) $\lambda \longmapsto (\lambda I - A)^{-1}$ is an analytic function on the open right complex halfplane and bounded on the closed right complex halfplane,

(iii) $\lambda \longmapsto (\lambda I - A)^{-1}$ is a bounded function on $j\,\mathbb{R}$ and $\sigma(A)$ lies in the open left complex halfplane.

Only an incomplete spectral characterization of the notion of strong asymptotic stability is known. The next theorem follows from the functional calculus for normal operators and the diagram obtained in [11, p. 88].

**Theorem 1.2.** *Let $A$ be an operator which is similar to a normal one. Then:*

(i) *$A$ generates a uniformly bounded semigroup iff* $\sup_{\lambda \in \sigma(A)} \operatorname{Re} \lambda \leq 0$*,*

(ii) *$A$ generates an **EXS** semigroup iff* $\sup_{\lambda \in \sigma(A)} \operatorname{Re} \lambda < 0$*,*

(iii) *Under the additional assumption that $A$ has a compact resolvent we have: $A$ generates a strongly asymptotically semigroup iff $\sigma(A)$ is contained in the left open complex half-plane.*

**Remark 1.2.** *The last statement appears also in* [16, Corollary 2.5/(i), p. 319].

**Remark 1.3.** *Levan* [21] *has proved that if $A$ is normal then $A$ is* strictly dissipative *(i.e.* $\operatorname{Re} \langle Af, f \rangle < 0$ *for all $f \in D(A)$, $f \neq 0$) iff the semigroup generated by $A$ is strongly asymptotically stable. However, his results are not explicitly expressed by the spectrum of $A$.*

## 2. HYBRID FEEDBACK OPERATORS

Let us consider a feedback control system consisting of a distributed parameter plant with boundary observation and boundary control worked out by a finite-dimensional controller (e.g. conventional controller), depicted in Figure 1. Here $\mathrm{P} \in \mathbf{L}(\mathbb{R}^n)$, $\mathrm{Q} \in \mathbf{L}(\mathbb{R}^r, \mathbb{R}^n)$, $\mathrm{R} \in \mathbf{L}(\mathbb{R}^m, \mathbb{R}^n)$, $\mathrm{D} \in \mathbf{L}(\mathbb{R}^r, \mathbb{R}^m)$, H is a Hilbert space; $L : (D(L) \subset \mathrm{H}) \longrightarrow \mathrm{H}$, is a linear closed operator with domain $D(L) \subset D(\Gamma_0)$, $D(L) \subset D(\Gamma_1)$ where $\Gamma_0$, $\Gamma_1$ are some boundary operators, e.g. Dirichlet or Neumann trace operators. The closed-loop system is naturally described on the space $\mathrm{X} = \mathbb{C}^n \oplus \mathrm{H}$ by a *hybrid linear operator*

$$\left\{ \begin{array}{ll} A \begin{bmatrix} v \\ u \end{bmatrix} & = \begin{bmatrix} \mathrm{P}v + \mathrm{Q}\Gamma_1 u \\ Lu \end{bmatrix}, \\ D(A) & = \left\{ \begin{bmatrix} v \\ u \end{bmatrix} \in \mathrm{X} : u \in D(L), \quad \mathrm{R}^* v + \mathrm{D}\Gamma_1 u = \Gamma_0 u \right\} \end{array} \right\}.$$

The problem is to recognize whether a closed-loop system operator $A$ generates a strongly continuous semigroup on X. As we know from Theorem 1.1, the spectral approach is an effective tool for establishing the well-posedness of the feedback system (i.e. generation of a semigroup by the closed-loop system operator) if we can prove that the operator describing the closed-loop system is a discrete operator, similar to a normal one.

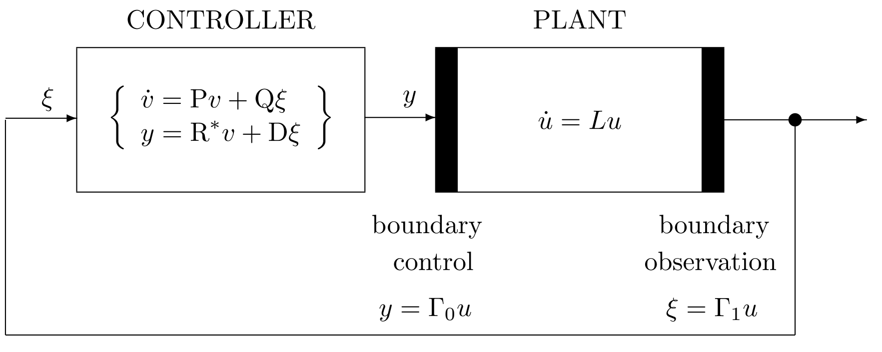


**Fig. 1.** The feedback control system

The eigenproblem for $A$ takes the form

$$\left\{ \begin{array}{rl} (\lambda \mathrm{I} - \mathrm{P})v = & \mathrm{Q}\Gamma_1 u \\ Lu = \lambda u, & u \in D(L) \\ \mathrm{R}^* v + \mathrm{D}\Gamma_1 u = & \Gamma_0 u \end{array} \right\} \tag{2.1}$$

and for $\lambda \notin \sigma(\mathrm{P})$ (2.1) reduces to

$$\left\{ \begin{array}{rl} Lu = \lambda u, & u \in D(L) \\ \mathrm{W}(\lambda)\Gamma_1 u = \Gamma_0 u & \\ \mathrm{W}(\lambda) = \mathrm{D} + \mathrm{R}^*(\lambda \mathrm{I} - \mathrm{P})^{-1}\mathrm{Q} & \end{array} \right\}. \tag{2.2}$$

The spectral parameter $\lambda$ rationally enters the *transfer function* W in (2.2), but after the multiplication of both sides of the boundary condition by the characteristic polynomial $\det(\lambda \mathrm{I}-\mathrm{P})$ of the matrix P, it enters the boundary condition polynomially. In the particular case of a proportional controller the transfer function W is constant and the spectral parameter does not enter these conditions. From the survey given in [11, Chapter I and references therein] we know that then the so-called *strict regularity of the boundary problem* decides about the existence of a Riesz basis of eigenvectors. Moreover, in this case there are some criteria based on determinants which allow to check the strict regularity of boundary problem in a simple way. The general case is much more involved and it will be discussed in the sequel of this paper.

## 3. SHKALIKOV'S THEORY

The theory concerns the Sturm–Liouville boundary-value problems, containing a spectral parameter in the boundary conditions,

$$\ell(y,\lambda) = y^{(n)} + \sum_{k=1}^{n} p_k(x,\lambda) y^{(n-k)} = 0 \tag{3.1}$$

$$U_j(y,\lambda) = \sum_{k=0}^{n-1} a_{jk}(\lambda) y^{(k)}(0) + b_{jk}(\lambda) y^{(k)}(1) = 0, \qquad j = 1,2,\ldots,n \tag{3.2}$$

where $p_s(x,\lambda) = \sum_{\nu=0}^{s} p_{\nu s}(x)\lambda^{\nu}$; $p_{ss}(x) = \text{const}$, $s = 1,2,\ldots,n$, $a_{jk}(\lambda)$, $b_{jk}(\lambda)$ are arbitrary polynomials of the spectral parameter $\lambda$.

**Definition 3.1.** *A nonnegative integer $\kappa_j$ is said to be the* order of the boundary condition *$U_j(y,\lambda)$ of the form* (3.2) *if the linear form $U_j(y,\lambda)$ contains the terms $\lambda^{\nu} y^{(k)}(0)$ or $\lambda^{\nu} y^{(k)}(1)$ for $\nu+k = \kappa_j$ and it does not contain such terms for $\nu+k > \kappa_j$. $\kappa = \kappa_1 + \kappa_2 + \ldots + \kappa_n$ is then called the* total order *of the boundary conditions* (3.2)*. If any $n$ boundary conditions equivalent to* (3.2)*, i.e. obtained from* (3.2) *by taking linear combinations, have the total order not less than $\kappa$ then we say that the boundary conditions* (3.2) *are* normalized.

For further considerations we assume without loss of generality that the boundary conditions (3.2) are normalized and that they are arranged in the decreasing orders, to be more precise: $\kappa_1 \geq \kappa_2 \geq \ldots \geq \kappa_n$.

Assume also that $p_{\nu s} \in \mathrm{W}_1^r(0,1)$, $r \geq 0$ and the characteristic polynomial of the problem (3.1), (3.2)

$$\omega^n + p_{11}\omega^{n-1} + \cdots + p_{nn} = 0 \tag{3.3}$$

has only simple roots: $\omega_1, \omega_2, \ldots, \omega_n$.

**Remark 3.1.** *Without loss of generality we may assume that $0 \leq \nu \leq s-1$. This implies $r + (\nu - s + 1) \leq r$, and $p_{\nu s} \in \mathrm{W}_1^{r-s+\nu+1}(0,1) \cap \mathrm{L}^1(0,1)$.*

Under the above assumptions the complex plane $\mathbb{C}$ can be decomposed into $2h$, $h \le n$ sectors $S_1, S_2, \ldots, S_{2h}$ and in each sector (3.1) has the fundamental system of solutions of the following asymptotic form as $|\lambda| \to \infty$ (the theory of Birkhoff [2, 3] and Tamarkin [35, 36]),

$$y_k^{(s-1)}(x,\lambda) = \omega_k^{s-1}\lambda^{s-1}e^{\omega_k \lambda x}\left[\sum_{\nu=0}^{r} \lambda^{-\nu}\eta_{ks\nu}(x) + O(\lambda^{-r-1})\right] \tag{3.4}$$

$k, s = 1, 2, \ldots, n$, $r \ge 0$, $r$ is arbitrary and fixed; $\eta_{ks\nu} \in \mathrm{W}_1^{r-\nu+1}(0,1)$, $\nu = 0, 1, \ldots, r$, $\eta_{ks0}$ does not depend on $s$, and $\eta_{ks\nu}$ does not depend on the choice of a sector.

Let $\mu_{J_k} = \sum_{\alpha \in J_k} \omega_\alpha$, where $J_k$ $(k = 1, 2, \ldots, n)$ denotes a $k$-element subset of $\{1, 2, \ldots, n\}$, for $k = 0$ we put $\mu_{J_0} = 0$. Let us consider the set of all complex numbers $\mu_{J_k}$ which can be obtained by variating over all possible selections of $J_k$ (in this way we get nothing more than the set of all possible sums which can be created from the set of complex numbers $\omega_1, \omega_2, \ldots, \omega_n$). Let $\mathcal{M}$ be the smallest convex polygon containing all points $\mu_{J_k}$. It may happen that $\mathcal{M}$ is an interval.

Further, we consider the characteristic determinant

$$\Delta(\lambda) = \det[U_j(y_k,\lambda)]_{j,k=1,2,\ldots,n}$$

with functions $y_k$ defined in sectors $S_1, S_2, \ldots, S_{2h}$ by (3.4).

This determinant may be expressed as

$$\Delta(\lambda) = \lambda^\kappa \sum_{J_k} [F^{J_k}]_r e^{\lambda \mu_{J_k}},$$

$$\left[F^{J_k}\right]_r = F_0^{J_k} + \lambda^{-1}F_1^{J_k} + \cdots + \lambda^{-r}F_r^{J_k} + O(\lambda^{-r-1}).$$

**Definition 3.2.** *The problem* (3.1), (3.2) *is said to be* regular *if the numbers* $F_0^{J_k}$ *in the resolutions of* $[F^{J_k}]_0$*, corresponding to the vertexes of* $\mathcal{M}$ *are nonzero. The problem* (3.1), (3.2) *is* strictly regular *if it is regular and additionally, the zeros of* $\Delta(\lambda)$ *are asymptotically simple and isolated one from another.*

In what follows, we assume without loss of generality that $p_{nn} = 1$ and for simplicity of notation we represent (3.1) in the form

$$\ell(y,\lambda) = \sum_{k=0}^{n-1} \lambda^k \ell_k(y) + \lambda^n y = 0. \tag{3.5}$$

For any fixed $r \ge 0$ let us denote $W_2^r := \bigoplus_{j=1}^{n} \mathrm{W}_2^{n-j+r}(0,1)$ and define an operator

$$W_2^r \ni \tilde{v} = \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{n-2} \\ v_{n-1} \end{bmatrix} \longmapsto H\tilde{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \\ -\sum_{\nu=0}^{n-1} \ell_\nu(v_\nu) \end{bmatrix} \in W_2^r,$$

where $v_0 = y$, $v_1 = \lambda v_0, \ldots, v_{n-1} = \lambda v_{n-2}$ and hence, $H^\nu \tilde{v} \in W_2^{r-\nu}$ ($\nu$-th power of $H$), $\nu = 0, 1, 2, \ldots$. In (3.2) we make substitutions according to the rule

$$\lambda^\nu y^{(k)}(w) = \left\{ \begin{array}{ll} (H^\nu \tilde{v})_0^{(k)}(w), & \nu + k < n + r \\ \lambda^{\nu+k-n-r+1}(H^{n+r-k-1}\tilde{v})_0^{(k)}(w), & \nu + k \geq n + r \end{array} \right\}, \tag{3.6}$$

where $w = 0$ or $w = 1$ and the subscript 0 means that we take the first component of an appropriate vector. As a result of these substitutions we represent the boundary conditions in a form

$$\tilde{U}_j(\tilde{v}, \lambda) = \sum_{i=0}^{\nu_j(r)} \lambda^i U_j^i(\tilde{v}), \qquad 1 \leq j \leq n, \tag{3.7}$$

where now the functionals $U_j^i$ do not depend on $\lambda$. Next, we make the following partition of indices $\nu_j(r)$:

$$\nu_1(r) \geq \nu_2(r) \geq \ldots \nu_q(r) > 0 = \nu_{q+1}(r) = \ldots = \nu_n(r).$$

Consider the space $W_{2,U}^r \oplus \mathbb{C}^{N_r}$ where

$$W_{2,U}^r := \Big\{ \tilde{v} \in W_2^r : \tilde{U}_j(H^k \tilde{v}, \lambda) = \tilde{U}_j(H^k \tilde{v}) = 0 \quad \text{for} \quad 0 \leq k \leq n + r - 2 \\ \text{and all boundary conditions of order} \quad \leq n + r - k - 2 \Big\} \tag{3.8}$$

$$N_r = \sum_{j=1}^{q} \nu_j(r) \tag{3.9}$$

(if all $\nu_j(r)$ are zero then $N_r := 0$). Let us define an operator

$$H_r : \left( D(H_r) \subset W_{2,U}^r \oplus \mathbb{C}^{N_r} \right) \longmapsto W_{2,U}^r \oplus \mathbb{C}^{N_r},$$

$$H_r \begin{bmatrix} \tilde{v} \\ \boxed{\begin{array}{c} U_1^{\nu_1(r)} \\ z_{12} \\ \cdots \\ z_{1(\nu_1(r)-1)} \\ z_{1\nu_1(r)} \end{array}} \\ \cdots \\ \text{similar blocks} \\ \text{of variables} \\ \text{for succesive} \\ \text{numbers } \nu_j(r), \\ j = 2, 3, \ldots, q \end{bmatrix} = \begin{bmatrix} H\tilde{v} \\ \boxed{\begin{array}{c} z_{12} - U_1^{\nu_1(r)-1}(\tilde{v}) \\ z_{13} - U_1^{\nu_1(r)-2}(\tilde{v}) \\ \cdots \\ z_{1\nu_1(r)} - U_1^1(\tilde{v}) \\ -U_1^0(\tilde{v}) \end{array}} \\ \cdots \\ \text{similar blocks} \\ \text{of variables} \\ \text{for succesive} \\ \text{numbers } \nu_j(r), \\ j = 2, 3, \ldots, q \end{bmatrix}, \tag{3.10}$$

$$D(H_r) = \left\{ \left( \tilde{v}, U_1^{\nu_1(r)}, z_{12}, \ldots, z_{1\nu_1(r)}, \ldots, U_q^{\nu_q(r)}, z_{q2}, \ldots, z_{q\nu_q(r)} \right) : \right.$$

$$\tilde{v} \in W_{2,U}^{r+1}, \quad z_{j\nu} \in \mathbb{C}, \quad 2 \le \nu \le \nu_j(r), \quad 1 \le j \le q \Big\}.$$

(3.10) will be called *Shkalikov's linearization* of the problem (3.1), (3.2) because the eigenvalue problem for $H_r$ in this space $W_{2,U}^r \oplus \mathbb{C}^{N_r}$ reduces clearly to (3.1), (3.2).

**Theorem 3.1 (Shkalikov** [33]**).** *Let the above assumptions hold and, additionally, let the problem* (3.1), (3.2) *be strictly regular. Then:*

(i) *There exists a system of generalized eigenvectors (only finitely many of them are not eigenvectors) of the operator* (3.10) *which forms a Riesz basis in* $W_{2,U}^r \oplus \mathbb{C}^{N_r}$.

(ii) *A necessary and sufficient condition for the existence of a system of generalized eigenvectors of the operator* (3.10) *which forms a Riesz basis in* $W_{2,U}^r$ *(the case of* $N_r = 0$*), is that all boundary conditions should have the order less or equal* $n + r - 1$. *If such a system of generalized eigenvectors exists, then only a finite number of them are not eigenvectors.*

## 4. APPLICATIONS TO WAVE EQUATIONS

### 4.1. RIDEAU'S FIRST PROBLEM

The system

$$\left\{ \begin{array}{lr} u_{tt} - u_{xx} = 0, & 0 \le x \le 1, \ t \ge 0 \\ u(0,t) = 0, & t \ge 0 \\ u_x(1,t) = -k u_t(1,t), & t \ge 0 \end{array} \right\} \tag{4.1}$$

has been investigated by Rideau [31]. In particular (4.1) is the mathematical model of dynamics of the system depicted in Figure 2.

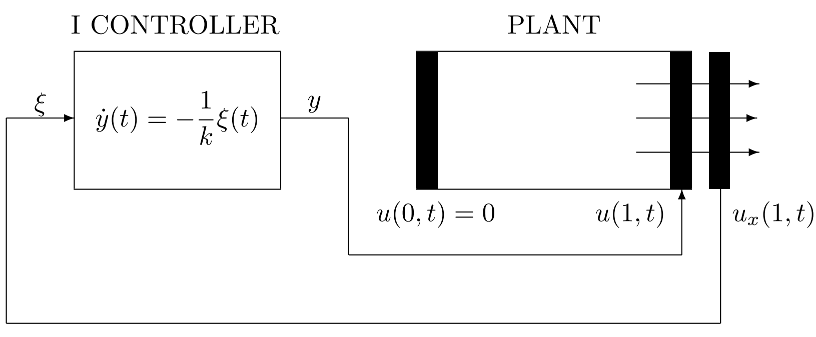


**Fig. 2.** Hybrid control system corresponding to (4.1)

The first order dynamics is

$$\left\{ \begin{array}{lr} u_t = v, & 0 \le x \le 1, \ t \ge 0 \\ v_t = u_{xx}, & t \ge 0 \\ u(0,t) = 0, & t \ge 0 \\ u_x(1,t) = -k v(1,t), & t \ge 0 \end{array} \right\}$$

whence the eigenproblem takes the form

$$\left\{\begin{array}{c} v = \lambda u \\ u'' = \lambda v \\ u(0) = 0 \\ u'(1) = -kv(1) \end{array}\right\}. \tag{4.2}$$

Eliminating $v$ we get

$$\left\{\begin{array}{c} u'' = \lambda^2 u \\ u(0) = 0 \\ u'(1) + k\lambda u(1) = 0 \end{array}\right\}. \tag{4.3}$$

The eigenproblem (4.3) is a particular case of the Sturm–Liouville boundary-value problem (3.1), (3.2) with $n = 2$, $p_2(x,\lambda) = -\lambda^2$, $p_1(x,\lambda) = 0$ and with the boundary conditions in ordered form

$$U_1(u,\lambda) = u'(1) + k\lambda u(1) = 0, \tag{4.4}$$

$$U_2(u,\lambda) = u(0) = 0. \tag{4.5}$$

By Definition 3.1 the order of (4.4) is $\kappa_1 = 1$, while the order of (4.5) is $\kappa_2 = 0$. The total order of boundary conditions is $\kappa = \kappa_1 + \kappa_2 = 1$.

The roots of the characteristic polynomial $\omega^2 - 1 = 0$ are $\omega_1 = -1$, $\omega_2 = 1$ and the polygon $\mathcal{M}$ reduces to the interval $[-1, 1]$.

Assuming a solution of (4.3) in the form $u(x) = C_1 e^{-\lambda x} + C_2 e^{\lambda x}$ we find the characteristic determinant of the problem,

$$\Delta(\lambda) = \lambda\left[e^{-\lambda}(k-1) - e^{\lambda}(k+1)\right]$$

and according to Definition 3.2 we assume $|k| \neq 1$ to ensure the regularity of the problem (4.3).

Now we check whether the problem (4.3) is strictly regular. For this zeros of $\Delta(\lambda)$ should be asymptotically simple and isolated one from another. This can be checked by directed examination of zeros. Observe that $\lambda = 0$ is not an eigenvalue, otherwise (4.3) yields

$$\left\{\begin{array}{c} u'' = 0 \\ u(0) = 0 \\ u'(1) = 0 \end{array}\right\},$$

and $u \equiv 0$. Finally all eigenvalues satisfy the equation

$$e^{-\lambda}(k-1) - e^{\lambda}(k+1) = 0.$$

1° $|k| > 1$. Then $\dfrac{k-1}{k+1} > 0$ and thus we get

$$\lambda_n = \ln\sqrt{\frac{k-1}{k+1}} + jn\pi, \qquad n \in \mathbb{Z};$$

2° $|k| < 1$. Then $\dfrac{k-1}{k+1} < 0$ and we obtain

$$\lambda_n = \ln\sqrt{\frac{1-k}{1+k}} + j\left(n\pi + \frac{\pi}{2}\right), \qquad n \in \mathbb{Z}.$$

We conclude that all eigenvalues are simple and isolated one from another. Moreover, they are located in the left open complex halfplane iff $k > 0$. We verified that the problem (4.3) is strictly regular. For further investigations we represent $\ell(u,\lambda)$ in the form (3.5)

$$\ell(u,\lambda) = \ell_0(u) + \lambda\ell_1(u) + \lambda^2 u = 0$$

and thus $\ell_0(u) = -u''$, $\ell_1 = 0$ (here $p_{22} = 1$).

Let $r = 0$ (it is possible to take $r \in \mathbb{N}$ but it leads to realizations of the semigroup generator on a smaller state space). Now $W_2^0 = \mathrm{W}^{1,2}(0,1) \oplus \mathrm{L}^2(0,1)$ and the operator $H$ takes the form

$$W_2^0 \ni \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} \longmapsto H\begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = \begin{bmatrix} v_1 \\ -\ell_0(v_0) \end{bmatrix} = \begin{bmatrix} v_1 \\ v_0'' \end{bmatrix}.$$

According to the rule (3.6) the term $\lambda u(1)$ in (4.4) condition should be replaced by

$$H\begin{bmatrix} v_0 \\ v_1 \end{bmatrix}_0^{(0)}(1) = v_1(1) = v(1)$$

(here $\nu + k < n + r$, $\nu = 1$ and $k = 0$) while other terms in boundary condition remain unchanged. The resulting boundary conditions (3.7) are

$$\left\{\begin{aligned} u'(1) + kv(1) = 0 \\ u(0) = 0 \end{aligned}\right\}$$

do not contain $\lambda$, so we get $\nu_1(0) = \nu_2(0) = 0$, $q = 0$. Since (3.9) yields $N_0 = 0$ it follows that $W_{2,U}^0$ is the state space. To identify its structure we use formula (3.8). Since $n = 2$, $r = 0$ we have $k = 0$ and exclusively the boundary conditions of the null order will participate in defining the space $W_{2,U}^0$. Hence the state space adequate for our problem is

$$W_{2,U}^0 = \left\{\begin{bmatrix} u \\ v \end{bmatrix} \in \mathrm{H}^1(0,1) \oplus \mathrm{L}^2(0,1) : \ u(0) = 0\right\} := \mathrm{H}_0^1(0,1) \oplus \mathrm{L}^2(0,1).$$

By (3.10) the domain of $H_0$ is $W_{2,U}^1$ where now $r = 1$, so we have $0 \le k \le n+r-2 = 1$. If $k = 0$ then all boundary condition of order $\le n + r - k - 2 = 2 + 1 - 0 - 2 = 1$ should be encountered, i.e., only the boundary condition $u(1) + kv(1) = 0$. If $k = 1$ then all boundary condition of order $\le n + r - k - 2 = 2 + 1 - 1 - 2 = 0$ should be encountered. Thus we take the boundary condition $u(0)=0$ with $u$ replaced by the first component of $H\left[\begin{smallmatrix}u\\v\end{smallmatrix}\right]$ i.e. by $v$. Finally,

$$D(H_0)=\left\{\begin{bmatrix}u\\v\end{bmatrix}\in \mathrm{H}_0^1(0,1)\oplus \mathrm{L}^2(0,1):\ u\in \mathrm{H}^2(0,1),\ v\in \mathrm{H}_0^1(0,1),\ u'(1)+kv(1)=0\right\}$$

Observe that the eigenproblem for $H_0$ on the state space $\mathrm{H}_0^1(0,1)\oplus \mathrm{L}^2(0,1)$ reduces to (4.2).

Summarizing the above facts we see that all boundary conditions have order $\leq n+r-1=2+0-1=1$, and there are no generalized eigenvectors which are not eigenvectors. By Theorem 3.1/(ii) there exists a system of eigenvectors of $H_0$ which form the Riesz basis in $W^0_{2,U}$. From Theorem 1.1 we know that $H_0$ generates a $\mathrm{C}_0$-group on $\mathrm{H}_0^1(0,1)\oplus \mathrm{L}^2(0,1)$. Moreover, by Theorem 1.2 this group is (positively) **EXS** for $k>0$. The same state space was considered by Rideau [31, pp. 16–17].

Recall that our results have been derived under assumption $|k|\neq 1$. For $|k|=1$ the resolvent is still compact, but $H_0$ has no spectrum. The cases $k=1$ and $k=-1$ should be analized separately.

**Lemma 4.1.** *For $k=1$ the operator*

$$\left\{\begin{aligned}&H_0\begin{bmatrix}u\\v\end{bmatrix}=\begin{bmatrix}v\\u''\end{bmatrix},\\ &D(H_0)=\left\{\begin{bmatrix}u\\v\end{bmatrix}\in \mathrm{H}_0^1(0,1)\oplus \mathrm{L}^2(0,1): v\in \mathrm{H}_0^1(0,1),\ u\in \mathrm{H}^2(0,1), u'(1)=-v(1)\right\}\end{aligned}\right\}$$

*generates a $\mathrm{C}_0$-semigroup of contractions.*

*Proof.* $H_0$ is dissipative as for $\left[\begin{smallmatrix}u\\v\end{smallmatrix}\right]\in D(H_0)$ we have

$$\left\langle\begin{bmatrix}u\\v\end{bmatrix},H_0\begin{bmatrix}u\\v\end{bmatrix}\right\rangle=\int_0^1 u'v'dx+\int_0^1 vu''dx=v(1)u'(1)-v(0)u'(0)=-v^2(1)\leq 0.$$

Its adjoint operator can be determined from identity

$$\left\langle H_0^*\begin{bmatrix}f\\g\end{bmatrix},\begin{bmatrix}u\\v\end{bmatrix}\right\rangle=\left\langle\begin{bmatrix}f\\g\end{bmatrix},H_0\begin{bmatrix}u\\v\end{bmatrix}\right\rangle=\int_0^1 f'v'dx+\int_0^1 gu''dx=$$

$$=-\int_0^1 f''vdx+f'(1)v(1)-f'(0)v(0)-\int_0^1 g'u'dx+g(1)u'(1)-g(0)u'(0)$$

$$\forall\begin{bmatrix}u\\v\end{bmatrix}\in D(H_0),\quad \forall\begin{bmatrix}f\\g\end{bmatrix}\in D(H_0^*).$$

Hence

$$\left\{\begin{aligned} &H_0^* \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} -g \\ -f'' \end{bmatrix} \\ &D(H_0^*) = \left\{ \begin{bmatrix} f \\ g \end{bmatrix} \in W^0_{2,U} : \ f \in \mathrm{H}^2(0,1), \ g \in \mathrm{H}^1_0(0,1), f'(1) = g(1) \right\} \end{aligned}\right\}.$$

The adjoint operator is also dissipative as for $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] D(H_0^*)$ we have

$$\left\langle \begin{bmatrix} f \\ g \end{bmatrix}, H_0^* \begin{bmatrix} f \\ g \end{bmatrix} \right\rangle = -\int_0^1 f'g'dx - \int_0^1 gf''dx = -g(1)f'(1) + g(0)f'(0) = -g^2(1) \leq 0.$$

By the Lummer–Phillips theorem [28, Corrollary 4.4, p. 15] the operator generates a semigroup of contractions. □

**Lemma 4.2.** *The operator $H_0^{-1}$ is compact, whence $H_0$ has a compact resolvent.*

*Proof.* Notice that

$$H_0^{-1} \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} -xf(1) - \int_0^x \int_s^1 g(\xi) d\xi ds \\ f \end{bmatrix}.$$

The identity operator acting from $\mathrm{H}^1_0(0,1)$ into $\mathrm{L}^2(0,1)$ can be regarded as a composition of the differentiation operator $\mathrm{H}^1_0(0,1) \ni f \longmapsto f' \in \mathrm{L}^2(0,1)$, which is bounded, and the compact operator of integration $\mathrm{L}^2(0,1) \ni f \longmapsto \int_0^x \in \mathrm{L}^2(0,1)$. Furthermore, the mapping $\mathrm{H}^1_0(0,1) \ni f \longmapsto xf(1) = x\int_0^1 f'(s)ds \in \mathrm{H}_0(0,1)$ is a first rank operator. Finally, the composition of operators (the first one is compact, second one is bounded)

$$\mathrm{L}^2(0,1) \ni g \longmapsto \int_x^1 g(s)ds \in \mathrm{L}^2(0,1) \longmapsto \int_0^x \int_s^1 g(\xi)d\xi ds \in \mathrm{H}^1_0(0,1)$$

is compact. By the above arguments all components of $H_0^{-1}$ are compact which ends the proof. □

**Lemma 4.3.** *For $k = -1$ the operator*

$$\left\{\begin{aligned} &H_0 \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u \\ v'' \end{bmatrix}, \\ &D(H_0) = \left\{ \begin{bmatrix} u \\ v \end{bmatrix} \in \mathrm{H}^1_0(0,1) \oplus \mathrm{L}^2(0,1) : \ v \in \mathrm{H}^1_0(0,1), \ u \in \mathrm{H}^2(0,1), \ u'(1) = v(1) \right\} \end{aligned}\right\}$$

*does not generate a* $\mathrm{C}_0$*-semigroup.*

*Proof.* Notice that

$$\left\{\begin{array}{l} H_0^* \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} -g \\ -f'' \end{bmatrix}, \\ D(H_0^*) = \left\{ \begin{bmatrix} f \\ g \end{bmatrix} \in W_{2,U}^0 : \ f \in \mathrm{H}^2(0,1), \ g \in \mathrm{H}_0^1(0,1), \ f'(1) = -g(1) \right\} \end{array}\right\}.$$

It follows from Lemma 4.1 that $-H_0^*$ coincide with $H_0$ for $k = 1$, and $H_0$ generates a $\mathrm{C}_0$-semigroup which is not the $\mathrm{C}_0$-group. Consequently, both $H_0^*$ and $H_0$ do not generate $\mathrm{C}_0$-semigroups. □

**Remark 4.1.** *The d'Alembert solution of the first equation of the system* (4.1) *is* $u(t,x) = \phi(t+x) + \psi(t-x)$ *where* $\phi$, $\psi$ *are sufficiently smooth functions. Substituting this expression into the second and third equation of* (4.1) *we obtain* $\psi = -\phi$, *and*

$$\frac{d}{dt}\left[(k+1)v(t) - (1-k)v(t-2)\right] = 0$$

*where* $v(t) := \phi(t+1)$. *It is known* [10] *that the above difference – differential equation is of neutral type if* $|k| \neq 1$ *and then it gives rise to a group on* $\mathbb{M}^2 = \mathrm{L}^2(-2,0) \oplus \mathbb{R}$. *If* $k = 1$ *the system is well-posed but leads to a* $\mathrm{C}_0$*-semigroup which is not a group. For* $k = -1$ *the system is not well-posed. Moreover, the positive semigroup related to the investigated system is* ***EXS*** *provided that* $\left|\frac{k-1}{k+1}\right| < 1$, *or equivalently,* $k > 0$. *This agrees with the results obtained for* (4.1) *with the aid of spectral methods.*

## 4.2. STABILIZATION OF A CABLE WITH A TIP MASS

Morgül, Rao and Conrad [26] have considered the stabilization problem of a cable with a tip mass. The problem leads to the system of equations

$$\left\{\begin{array}{rl} u_{tt} - u_{xx} = 0, & 0 \le x \le 1, \ t \ge 0 \\ u(0,t) = 0, & 0 \le x \le 1, \ t \ge 0 \\ u_x(1,t) + m u_{tt}(1,t) + a u_{xt}(1,t) + \alpha u_t(1,t) = 0, & t \ge 0 \end{array}\right\}, \ m > 0.$$

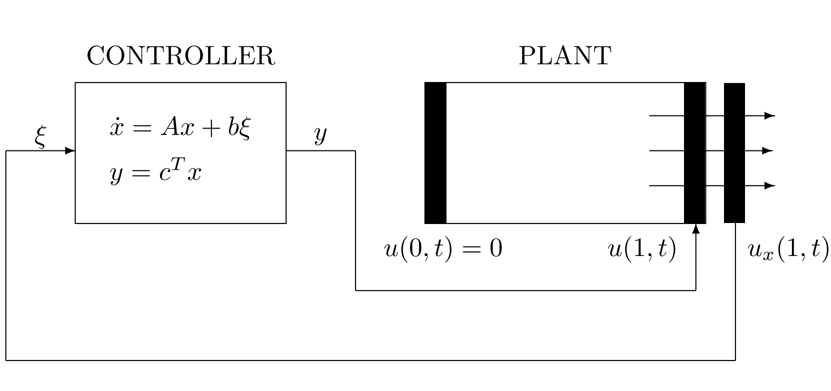


**Fig. 3.** The hybrid control system

It also describes the hybrid feedback system depicted in Figure 3. In the latter case we have:

$$x = \begin{bmatrix} u(1,t) \\ u_t(0,t) + \dfrac{a}{m}u_x(1,t) + \dfrac{\alpha}{m}u(1,t) \end{bmatrix}, \qquad \xi = u_x(1,t), \qquad y = u(1,t),$$
$$A = \frac{1}{m}\begin{bmatrix} -\alpha & m \\ 0 & 0 \end{bmatrix}, \qquad b = \frac{1}{m}\begin{bmatrix} -a \\ -1 \end{bmatrix}, \qquad c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The first order dynamics is

$$\left\{ \begin{array}{rlr} u_t = v, & & 0 \le x \le 1,\ t \ge 0 \\ v_t = u_{xx}, & & 0 \le x \le 1,\ t \ge 0 \\ u(0,t) = 0, & & t \ge 0 \\ u_x(1,t) + mu_{tt}(1,t) + au_{xt}(1,t) + \alpha u_t(1,t) = 0, & & t \ge 0 \end{array} \right\}.$$

The eigenproblem takes the form

$$\left\{ \begin{array}{r} v = \lambda u \\ u'' = \lambda v \\ u(0) = 0 \\ u'(1) + m\lambda^2 u(1) + a\lambda u'(1) + \alpha\lambda u(1) = 0 \end{array} \right\}.$$

Eliminating $v$ we get

$$\left\{ \begin{array}{rl} u''(x) = & \lambda^2 u(x) \\ (m\lambda^2 + \alpha\lambda)u(1) + (1 + a\lambda)u'(1) = & 0 \\ u(0) = & 0 \end{array} \right\}. \tag{4.6}$$

The eigenproblem (4.6) is a particular case of the Sturm–Liouville boundary-value problem (3.1), (3.2) with $n = 2$, $p_2(x,\lambda) = -\lambda^2$, $p_1(x,\lambda) = 0$. Since the boundary conditions are already in ordered form we find easily their orders: $\kappa_1 = 2$, $\kappa_2 = 0$. The total order of boundary conditions is $\kappa = \kappa_1 + \kappa_2 = 2$.

The roots of the characteristic polynomial $\omega^2 - 1 = 0$ are $\omega_1 = -1$, $\omega_2 = 1$ and the polygon $\mathcal{M}$ reduces to the interval $[-1, 1]$.

Assuming a solution of (4.6) in the form $u(x) = C_1 e^{-\lambda x} + C_2 e^{\lambda x}$ we find the characteristic determinant of the problem,

$$\left[(m-a)\lambda^2 + (\alpha - 1)\lambda\right] e^{-\lambda} - \left[(m+a)\lambda^2 + (\alpha+1)\lambda\right] e^{\lambda} =$$
$$= \lambda^2 \left\{ \left[(m-a) + \frac{\alpha - 1}{\lambda}\right] e^{-\lambda} - \left[(m+a) + \frac{\alpha+1}{\lambda}\right] e^{\lambda} \right\}.$$

Hence the problem (4.6) is regular iff

$$m - a \neq 0, \qquad m + a \neq 0.$$

Writing the characteristic equation in the form

$$e^{2\lambda} = g(\lambda), \qquad g(\lambda) := \frac{(m-a)\lambda + (\alpha - 1)}{(m+a)\lambda + (\alpha + 1)} \longrightarrow 0 \quad \text{as} \quad |\lambda| \to \infty$$

we conclude that asymptotic eigenvalues are the roots of the equation $e^{2\lambda} = \dfrac{m-a}{m+a}$.

1° $\dfrac{m-a}{m+a} > 0$. Then the asymptotic eigenvalues are

$$\lambda_n = \ln\sqrt{\frac{m-a}{m+a}} + jn\pi, \qquad n \in \mathbb{Z}\,.$$

2° $\dfrac{m-a}{m+a} < 0$. Then the asymptotic eigenvalues are

$$\lambda_n = \ln\sqrt{\frac{a-m}{a+m}} + j\left(n\pi + \frac{\pi}{2}\right), \qquad n \in \mathbb{Z}\,.$$

Moreover, the asymptotic eigenvalues are located in the left open complex half-plane iff

$$\left|\frac{m-a}{m+a}\right| < 1 \Longleftrightarrow ma > 0 \Longleftrightarrow a > 0.$$

Observe that for a nonsimple eigenvalue $\lambda$ we have

$$\left\{ \begin{array}{c} e^{2\lambda} = g(\lambda) \\ 2e^{2\lambda} = g'(\lambda) \end{array} \right\}.$$

Eliminating $e^{2\lambda}$ we get

$$2g(\lambda) - g'(\lambda) = 0. \tag{4.7}$$

Since $g$ is a rational function being the quotient of two polynomials of the same degree, we have

$$\lim_{|\lambda|\to\infty} g(\lambda) = \frac{m-\beta}{m+\beta}, \qquad \lim_{|\lambda|\to\infty} g'(\lambda) = 0.$$

Hence (4.7) cannot hold for asymptotic eigenvalues and therefore they are asymptotically simple.

We proved that the problem (4.6) is strictly regular.

Let $r = 0$. By Theorem 3.1 there exists a system of generalized eigenvectors which forms a Riesz basis of the appropriately chosen Shkalikov space $W_2^0$. To construct it we take $W_2^0 = \mathrm{W}^{1,2}(0,1) \oplus \mathrm{L}^2(0,1)$ and consider the operator $H$,

$$W_2^0 \ni \begin{bmatrix} u \\ v \end{bmatrix} \longmapsto H \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} v \\ -\ell_0(u) \end{bmatrix} = \begin{bmatrix} v \\ u'' \end{bmatrix}.$$

Observe that

$$H^2 \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u'' \\ v'' \end{bmatrix}.$$

**Table 1.** Results of applying the rule (3.6)

| Original term | $\nu$ | $k$ | $\nu + k < 2 + r$ | Replaced term |
|---|---|---|---|---|
| $\lambda^2 u(1)$ | 2 | 0 | not | $\lambda v(1)$ |
| $\lambda u(1)$ | 1 | 0 | yes | $v(1)$ |
| $\lambda u'(1)$ | 1 | 1 | not | $\lambda u'(1)$ |
| $u'(1)$ | 0 | 1 | yes | $u'(1)$ |
| $u(0)$ | 0 | 0 | yes | $u(0)$ |

The results of interchanging terms in the boundary conditions according to the rule (3.6), are presented in Table 1. The modified boundary conditions are

$$\tilde{U}_1 = [u'(1) + \alpha v(1)] + \lambda [au'(1) + mv(1)] = U_1^0 + \lambda U_0^1,$$
$$\tilde{U}_2 = u(0) = U_2^0.$$

Hence $\nu_1(0) = 1$, $\nu_2(0) = 0$, $q = 1$. Since (3.9) yields $N_0 = 1$ it follows that $W_{2,U}^0 \oplus \mathbb{C}$ is the state space. To identify its structure we use formula (3.8). Since $n = 2$, $r = 0$ we have $k = 0$ and exclusively the boundary conditions of the null order will participate in defining the space $W_{2,U}^0$. Thus the state space adequate for our problem is

$$W_{2,U}^0 = \left\{ \begin{bmatrix} u \\ v \end{bmatrix} \in \mathrm{H}^1(0,1) \oplus \mathrm{L}^2(0,1) : \; u(0) = 0 \right\}.$$

Now we can write down, using (3.10), the particular form of Shkalikov's linearization operator $H_0$,

$$H_0 \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} v \\ u'' \\ -u'(1) - \alpha v(1) \end{bmatrix}$$

with the domain

$$D(H_0) = \left\{ \begin{bmatrix} u \\ v \\ w \end{bmatrix} \in \mathrm{H}^2(0,1) \oplus \mathrm{H}^1(0,1) \oplus \mathbb{C} : u(0) = 0, \; v(0) = 0, \; w = au'(1) + mv(1) \right\}.$$

The time-domain description of the original model involves a real state space. Thus $\mathbb{C}$ should we replaced by $\mathbb{R}$ as we complexified the state space to carry on the spectral analysis. The same state space as above has been introduced in [26].

**Remark 4.2.** *Proceeding as in Remark 4.1 we observe that the d'Alembert solution of the first equation of the system is* $u(t, x) = \phi(t + x) + \psi(t - x)$ *where* $\phi$, $\psi$ *are* *sufficiently smooth functions. Substituting this expression into the second and third equation of the dynamical model we obtain* $\psi = -\phi$, *and*

$$\frac{d}{dt}\left[(m+a)v(t)+(m-a)v(t-2)\right] = -(\alpha+1)v(t)+(1-\alpha)v(t-2)$$

*where* $v(t) := \phi'(t+1)$. *The above difference – differential equation is of neutral type* [10] *if* $m \neq |a|$ *and then it gives rise to group on* $\mathbb{M}^2 = \mathrm{L}^2(-2,0) \oplus \mathbb{R}$. *Notice that the regularity of the problem* (4.6) *corresponds to atomicity of the difference operator at* 0 *and* $-2$. *If* $m = a$ *the system is well-posed but leads to a* $\mathrm{C}_0$*-semigroup which is not a group. For* $m = -a$ *the system is not well-posed. Moreover, the positive semigroup related to the investigated system is* ***EXS*** *provided that* $\left|\dfrac{m-a}{m+a}\right| < 1$. *These results agree with those obtained via the spectral methods.*

4.3. DYNAMICAL MODEL OF THE CRANE

The system

$$\left\{\begin{array}{lr} u_{tt} - u_{xx} = 0, & 0 \le x \le 1,\ t \ge 0 \\ u_x(0,t) - m u_{tt}(0,t) = \alpha u(0,t) + \alpha\beta u_t(0,t) - \beta u_{xt}(0,t), & t \ge 0 \\ u_x(1,t) + M u_{tt}(1,t) = 0, & t \ge 0 \end{array}\right\}$$

has been investigated by Rao, Conrad and Mifdal [23] as a mathematical model of the crane dynamics. It can be interpreted as the hybrid feedback system depicted in Figure 4.

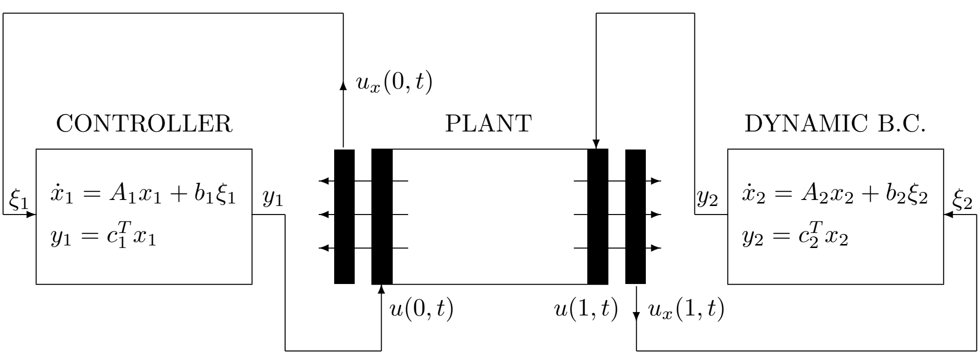


**Fig. 4.** The hybrid control system of the crane

Then we have:

$$x_1(t) = \begin{bmatrix} u(0,t) \\ u_t(0,t) + \dfrac{\alpha\beta}{m}u(0,t) - \dfrac{\beta}{m}u_x(0,t) \end{bmatrix}, \qquad \xi_1(t) = u_x(0,t), \qquad y_1(t) = u(0,t),$$

$$A_1 = \frac{1}{m}\begin{bmatrix} -\alpha\beta & m \\ -\alpha & 0 \end{bmatrix}, \qquad b_1 = \frac{1}{m}\begin{bmatrix} \beta \\ 1 \end{bmatrix}, \qquad c_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$x_2(t) = \begin{bmatrix} u(1,t) \\ u_t(1,t) \end{bmatrix}, \qquad \xi_2(t) = u_x(1,t), \qquad y_2(t) = u(1,t),$$

$$A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad b_2 = \frac{1}{M}\begin{bmatrix} 0 \\ -1 \end{bmatrix}, \qquad c_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The first order dynamics is

$$\left\{ \begin{array}{lr} u_t = v, & 0 \le x \le 1,\ t \ge 0 \\ v_t = u_{xx}, & 0 \le x \le 1,\ t \ge 0 \\ u_x(0,t) - m u_{xx}(0,t) = \alpha u(0,t) + \alpha\beta v(0,t) - \beta v_x(0,t), & t \ge 0 \\ u_x(1,t) + M u_{xx}(1,t) = 0 & t \ge 0 \end{array} \right\}$$

whence the eigenproblem takes the form

$$\left\{ \begin{array}{l} v = \lambda u \\ u'' = \lambda v \\ u'(0) - m u''(0) = \alpha u(0) + \alpha\beta v(0) - \beta v'(0) \\ u'(1) + M u''(1) = 0 \end{array} \right\}.$$

Eliminating $v$ we get

$$\left\{ \begin{array}{r} u'' = \lambda^2 u \\ (\beta\lambda + 1)u'(0) - (\lambda^2 m + \alpha\beta\lambda + \alpha)u(0) = 0 \\ u'(1) + M\lambda^2 u(1) = 0 \end{array} \right\}. \tag{4.8}$$

**Remark 4.3.** *Observe that there are many possibilities to express the boundary conditions in* (4.8) *employing the following relationships*

$$u''(0) = \lambda^2 u(0) = \lambda v(0), \quad v(0) = \lambda u(0), \quad v'(0) = \lambda u'(0), \quad u''(1) = \lambda^2 u(1) = \lambda v(1).$$

The eigenproblem (4.8) is a particular case of the Sturm–Liouville boundary-value problem (3.1), (3.2) with $n = 2$, $p_2(x,\lambda) = -\lambda^2$, $p_1(x,\lambda) = 0$ and with the boundary conditions in ordered form

$$U_1(u,\lambda) = -(\lambda^2 m + \alpha\beta\lambda + \alpha)u(0) + (\beta\lambda + 1)u'(0) = 0, \tag{4.9}$$

$$U_2(u,\lambda) = M\lambda^2 u(1) + u'(1) = 0. \tag{4.10}$$

By Definition 3.1 the order of (4.9) is $\kappa_1 = 2$, while the order of (4.10) is $\kappa_2 = 2$. The total order of boundary conditions is $\kappa = \kappa_1 + \kappa_2 = 4$.

The roots of the characteristic polynomial $\omega^2 - 1 = 0$ are $\omega_1 = -1$, $\omega_2 = 1$ and the polygon $\mathcal{M}$ reduces to the interval $[-1, 1]$.

Assuming a solution of (4.8) in the form $u(x) = C_1 e^{-\lambda x} + C_2 e^{\lambda x}$ we find the characteristic determinant of the problem,

$$\begin{aligned} &\Delta(\lambda) = \\ &= \lambda^4 e^{-\lambda}\left\{(m-\beta)M + \lambda^{-1}\left[\beta - m + M(\alpha\beta - 1)\right] + \lambda^{-2}\left[1 - \alpha\beta + \alpha M\right] - \alpha\lambda^{-3}\right\} - \\ &\quad - \lambda^4 e^{\lambda}\left\{(m+\beta)M + \lambda^{-1}\left[m + \beta + M(\alpha\beta + 1)\right] + \lambda^{-2}\left[\alpha\beta + 1 + \alpha M\right] + \alpha\lambda^{-3}\right\}, \end{aligned}$$

and according to Definition 3.2 we assume $m \pm \beta \neq 0$ to ensure regularity of the problem (4.8).

Now we check whether the problem (4.8) is strictly regular. Writing the characteristic equation in the equivalent form

$$e^{2\lambda} = g(\lambda), \qquad g(\lambda) := \frac{1 - M\lambda}{1 + M\lambda} \cdot \frac{(\beta - m)\lambda^2 + (1 - \alpha\beta)\lambda - \alpha}{(\beta + m)\lambda^2 + (1 + \alpha\beta)\lambda + \alpha}$$

we establish that the asymptotic eigenvalues are the roots of the equation $e^{2\lambda} = \frac{m - \beta}{m + \beta}$.

1° $\dfrac{m - \beta}{m + \beta} > 0$. Then asymptotic eigenvalues are

$$\lambda_n = \ln \sqrt{\frac{m - \beta}{m + \beta}} + jn\pi, \qquad n \in \mathbb{Z}.$$

2° $\dfrac{m - \beta}{m + \beta} < 0$. Then asymptotic eigenvalues are

$$\lambda_n = \ln \sqrt{\frac{\beta - m}{\beta + m}} + j\left(n\pi + \frac{\pi}{2}\right), \qquad n \in \mathbb{Z}.$$

By the same arguments as in Subsection 4.2 we conclude that all eigenvalues are asymptotically simple and isolated one from another which proves that the problem (4.8) is strictly regular. Moreover, they are asymptotically located in the left open complex halfplane iff $\left|\dfrac{m - \beta}{m + \beta}\right| < 1$. For further investigations we represent $\ell(u, \lambda)$ in the form (3.5)

$$\ell(u, \lambda) = \ell_0(u) + \lambda\ell_1(u) + \lambda^2 u = 0$$

and thus $\ell_0(u) = -u''$, $\ell_1 = 0$ (here $p_{22} = 1$).

Let $r = 0$. By Theorem 3.1 there exists a system of generalized eigenvectors which forms a Riesz basis of the appropriately chosen Shkalikov space $W_2^0$. To construct it we take $W_2^0 = \mathrm{W}^{1,2}(0,1) \oplus \mathrm{L}^2(0,1)$ and consider the operator $H$,

$$W_2^0 \ni \begin{bmatrix} u \\ v \end{bmatrix} \longmapsto H \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} v \\ -\ell_0(u) \end{bmatrix} = \begin{bmatrix} v \\ u'' \end{bmatrix}.$$

Observe that

$$H^2 \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u'' \\ v'' \end{bmatrix}.$$

The results of interchanging terms in the boundary conditions 4.9, (4.10), according to the rule (3.6), are presented in Table 2.

**Table 2.** Results of applying the rule (3.6)

| Original term | $\nu$ | $k$ | $\nu+k<2+r$ | Replaced term |
|---|---|---|---|---|
| $\lambda^2 u(0)$ | 2 | 0 | not | $\lambda v(0)$ |
| $\lambda u(0)$ | 1 | 0 | yes | $v(0)$ |
| $u(0)$ | 0 | 0 | yes | $u(0)$ |
| $\lambda u'(0)$ | 1 | 1 | not | $\lambda u'(0)$ |
| $u'(0)$ | 0 | 1 | yes | $u'(0)$ |
| $\lambda^2 u(1)$ | 2 | 0 | not | $\lambda v(1)$ |
| $u'(1)$ | 0 | 1 | yes | $u'(1)$ |

The modified boundary conditions are

$$\tilde{U}_1 = [u'(0) - \alpha u(0) - \alpha\beta v(0)] + \lambda\,[mv(0) + \beta u'(0)] = U_1^0 + \lambda U_0^1,$$
$$\tilde{U}_2 = u'(1) + \lambda M v(1) = U_2^0 + \lambda U_2^1.$$

Hence $\nu_1(0) = \nu_2(0) = 1$, $q = 2$. Since (3.9) yields $N_0 = 2$ it follows that $W_{2,U}^0 \oplus \mathbb{C}^2$ is the state space. To identify its structure we use formula (3.8). Since $n = 2$, $r = 0$ we have $k = 0$ and exclusively the boundary conditions of the null order will participate in defining the space $W_{2,U}^0$ but there are no such boundary conditions. Finally the state space adequate for our problem is

$$\mathrm{X} = \mathrm{H}^1(0,1) \oplus \mathrm{L}^2(0,1) \oplus \mathbb{C}^2\,.$$

Now we can write down using (3.10) the particular form of Shkalikov's linearization operator $H_0$,

$$H_0 \begin{bmatrix} u \\ v \\ w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} v \\ u'' \\ -u'(0) + \alpha u(0) + \alpha\beta v(0) \\ -u'(1) \end{bmatrix}$$

with the domain

$$D(H_0) = \left\{ \begin{bmatrix} u \\ v \\ w_1 \\ w_2 \end{bmatrix} \in \mathrm{H}^2(0,1) \oplus \mathrm{H}^1(0,1) \oplus \mathbb{C}^2 :\ w_1 = -mv(0) + \beta u'(0),\ w_2 = Mv(1) \right\}$$

**Remark 4.4.** *For $r = 1$ we get the Shaklikov's state space $W_{2,U}^1 = \mathrm{H}^2(0,1) \oplus \mathrm{H}^1(0,1)$. The operator $H_1$ takes the form*

$$H_1 \begin{bmatrix} u \\ v \end{bmatrix} = H \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} v \\ u'' \end{bmatrix}, \qquad D(H_1) = W_{2,U}^2 = \left\{ \begin{bmatrix} u \\ v \end{bmatrix} \in \mathrm{H}^3(0,1) \oplus \mathrm{H}^2(0,1) : \right.$$
$$\left. u'(0) + \beta v'(0) - \alpha u(0) - \alpha\beta v(0) - mu''(0) = 0,\ u'(1) + Mu''(1) = 0 \right\}.$$

*This result agrees with that of Mifdal* [23, Subsection 3.2, pp. 72–77].

4.4. A DYNAMIC CONTROL LAW FOR THE WAVE EQUATION

The control system depicted in Figure 5 is governed by the system of equations

$$\left\{\begin{array}{rlr} \dot{w} = Aw + bu_t(1,t), & & t \ge 0 \\ -u_x(1,t) = c^T w + du_t(1,t) + ku(1,t), & & t \ge 0 \\ u_{tt} = u_{xx}, & 0 \le x \le 1, & t \ge 0 \\ u(0,t) = 0, & & t \ge 0 \end{array}\right\}.$$

Here $A \in \mathbf{L}(\mathbb{R}^n)$, $b, c \in \mathbb{R}^n$ and $d, k \in \mathbb{R}$. The operator describing the closed-loop feedback system in the space $\mathrm{X} = \mathrm{H}^1_0(0,1) \oplus \mathrm{L}^2(0,1) \oplus \mathbb{R}^n$ is

$$L\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} v \\ u'' \\ Aw + bv(1) \end{bmatrix}, \qquad D(L) = \left\{ \begin{bmatrix} u \\ v \\ w \end{bmatrix} \in \mathrm{H}^1_0 \oplus \mathrm{L}^2(0,1) \oplus \mathbb{R}^n : \right.$$

$$\left. u \in \mathrm{H}^2(0,1),\ v \in \mathrm{H}^1_0(0,1),\ c^T w + dv(1) + u'(1) + ku(1) = 0 \right\}.$$

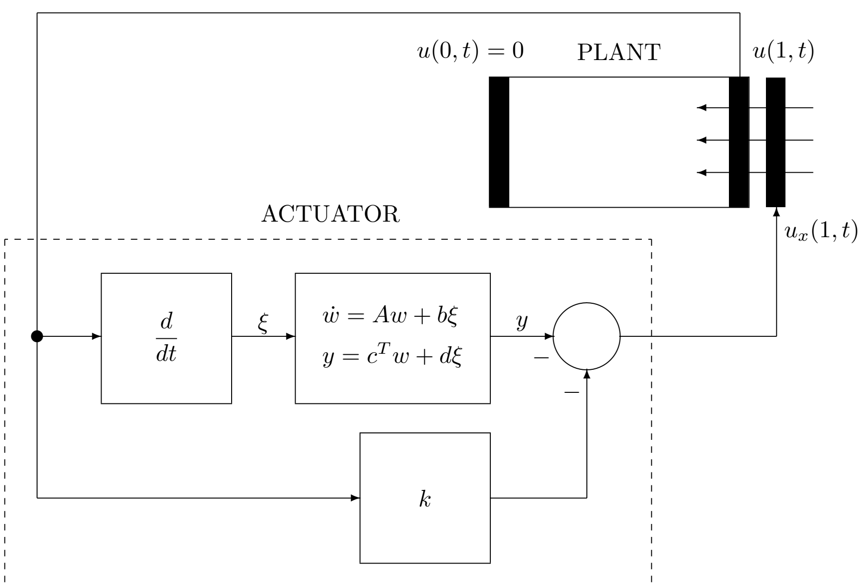


**Fig. 5.** The feedback control system

Using the energy estimate obtained with the aid of the Kalman–Yacubovich lemma, Morgül [25] proved that:

(i) $L$ generates a $\mathrm{C}_0$-semigroup of contractions on X. This semigroup is strongly asymptotically stable, provided that:

— $\operatorname{Re}\lambda(A) < 0$,

— the pair $(A, b)$ is controllable and the pair $(A, c^T)$ is observable,

— $d \geq 0$, $k \geq 0$ and the transfer function $d + c^T(sI - A)^{-1}b$ is strictly positive real.

(ii) The above semigroup is **EXS** provided that $d > 0$.

We shall generalize these results employing the Shkalikov theory. If $\lambda$ is not an eigenvalue of $A$ then the eigenproblem for $L$ reduces to

$$\left\{ \begin{array}{rl} u'' = & \lambda^2 u \\ \left[\lambda \hat{g}(\lambda) + k\right] u(1) + u'(1) = & 0 \\ u(0) = & 0 \end{array} \right\} \tag{4.11}$$

where $\hat{g}(\lambda) := c^T(\lambda I - A)^{-1}b + d$ is the rational transfer function of the $n$-th dimensional part of the actuator. Its numerator and denominator are respectively,

$$l(\lambda) = c^T \mathrm{adj}(\lambda I - A)b + d \det(\lambda I - A), \qquad m(\lambda) = \det(\lambda I - A).$$

Multiplying the first boundary condition by $m(\lambda)$ we get a particular case of the Sturm–Liouville boundary-value problem (3.7) with

$$U_1(u, \lambda) = \left[\lambda l(\lambda) + k m(\lambda)\right] u(1) + m(\lambda) u'(1),$$
$$U_2(u, \lambda) = u(0).$$

The boundary conditions are ordered with orders $\kappa_1 = n+1$ and $\kappa_2 = 0$ respectively. The total order of boundary conditions is $\kappa = \kappa_1 + \kappa_2 = n + 1$.

The roots of the characteristic polynomial $\omega^2 - 1 = 0$ are $\omega_1 = -1$, $\omega_2 = 1$ and the polygon $\mathcal{M}$ reduces to the interval $[-1, 1]$.

Assuming a solution in the form $u(x) = C_1 e^{-\lambda x} + C_2 e^{\lambda x}$ we find the characteristic determinant of the problem,

$$\begin{aligned} \Delta(\lambda) =& e^{\lambda} \left[\lambda l(\lambda) + (k + \lambda) m(\lambda)\right] - e^{-\lambda} \left[\lambda l(\lambda) + (k - \lambda) m(\lambda)\right] = \\ =& \lambda^{n+1} \Big\{ e^{\lambda} \left[(d + 1) + \text{ terms with negative powers of } \lambda\right] - \\ & - e^{-\lambda} \left[(d - 1) + \text{ terms with negative powers of } \lambda\right] \Big\} \end{aligned}$$

and according to Definition 3.2 we assume $|d| \neq 1$ to ensure the regularity of the problem (4.11).

Now we check whether the problem (4.11) is strictly regular. The asymptotic roots of $\Delta$ coincide with the roots of the equation $e^{2\lambda} = \dfrac{d-1}{d+1}$ and therefore they are

$$\lambda_n = \left\{ \begin{array}{ll} \ln \sqrt{\dfrac{d-1}{d+1}} + jn\pi, & \text{if } \dfrac{d-1}{d+1} > 0 \\ \ln \sqrt{\dfrac{1-d}{1+d}} + j\left(n\pi + \dfrac{\pi}{2}\right), & \text{otherwise} \end{array} \right\}, \qquad n \in \mathbb{Z}.$$

The asymptotic eigenvalues are therefore uniformly separated, simple and they have negative real parts iff $d > 0$. Simplicity follows by the same reasoning as in Subsection 4.2.

The first boundary condition contains the term $\lambda^n u'(1)$. According to the rule (3.6) (here $\nu = n$, $k = 1$ and we have $\nu + k = n + 1 \geq 2$ as $n \geq 1$) this term remains unchanged. Thus in (3.7) we have $\nu_1(0) = n$. The second boundary condition remains also unchanged after applying the rule (3.6) (here $\nu = k = 0$ and therefore $\nu + k = 0 < 2$). From (3.7) we find $\nu_2(0) = 0$. Now $q = 1$ and $N_0 = n$. The state space is $\mathrm{X} := W^0_{2,U} \oplus \mathbb{C}^n$ and it coincides with the state space introduced by Morgül [25] as by (3.8)

$$W^0_{2,U} = \left\{ \begin{bmatrix} u \\ v \end{bmatrix} \in \mathrm{H}^1(0,1) \oplus \mathrm{L}^2(0,1) : \ u(0) = 0 \right\} = \mathrm{H}^1_0(0,1) \oplus \mathrm{L}^2(0,1).$$

By Theorem 3.1 there exists a system of generalized eigenvectors of the operator $L$ which forms a Riesz basis in $\mathrm{X} = \mathrm{H}^1_0(0,1) \oplus \mathrm{L}^2(0,1) \oplus \mathbb{C}^n$. Moreover, only finitely many of generalized eigenvectors are not eigenvectors. Applying Theorem 1.2 we conclude that $L$ generates a group on X. This group is positively **EXS** iff $\sigma(L)$ is contained in the open left complex halfplane. Equivalently, all roots of $\Delta$ have negative real parts. This can be verified using the Pontriagin criterion [10, Theorem 6.4.7, p. 198].

## 5. APPLICATIONS TO ELASTIC BEAM EQUATIONS

### 5.1. RIDEAU'S SECOND PROBLEM

Rideau has examined [31] the system

$$\left\{ \begin{array}{rlr} u_{tt} + a u_{xxxx} = 0, & & 0 \leq x \leq 1, \ t \geq 0 \\ u(0,t) = 0, & & t \geq 0 \\ u_x(0,t) = 0, & & t \geq 0 \\ EI u_{xxx}(1,t) = \alpha u_t(1,t) + \beta u_{xt}(1,t), & & t \geq 0 \\ -EI u_{xx}(1,t) = \gamma u_t(1,t) + \delta u_{xt}(1,t), & & t \geq 0 \end{array} \right\}. \tag{5.1}$$

The particular cases of (5.1) were discussed more extensively in subsequent papers. Chen et al. [4] and Krall [20] considered (5.1) with $\alpha = EIk_1$, $\beta = 0$, $\gamma = 0$, $\delta = EIk_2$. Simplifying the feedback boundary control proposed by them, Conrad [5] has investigated the system (5.1) with $\beta = \gamma = \delta = 0$, $\alpha = EIk$. The same case, but with additional structural damping term $-ku_{txx}$ entering the wave equation has been studied by Rebarber [30]. Here we shall discuss the results of Conrad [5] from the viewpoint of Skalikov's theory. Starting from now the following simplification of

(5.1) due to Conrad will be examined,

$$\left\{\begin{array}{rlr} u_{tt}+u_{xxxx}=0, & 0\le x\le 1, & t\ge 0\\ u(0,t)=0, & & t\ge 0\\ u_x(0,t)=0, & & t\ge 0\\ u_{xxx}(1,t)-ku_t(1,t)=0, & & t\ge 0\\ u_{xx}(1,t)=0, & & t\ge 0 \end{array}\right\}. \tag{5.2}$$

By substituting $\lambda=\mu^2$ the eigenproblem corresponding to (5.2) reduces to

$$\left\{\begin{array}{r} u''''+\mu^4u=0\\ u'''(1)-k\mu^2u(1)=0\\ u''(1)=0\\ u'(0)=0\\ u(0)=0 \end{array}\right\}. \tag{5.3}$$

Here the boundary conditions are ordered and have orders $\kappa_1=3$, $\kappa_2=2$, $\kappa_3=1$, $\kappa_4=0$, respectively. The total order of boundary conditions is $\kappa=6$. We have $n=4$, $p_1=p_2=p_3=0$, $p_4=\mu^4$ and $p_{44}=1\neq 0$ in (3.1). By (3.3) the characteristic polynomial of the problem takes the form

$$\omega^4+1=\left(\omega^2-\sqrt{2}\omega+1\right)\left(\omega^2+\sqrt{2}\omega+1\right). \tag{5.4}$$

Its roots and the polygon $\mathcal{M}$ are depicted in Figure 6.

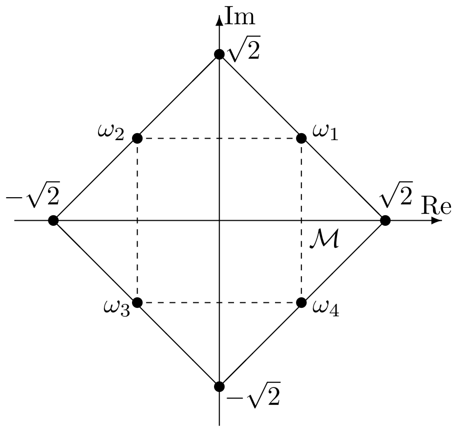


**Fig. 6.** Geometry of roots of the polynomial (5.4) and the polygon $\mathcal{M}$

Assuming a solution of (5.3) in the form

$$u(x)=C_1e^{\mu\omega_1x}+C_2e^{\mu\omega_2x}+C_3e^{\mu\omega_3x}+C_4e^{\mu\omega_4x}$$

we get

$$\begin{bmatrix} (\mu^3\omega_1^3 - k\mu^2)\, e^{\mu\omega_1} & (\mu^3\omega_2^3 - k\mu^2)\, e^{\mu\omega_2} & (\mu^3\omega_3^3 - k\mu^2)\, e^{\mu\omega_3} & (\mu^3\omega_4^3 - k\mu^2)\, e^{\mu\omega_4} \\ \mu^2\omega_1^2 e^{\mu\omega_1} & \mu^2\omega_2^2 e^{\mu\omega_2} & \mu^2\omega_3^2 e^{\mu\omega_3} & \mu^2\omega_4^2 e^{\mu\omega_4} \\ \mu\omega_1 & \mu\omega_2 & \mu\omega_3 & \mu\omega_4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Hence the characteristic determinant of the problem is

$$\Delta(\mu) = \mu^5 \det \begin{bmatrix} (\mu\omega_1^3 - k)\, e^{\mu\omega_1} & (\mu\omega_2^3 - k)\, e^{\mu\omega_2} & (\mu\omega_3^3 - k)\, e^{\mu\omega_3} & (\mu\omega_4^3 - k)\, e^{\mu\omega_4} \\ \omega_1^2 e^{\mu\omega_1} & \omega_2^2 e^{\mu\omega_2} & \omega_3^2 e^{\mu\omega_3} & \omega_4^2 e^{\mu\omega_4} \\ \omega_1 & \omega_2 & \omega_3 & \omega_4 \\ 1 & 1 & 1 & 1 \end{bmatrix} =$$

$$= \mu^5 \left\{ \mu \left[ e^{\mu j\sqrt{2}}(-2) + e^{\mu\sqrt{2}}(-2) + e^{-\mu\sqrt{2}}(-2) + e^{-\mu j\sqrt{2}}(-2) - 8 \right] - \right.$$
$$\left. -k \left[ 2\sqrt{2} j e^{j\mu\sqrt{2}} + 2\sqrt{2} e^{\mu\sqrt{2}} - 2\sqrt{2} e^{-\mu\sqrt{2}} - 2\sqrt{2} j e^{-j\mu\sqrt{2}} \right] \right\} =$$
$$= \mu^6 \left\{ -2e^{j\sqrt{2}\mu} - 2e^{\mu\sqrt{2}} - 2e^{-\mu\sqrt{2}} - 2e^{-\mu j\sqrt{2}} - 8 - \right.$$
$$\left. -\frac{k}{\mu} \left[ 2\sqrt{2} j e^{j\mu\sqrt{2}} + 2\sqrt{2} e^{\mu\sqrt{2}} - 2\sqrt{2} e^{-\mu\sqrt{2}} - 2j\sqrt{2} e^{-j\mu\sqrt{2}} \right] \right\} =$$
$$= \mu^6 \left\{ e^{j\mu\sqrt{2}} \left[ -2 - \frac{2k\sqrt{2}j}{\mu} \right] + e^{-\mu\sqrt{2}} \left[ -2 + \frac{2k\sqrt{2}}{\mu} \right] + \right.$$
$$\left. + e^{\mu\sqrt{2}} \left[ -2 - \frac{2k\sqrt{2}}{\mu} \right] + e^{-j\mu\sqrt{2}} \left[ -2 + \frac{2k\sqrt{2}j}{\mu} \right] - 8 \right\}.$$

The dominating terms in square brackets are $(-2)$ and thus the problem (5.3) is regular. Notice that $\mu$ is a root of the characteristic equation

$$\mu^5 \left\{ \mu \left[ -2e^{\mu j\sqrt{2}} - 2e^{\mu\sqrt{2}} - 2e^{-\mu\sqrt{2}} - 2e^{-j\mu\sqrt{2}} - 8 \right] - \right.$$
$$\left. -2\sqrt{2}k \left[ j e^{j\mu\sqrt{2}} + e^{\mu\sqrt{2}} - e^{-\mu\sqrt{2}} - j e^{-j\mu\sqrt{2}} \right] \right\} = 0$$

iff $\mu$ is a root of the equation

$$\mu \left[ e^{j\mu\sqrt{2}} + e^{\mu\sqrt{2}} + e^{-\mu\sqrt{2}} + e^{-j\mu\sqrt{2}} + 4 \right] +$$
$$+ \sqrt{2}k \left[ j e^{j\mu\sqrt{2}} + e^{\mu\sqrt{2}} - e^{-\mu\sqrt{2}} - j e^{-j\mu\sqrt{2}} \right] = 0$$

or, equivalently, iff $\omega = \dfrac{1-j}{\sqrt{2}}\mu$ satisfies the equation

$$F(\omega) = \omega[1 + \cos\omega \cosh\omega] - jk[\cos\omega \sinh\omega - \cosh\omega \sin\omega] = 0.$$

Writing down this equation in the form

$$\omega\cosh\omega\left\{\left[\frac{1}{\cosh\omega}+\cos\omega\right]-jk\left[\frac{\cos\omega}{\omega}\frac{\sinh\omega}{\cosh\omega}-\frac{\sin\omega}{\omega}\right]\right\}=0$$

we can see that the asymptotic roots of $F$ are real and they approximately coincide with roots of the equation $\cos\omega = 0$. This is because

$$\lim_{|\omega|\to\infty,\ \omega\in\mathbb{R}}\frac{1}{\cosh\omega}=0,\qquad \lim_{|\omega|\to\infty,\ \omega\in\mathbb{R}}\frac{\cos\omega}{\omega}=0,\qquad \lim_{|\omega|\to\infty,\ \omega\in\mathbb{R}}\frac{\sin\omega}{\omega}=0.$$

Hence $\omega_n \approx \frac{\pi}{2}+n\pi$, $n\in\mathbb{Z}$. Substituting $\eta = \frac{1+j}{\sqrt{2}}\mu$ and repeating the analysis above one gets

$$G(\eta)=\eta\left[1+\cos\eta\cosh\eta\right]+jk\left[\cos\eta\sinh\eta-\cosh\eta\sin\eta\right]=0.$$

Hence again $\eta_n = \frac{\pi}{2}+n\pi$, $n\in\mathbb{Z}$. Finally the asymptotic roots are distributed as shown at Figure 7. The roots of the characteristic equation are asymptotically equally spaced.

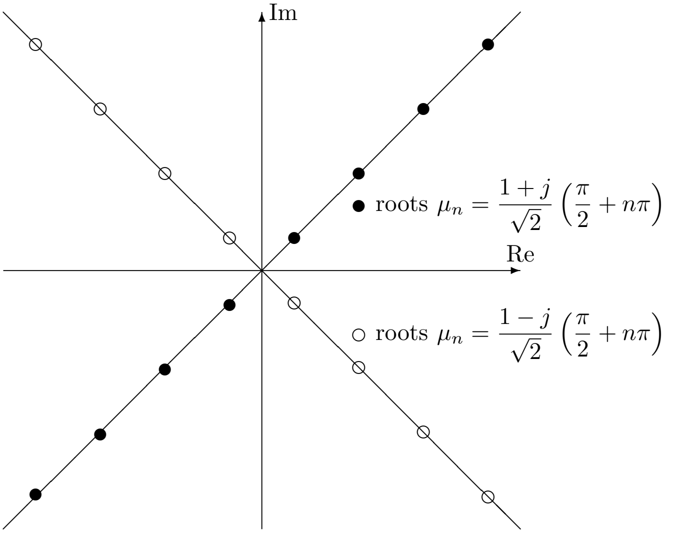


**Fig. 7.** Distribution of asymptotic roots

Observe that

$$F'(\omega)=1+\cos\omega\cosh\omega+\omega[\cos\omega\sinh\omega-\sin\omega\cosh\omega]+2jk\sin\omega\sinh\omega.$$

The nonsimple roots can exist iff

$$\left\{\begin{array}{c} F(\omega)=0\\ F'(\omega)=0\end{array}\right\}. \tag{5.5}$$

Now (5.5) is equivalent to the system

$$\begin{bmatrix} 1+\cos\omega\cosh\omega & -j(\cos\omega\sinh\omega-\cosh\omega\sin\omega) \\ \cos\omega\sinh\omega-\cosh\omega\sin\omega & 2j\sin\omega\sinh\omega \end{bmatrix}\begin{bmatrix}\omega\\ k\end{bmatrix}= \\ =\begin{bmatrix} 0 \\ -1-\cos\omega\cosh\omega\end{bmatrix}$$

which yields

$$\omega=\frac{\cosh^2\omega\left(\dfrac{1}{\cosh\omega}+\cos\omega\right)(\sin\omega-\tanh\omega\cos\omega)}{\sinh^2\omega\left(1+\dfrac{\sin\omega}{\sinh\omega}\right)^2}.$$

The right-hand side is bounded for large $|\omega|$ or even tends to zero on $\omega_n$ (as then $\cos\omega_n \longrightarrow 0$). Hence, the above equation cannot have real roots of large moduli. Consequently, the characteristic function of the problem cannot have asymptotic roots which are nonsingle. Therefore the problem (5.3) is strictly regular.

By Theorem 3.1 Shkalikov linearization of our boundary-value problem has a system of generalized eigenvectors which constitutes a Riesz basis of the Shkalikov space $W^0_{2,U}$ as all boundary condition are of order $\le n+r-1=3$.

For further investigations we represent $\ell(u,\mu)$ in the form (3.5),

$$\ell(u,\mu)=\ell_0(u)+\mu\ell_1(u)+\mu^2\ell_2(u)+\mu^3\ell_3(u)+\mu^4 u$$

where $\ell_0(u)=u''''$, $\ell_1(u)=\ell_2(u)=\ell_3(u)=0$. Here $n=4$, $r=0$ and thus

$$W^0_2 := \mathrm{H}^3(0,1)\oplus\mathrm{H}^2(0,1)\oplus\mathrm{H}^1(0,1)\oplus\mathrm{L}^2(0,1),$$

$$\tilde v=\begin{bmatrix}v_0\\ v_1\\ v_2\\ v_3\end{bmatrix},\quad H\begin{bmatrix}v_0\\ v_1\\ v_2\\ v_3\end{bmatrix}=\begin{bmatrix}v_1\\ v_2\\ v_3\\ -v_0''''\end{bmatrix},\quad H^2\begin{bmatrix}v_0\\ v_1\\ v_2\\ v_3\end{bmatrix}=\begin{bmatrix}v_2\\ v_3\\ -v_0''''\\ -v_1''''\end{bmatrix},\quad H^3\begin{bmatrix}v_0\\ v_1\\ v_2\\ v_3\end{bmatrix}=\begin{bmatrix}v_3\\ -v_0''''\\ -v_1''''\\ -v_2''''\end{bmatrix}.$$

The results of interchanging terms in the boundary conditions according to the rule (3.6) are presented in Table 3.

The resulting boundary conditions are

$$\tilde U_1(\tilde v,\mu)=v_0'''(1)-kv_2(1),\quad \tilde U_2(\tilde v,\mu)=v_0''(1),\quad \tilde U_3(\tilde v,\mu)=v_0'(0),\quad \tilde U_4(\tilde v,\mu)=v_0(0),$$

and consequently $\nu_1=\nu_2=\nu_3=\nu_4=0$, $q=0$, $N_0=0$. By (3.8) the state space is

$$W^0_{2,U}=\left\{\begin{bmatrix}v_0\\ v_1\\ v_2\\ v_3\end{bmatrix}\in \mathrm{H}^3(0,1)\oplus\mathrm{H}^2(0,1)\oplus\mathrm{H}^1(0,1)\oplus\mathrm{L}^2(0,1):\right. \\ \left. v_0(0)=0,\ v_0'(0)=0,\ v_0''(1)=0,\ v_1(0)=0,\ v_1'(0)=0,\ v_2(0)=0\right\}.$$

**Table 3.** Results of applying the rule (3.6)

| Original term | $\nu$ | $k$ | $\nu+k<4+r$ | Replaced term |
|---|---|---|---|---|
| $u'''(1)$ | 0 | 3 | yes | $v_0'''(1)$ |
| $\mu^2 u(1)$ | 2 | 0 | yes | $v_2(1)$ |
| $u''(1)$ | 0 | 2 | yes | $v_0''(1)$ |
| $u'(0)$ | 0 | 1 | yes | $v_0'(0)$ |
| $u(0)$ | 0 | 0 | yes | $v_0(0)$ |

Indeed, if $k=0$ then the boundary conditions of order less than or equal to 2 should be taken into account, i.e.,

$$\tilde{U}_4(\tilde{v}) = v_0(0) = 0, \qquad \tilde{U}_3(\tilde{v}) = v_0'(0) = 0, \qquad \tilde{U}_2(\tilde{v}) = v_0''(1) = 0.$$

If $k=1$ then the boundary conditions of order less than or equal to 1 should be encountered into the definition of the state space, i.e.,

$$\tilde{U}_4(H^1\tilde{v}) = v_1(0) = 0, \qquad \tilde{U}_3(H^1\tilde{v}) = v_1'(0) = 0.$$

If $k=2$ then only null order boundary conditions should be taken into account, i.e.,

$$\tilde{U}_4(H^2\tilde{v}) = v_2(0) = 0.$$

From (3.10) and (3.8) we determine the domain of $H_0$,

$$\begin{aligned} D(H_0) = \quad & W^1_{2,U} = \Big\{ \tilde{v} \in \mathrm{H}^4(0,1) \oplus \mathrm{H}^3(0,1) \oplus \mathrm{H}^2(0,1) \oplus \mathrm{H}^1(0,1) : \\ & \tilde{U}_j(H^k\tilde{v}) = 0 \quad \text{for} \quad 0 \le k \le 3 \quad \text{and all boundary conditions} \\ & \text{of order less than or equal to } \ 3-k \Big\}. \end{aligned}$$

If $k=0$ then the boundary conditions of order less than or equal to 3 should be taken into account, i.e.,

$$\tilde{U}_1(\tilde{v}) = v_0'''(1) - k v_2(1) = 0,\ \tilde{U}_2(\tilde{v}) = v_0''(1) = 0,\ \tilde{U}_3(\tilde{v}) = v_0'(0) = 0,\ \tilde{U}_4(\tilde{v}) = v_0(0) = 0.$$

If $k=1$ then the boundary conditions of order less than or equal to 2 should be encountered, i.e.,

$$\tilde{U}_2(H\tilde{v}) = v_1''(1) = 0, \qquad \tilde{U}_3(H\tilde{v}) = v_1'(0) = 0, \qquad \tilde{U}_4(H\tilde{v}) = v_1(0) = 0.$$

If $k=2$ then the bounadry conditions of order less than or equal to 1 should be considered, i.e.,

$$\tilde{U}_3(H^2\tilde{v}) = v_2'(0) = 0, \qquad \tilde{U}_4(H^2\tilde{v}) = v_2(0) = 0.$$

If $k=3$ the boundary conditions of null order should be encountered, which yields, $\tilde{U}_4(H^3\tilde{v}) = v_3(0) = 0$. Finally we obtain

$$D(H_0) = W^1_{2,U} = \left\{ \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathrm{H}^4(0,1) \oplus \mathrm{H}^3(0,1) \oplus \mathrm{H}^2(0,1) \oplus \mathrm{H}^1(0,1) : \right.$$

$$\left. \begin{array}{c} v_0(0)=0,\ v_1(0)=0,\ v_2(0)=0,\ v_3(0)=0,\ v_0'(0)=0, \\ v_1'(0)=0,\ v_2'(0)=0,\ v_0''(1)=0,\ v_1''(1)=0, v_0'''(1)-kv_2(1)=0 \end{array} \right\}.$$

Since all boundary conditions are of order less than or equal to $n+r-1=3$ the second assertion of Theorem 3.1 applies. The problem (5.3) is strictly regular, which implies the existence of a system of generalized eigenvectors which constitutes a Riesz basis in the space $W^0_{2,U}$.

Since we made the substitution $\lambda = \mu^2$ the question is whether the above results apply to the original problem (5.2). Recall that $\mu$ is an eigenvalue of $H_0$, so $\lambda = \mu^2$ is an eigenvalue of $H_0^2$ while eigenvectors correspondig to these eigenvalues are the same. Thus eigenvectors of $H_0^2$ still generate a Riesz basis for $W^0_{2,U}$. This suggests to look at $H_0^2$,

$$H_0^2\tilde{v} = H^2\tilde{v} = \begin{bmatrix} v_2 \\ v_3 \\ -v_0'''' \\ -v_1'''' \end{bmatrix}, \quad D(H_0^2) = \{\tilde{v} \in W^1_{2,U} : \ H_0\tilde{v} \in D(H_0)\} =$$

$$= \left\{ \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathrm{H}^5(0,1) \oplus \mathrm{H}^4(0,1) \oplus \mathrm{H}^3(0,1) \oplus \mathrm{H}^2(0,1) : \right.$$

$$\left. \begin{array}{c} \text{all boundary conditions defining } D(H_0) \text{ are satisfied and additionally:} \\ v_1'''(1)-kv_3(1)=0, \quad v_2''(1)=0, \quad v_3'(0)=0, \quad v_0''''(0)=0 \end{array} \right\}.$$

Now observe that $H_0^2$ decomposes into two operators. One of them,

$$H_B \begin{bmatrix} v_1 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_3 \\ -v_1'''' \end{bmatrix}, \quad D(H_B) = \left\{ \begin{bmatrix} v_1 \\ v_3 \end{bmatrix} \in \mathrm{H}^4(0,1) \oplus \mathrm{H}^2(0,1) : \right.$$

$$\left. v_1''(1)=0,\ v_3(0)=0,\ v_3'(0)=0,\ v_1'''-kv_3(1)=0 \right\}$$

acts in the state space

$$\mathrm{X}_B := \left\{ \begin{bmatrix} v_1 \\ v_3 \end{bmatrix} \in \mathrm{H}^2(0,1) \oplus \mathrm{L}^2(0,1) : \; v_1(0) = 0, \; v_1'(0) = 0 \right\}.$$

Chen *at al.* [4], Conrad [5] and Rebarber [30] have used this space as the underlying Hilbert space for the problem. The suboperator $H_B$ is a part of $H_0^2$ in $\mathrm{X}_B$ being an invariant subspace of $W^0_{2,U}$. Thus $H_B$ also possesses a system of generalized eigenvectors which forms a Riesz basis of the subspace $\mathrm{X}_B$. It is very easy to see that the eigenproblem for $H_B$ is equivalent to the eigenproblem for our original elastic system (5.2).

Rideau [31, pp. 73, 76] has established the following asymptotic formulae for eigenvalues of the system (5.1)

$$\lambda_n = \left\{ \begin{array}{lr} \left( -\frac{\alpha}{m} + \frac{\beta\gamma}{m\delta} - \frac{EI}{\delta} \right) + ja^2 \left[ \frac{\beta+\gamma}{\delta} + \left( n\pi - \frac{\pi}{4} \right)^2 \right] + O\left(\frac{1}{n}\right) & \text{if} \quad \delta \neq 0 \\ -2\frac{EI\alpha}{mEI - \beta\gamma} + ja^2 \left( n\pi + \frac{\pi}{2} \right)^2 + \frac{2jmEI\alpha^2 a^2}{\pi(mEI-\beta\gamma)^2} \frac{1}{n} + O\left(\frac{1}{n^2}\right) & \text{if} \quad \delta = 0 \end{array} \right\}.$$

In particular, for system (5.2) this yields

$$\lambda_n = -2k \pm j \left( n\pi + \frac{\pi}{2} \right)^2, \qquad n \in \mathbb{N}.$$

The above formulae have been derived, using first and second order approximation of the asymptotic zeros of $F(\omega)$.

By Theorem 1.2, $H_B$ generates a group on $\mathrm{X}_B$. Moreover only finitely many of generalized eigenvectors are not eigenvectors of $H_B$.

Comparying our results with those of [5] one can see that here we proved, using Shalikov's theory, that the operator describing the closed-loop feedback system possesses a system of generalized eigenvectors which forms a Riesz basis for all $k > 0$, i.e., not only for sufficiently small $k > 0$ as it was established by Conrad [5]. Conrad used the concept of the quadratic closeness for the system of generalized eigenvectors, i.e., he has proved that they create a Bari basis. In that sense, our result is weaker, however the Riesz basis property holds for all $k > 0$. Rebarber [30] proved the existence of a Bari basis for sufficiently small $k > 0$ under the assumptions that there is a structural damping term in the system. In the recent paper Conrad and Morgül [6, Theorem 3] improved the technique of the quadratic closeness used in [5] to show that there exists a system of eigenvectors which forms a Bari basis for almost all $k > 0$. Recall that the Bari basis property is equivalent to the existence of a similarity transformation of the semigroup generator to a normal operator, in the class of isomorphisms of the form "identity plus a Hilbert–Schmidt operator" [9].

## 5.2. THE TIMOSHENKO BEAM

In this section we shall discuss the Timoshenko beam equations from the viewpoint of the Shkalikov theory,

$$\left\{\begin{array}{rr} \rho w_{tt} - K w_{xx} + K\phi_x = 0, & 0 \le x \le 1,\ t \ge 0 \\ I_\rho \phi_{tt} - EI\phi_{xx} + K\phi - K w_x = 0, & 0 \le x \le 1,\ t \ge 0 \\ w(0,t) = 0, & t \ge 0 \\ \phi(0,t) = 0, & t \ge 0 \\ K\phi(1,t) - K w_x(1,t) - \alpha w_t(1,t) = 0, & t \ge 0 \\ EI\phi_x(1,t) + \beta\phi_t(1,t) = 0, & t \ge 0 \end{array}\right\}$$

$E$, $I$, $K$, $I_\rho$, $\rho > 0$ are physical constants, $\alpha$, $\beta$ are assumed to be positive parameters.

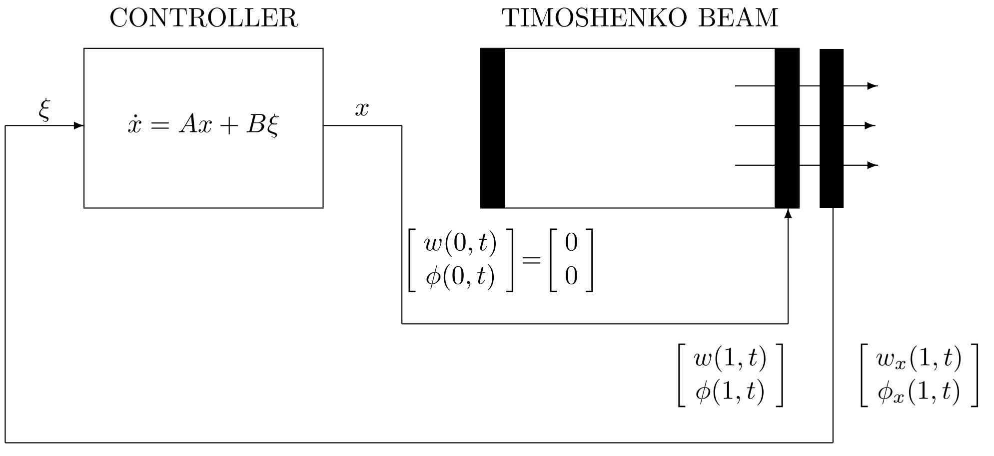


**Fig. 8.** The boundary control feedback system of the Timoshenko beam

The block diagram of this boundary control feedback system is depicted in Figure 8. In the latter case we have:

$$x = \begin{bmatrix} w(1,t) \\ \phi(1,t) \end{bmatrix}, \qquad \xi = \begin{bmatrix} w_x(1,t) \\ \phi_x(1,t) \end{bmatrix},$$

$$A = \begin{bmatrix} 0 & \dfrac{K}{\alpha} \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} -\dfrac{K}{\alpha} & 0 \\ 0 & -\dfrac{EI}{\beta} \end{bmatrix}.$$

The first order dynamics is

$$
\left\{
\begin{array}{ll}
w_t = z, & 0 \le x \le 1,\ t \ge 0 \\
z_t = \dfrac{1}{d^2} w_{xx} - \dfrac{1}{d^2} \phi_x, & 0 \le x \le 1,\ t \ge 0 \\
\phi_t = \psi, & 0 \le x \le 1,\ t \ge 0 \\
\psi_t = \dfrac{1}{a^2} \phi_{xx} - \dfrac{K}{I_\rho} \phi + \dfrac{K}{I_\rho} w_x, & 0 \le x \le 1,\ t \ge 0 \\
\phi(1,t) - w_x(1,t) - g z(1,t) = 0, & t \ge 0 \\
w(0,t) = 0, & t \ge 0 \\
\phi_x(1,t) + b\psi(1,t) = 0, & t \ge 0 \\
\phi(0,t) = 0, & t \ge 0
\end{array}
\right\}
\tag{5.6}
$$

where $d = \sqrt{\dfrac{\rho}{K}}$, $a = \sqrt{\dfrac{I_\rho}{EI}}$, $b = \dfrac{\beta}{EI}$, $g = \dfrac{\alpha}{K}$. To simplify the analysis of the corresponding eigenproblem we neglect the boxed terms

$$
\left\{
\begin{array}{c}
z = \lambda w \\
\dfrac{1}{d^2} w'' \boxed{-\dfrac{1}{d^2} \phi'} = \lambda z = \lambda^2 w \\
\psi = \lambda \phi \\
\dfrac{1}{d^2} \phi'' \boxed{-\dfrac{K}{I_\rho} \phi + \dfrac{K}{I_\rho} w'} = \lambda \psi = \lambda^2 \phi \\
w(0) = 0 \\
\phi(0) = 0 \\
\phi(1) - w'(1) - g z(1) = \phi(1) - w'(1) - g\lambda w(1) = 0 \\
\phi'(1) + b\psi(1) = \phi'(1) + b\lambda\phi(1) = 0
\end{array}
\right\}
$$

getting the simplified eigenproblem

$$
\left\{
\begin{array}{c}
z = \lambda w \\
w'' = \lambda^2 d^2 w \\
\psi = \lambda \phi \\
\boxed{\begin{array}{r}
\phi' = \lambda^2 a^2 \phi \\
\phi(0) = 0 \\
\phi'(1) + b\lambda\phi(1) = 0
\end{array}} \\
w(0) = 0 \\
\phi(1) - w'(1) - g\lambda w(1) = 0
\end{array}
\right\}.
$$

The following two cases are possible:

1° $\lambda$ is not an eigenvalue of the boxed subproblem. Then $\phi \equiv 0$ and consequently $\psi \equiv 0$. Now

$$\left\{\begin{array}{r} w'' = \lambda^2 d^2 w \\ w(0) = 0 \\ w'(1) + g\lambda w(1) = 0 \end{array}\right\}. \tag{5.7}$$

Assuming a solution in the form $w(x) = C_1 e^{-\lambda dx} + C_2 e^{\lambda dx}$ we find the characteristic determinant of the problem $\lambda\left\{(g+d)e^{\lambda d} + (d-g)e^{-\lambda d}\right\} = 0$. If

$$g + d \neq 0, \qquad d - g \neq 0 \tag{5.8}$$

then the problem (5.7) is regular. In this case the characteristic equation can equivalently be written as $\dfrac{g-d}{g+d} = e^{2\lambda}$. Its roots,

$$\lambda_n^d = \left\{\begin{array}{ll} \dfrac{1}{d}\ln\sqrt{\dfrac{g-d}{g+d}} + j\dfrac{n\pi}{d}, & \text{if} \quad g^2 > d^2 \\ \dfrac{1}{d}\ln\sqrt{\dfrac{d-g}{d+g}} + j\left(\dfrac{n\pi}{d} + \dfrac{\pi}{2d}\right), & \text{otherwise} \end{array}\right\}, \qquad n \in \mathbb{Z}$$

are simple and uniformly separated. They create the $d$-series of eigenvalues of the simplified eigenproblem. By Theorem 3.1 the system of corresponding eigenvectors

$$\left\{\begin{array}{ll} w_n^d(x) = \dfrac{1}{\lambda_n^d}\sinh\lambda_n^d x, & 0 \leq x \leq 1 \\ z_n^d(x) = \sin\lambda_n^d x, & 0 \leq x \leq 1 \end{array}\right\}, \qquad n \in \mathbb{Z} \tag{5.9}$$

forms a Riesz basis in $\mathrm{H}_0^1(0,1) \oplus \mathrm{L}^2(0,1)$ where $\mathrm{H}_0^1(0,1) := \left\{\mathrm{H}^1(0,1) : \ u(0) = 0\right\}$.

2° $\lambda$ is an eigenvalue of the boxed subproblem. Then $\phi$ does not vanish identically and $\phi$ is an eigenfunction satisfying the system of equations

$$\left\{\begin{array}{r} \phi'' = \lambda^2 a^2 \phi \\ \phi(0) = 0 \\ \phi'(1) + b\lambda\phi(1) = 0 \end{array}\right\}.$$

Repeating the analysis done in step 1° we obtain the $a$-series of eigenvalues

$$\lambda_n^a = \left\{\begin{array}{ll} \dfrac{1}{a}\ln\sqrt{\dfrac{b-a}{b+a}} + j\dfrac{n\pi}{a}, & \text{if} \quad b^2 > a^2 \\ \dfrac{1}{a}\ln\sqrt{\dfrac{a-b}{a+b}} + j\left(\dfrac{n\pi}{a} + \dfrac{\pi}{2a}\right), & \text{otherwise} \end{array}\right\}, \qquad n \in \mathbb{Z}$$

of the simplified eigenproblem.

Hence, again by the Shkalikov's theory, the system of eigenvectors

$$\left\{\begin{array}{ll} \phi_n^a(x) = \dfrac{1}{\lambda_n^a}\sinh\lambda_n^a a x, & 0 \leq x \leq 1 \\ \psi_n^a(x) = \sinh\lambda_n^a a x, & 0 \leq x \leq 1 \end{array}\right\}, \qquad n \in \mathbb{Z} \tag{5.10}$$

is a Riesz basis in $\mathrm{H}_0^1(0,1) \oplus \mathrm{L}^2(0,1)$ provided that

$$b + a \neq 0, \qquad b - a \neq 0. \tag{5.11}$$

Notice that the systems (5.9), (5.10) are quasinormalized. This is due to the uniform boundedness of $\cosh z$, $\sinh z$ and holds in any vertical strip parallel to $j\,\mathbb{R}$. Now

$$\left\{ \begin{array}{ll} w_n^a(x) = D\dfrac{1}{\lambda_n^a}\sinh \lambda_n^a dx, & 0 \leq x \leq 1 \\ z_n^a(x) = D\sinh \lambda_n^a dx, & 0 \leq x \leq 1 \end{array} \right\}, \qquad n \in \mathbb{Z},$$

and $\phi_n^a(1) - \left.\dfrac{d[w_n^a(x)]}{dx}\right|_{x=1} - g\lambda_n^a w_n^a(1) = 0$ or equivalently, $\dfrac{1}{\lambda_n^a}\sinh \lambda_n^a a - Dd\cosh \lambda_n^a d - Dg\sinh \lambda_n^a d = 0$ which gives

$$D_n = \frac{\sinh \lambda_n^a}{\lambda_n^a\left[d\cosh \lambda_n^a d + g\sinh \lambda_n^a d\right]}.$$

**Remark 5.1.**

$$d\cosh \lambda_n^a d + g\sinh \lambda_n^a d = \frac{(d+g)e^{\lambda_n^a d} + (d-g)e^{-\lambda_n^a d}}{2} = 0$$

*iff* $\lambda_n^a$ *coincides with* $\lambda_n^d$, $n \in \mathbb{Z}$.

Finally,

$$\left\{ \begin{array}{lll} w_n^a(x) = \dfrac{\sinh \lambda_n^a}{(\lambda_n^a)^2\left[d\cosh \lambda_n^a d + g\sinh \lambda_n^a d\right]}\sinh \lambda_n^a\, dx, & 0 \leq x \leq 1 \\ z_n^a(x) = \dfrac{\sinh \lambda_n^a}{\lambda_n^a\left[d\cosh \lambda_n^a d + g\sinh \lambda_n^a d\right]}\sinh \lambda_n^a\, dx, & 0 \leq x \leq 1 \end{array} \right\}, \quad n \in \mathbb{Z}\,.$$

We shall show that this system can be represented in the form

$$\begin{bmatrix} w_n^a \\ z_n^a \end{bmatrix} = \mathcal{L}\begin{bmatrix} \phi_n^a \\ \psi_n^a \end{bmatrix}, \tag{5.12}$$

with $\mathcal{L} \in \mathbf{L}(\mathrm{X})$, $\mathrm{X} = \mathrm{H}_0^1(0,1) \oplus \mathrm{L}^2(0,1)$, $\mathrm{H}_0^1(0,1) = \{u \in \mathrm{H}^1(0,1) : \; u(0) = 0\}$.

**Lemma 5.1.**

$$\left[(d+g)T(d) + (d-g)T(-d)\right]^{-1}\begin{bmatrix} \phi_n^a \\ \psi_n^a \end{bmatrix} = \frac{1}{2\left[d\cosh \lambda_n^a d + g\sinh \lambda_n^a d\right]}\begin{bmatrix} \phi_n^a \\ \psi_n^a \end{bmatrix} \tag{5.13}$$

*where* $\{T(t)\}_{t\in\mathbb{R}}$ *denotes the* $\mathrm{C}_0$*-group generated on* X *by the operator*

$$A\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} v \\ \dfrac{1}{a^2}u'' \end{bmatrix},$$

$$D(A) = \left\{ \begin{bmatrix} u \\ v \end{bmatrix} \in \mathrm{H}^2(0,1) \oplus \mathrm{H}^1(0,1) : \; v \in \mathrm{H}_0^1(0,1), \; u'(1) + bv(1) = 0 \right\}.$$

*Proof.* Since $A$ possesses a system of eigenvectors which forms a Riesz basis in X and the spectrum is located in a vertical strip parallel to $j\,\mathbb{R}$, $A$ generates the $C_0$-group $\{T(t)\}_{t\in\mathbb{R}}$ on X. Moreover,

$$T(t)\begin{bmatrix}\phi_n^a\\ \psi_n^a\end{bmatrix} = e^{t\lambda_n^a}\begin{bmatrix}\phi_n^a\\ \psi_n^a\end{bmatrix}, \qquad n\in\mathbb{Z}, \quad t\in\mathbb{R}\,.$$

To $A$, being similar to a normal operator on X, we can apply the functional calculus getting $[(d+g)T(d)+(d-g)T(-d)]^{-1}$. If we consider the function

$$\lambda \longmapsto \frac{1}{(d+g)e^{d\lambda}+(d-g)e^{-d\lambda}}$$

then replacing $\lambda$ by $A$ one obtains (5.13). □

The same arguments lead to the next lemma.

**Lemma 5.2.**

$$[T(dx)-T(-dx)]\begin{bmatrix}\phi_n^a\\ \psi_n^a\end{bmatrix} = \left(e^{d\lambda_n^a x}-e^{-d\lambda_n^a x}\right)\begin{bmatrix}\phi_n^a\\ \psi_n^a\end{bmatrix} = 2\sinh\lambda_n^a dx\begin{bmatrix}\phi_n^a\\ \psi_n^a\end{bmatrix}.$$

Now we are in position to define $\mathcal{L}$ precisely.

**Lemma 5.3.** *Let*

$$\mathcal{L}\begin{bmatrix}\phi\\ \psi\end{bmatrix}(x) = \begin{bmatrix} r[T(dx)-T(-dx)]W^{-1}\left(A^{-1}\begin{bmatrix}\phi\\ \psi\end{bmatrix}\right)\\ r[T(dx)-T(-dx)]W^{-1}\begin{bmatrix}\phi\\ \psi\end{bmatrix}\end{bmatrix}, \qquad 0\le x\le 1$$

*where* $W := (d+g)T(d)+(d-g)T(-d)\in\mathbf{L}(\mathrm{X})$ *and* $r$ *stands for the projections of* X *onto* $\mathrm{L}^2(0,1)$. *Then* (5.12) *holds.*

*Proof.* By Lemmas 5.1, 5.2 and the functional calculus for $A$ we get

$$\begin{bmatrix}\phi_n^a\\ \psi_n^a\end{bmatrix} \xmapsto{A^{-1}} \frac{1}{\lambda_n^a}\begin{bmatrix}\phi_n^a\\ \psi_n^a\end{bmatrix} \xmapsto{W^{-1}} \frac{1}{\lambda_n^a 2\,[d\cosh\lambda_n^a d+g\sinh\lambda_n^a d]}\begin{bmatrix}\phi_n^a\\ \psi_n^a\end{bmatrix} \xmapsto{T(dx)-T(-dx)}$$

$$\xmapsto{T(dx)-T(-dx)} \frac{2\sinh\lambda_n^a dx}{2\lambda_n^a\,[d\cosh\lambda_n^a d+g\sinh\lambda_n^a d]}\begin{bmatrix}\phi_n^a\\ \psi_n^a\end{bmatrix} \xmapsto{r}$$

$$\xmapsto{r} \frac{\sinh\lambda_n^a}{(\lambda_n^a)^2\,[d\cosh\lambda_n^a+g\sinh\lambda_n^a d]}\sinh\lambda_n^a dx = w_n^a(x).$$

For the second component the arguments are the same. □

**Lemma 5.4.** *The operator* $\mathcal{L}$ *is bounded, i.e.,* $\mathcal{L}\in\mathbf{L}(\mathrm{X})$.

*Proof.* The operators $W$ and $A$ commute. Hence $W^{-1}$, $A^{-1}$ commute too. This implies that $W^{-1}A^{-1}\begin{bmatrix}\phi\\\psi\end{bmatrix} \in D(A)$. Hence $x \longmapsto r\,[T(dx) - T(-dx)]\,W^{-1}A^{-1}\begin{bmatrix}\phi\\\psi\end{bmatrix}$ is continuously differentiable as $\{T(t)\}_{t\in\mathbb{R}}$ is a $\mathrm{C}_0$-group on X. We proved that the first component of $\mathcal{L}\begin{bmatrix}\phi\\\psi\end{bmatrix}$ is in $\mathrm{H}^1(0,1)$. Actually, the mapping $\begin{bmatrix}\phi\\\psi\end{bmatrix} \longmapsto$ first component of $\mathcal{L}\begin{bmatrix}\phi\\\psi\end{bmatrix}$ is continuous by continuity of all operators defining that component, and by continuity of its derivative with respect to x. For the second component of $\mathcal{L}\begin{bmatrix}\phi\\\psi\end{bmatrix}$ the arguments are similar. □

From the above analysis we know that the system $\left\{\begin{bmatrix}w_n^d\\z_n^d\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\\phi_n^a\\\psi_n^a\end{bmatrix}\right\}_{n\in\mathbb{Z}}$ forms a Riesz basis on $\mathrm{X}\oplus\mathrm{X}$, provided that (5.8), (5.11) and

$$\inf_{n\in\mathbb{Z}} |d\cosh\lambda_n^a d + g\sin\lambda_n^a d| > 0 \tag{5.14}$$

hold. The condition (5.14) ensures invertibility of $W$ and it geometrically means that $a$-series and $d$-series of eigenvalue are strictly isolated. Now observe that

$$\begin{bmatrix}w_n^d\\z_n^d\\0\\0\end{bmatrix} = \begin{bmatrix}I & \mathcal{L}\\0 & I\end{bmatrix}\begin{bmatrix}w_n^d\\z_n^d\\0\\0\end{bmatrix}, \qquad \begin{bmatrix}w_n^a\\z_n^a\\\phi_n^a\\\psi_n^a\end{bmatrix} = \begin{bmatrix}\mathcal{L}\begin{bmatrix}\phi_n^a\\\psi_n^a\end{bmatrix}\\\phi_n^a\\\psi_n^a\end{bmatrix} = \begin{bmatrix}I & \mathcal{L}\\0 & I\end{bmatrix}\begin{bmatrix}0\\0\\\phi_n^a\\\psi_n^a\end{bmatrix}.$$

Since the matrix of operators $\left[\begin{smallmatrix}I & \mathcal{L}\\0 & I\end{smallmatrix}\right]$ belongs to $\mathbf{L}(\mathrm{X}\oplus\mathrm{X})$ and is boundedly invertible (the inverse is the matrix of operators $\left[\begin{smallmatrix}I & -\mathcal{L}\\0 & I\end{smallmatrix}\right]$), the system $\left\{\begin{bmatrix}w_n^d\\z_n^d\\0\\0\end{bmatrix}, \begin{bmatrix}w_n^a\\z_n^a\\\phi_n^a\\\psi_n^a\end{bmatrix}\right\}_{n\in\mathbb{Z}}$ is also a Riesz in $\mathrm{X}\oplus\mathrm{X}$. However, the last system is a system of eigenvectors of the operator

$$\Lambda_0\begin{bmatrix}w\\z\\\phi\\\psi\end{bmatrix} = \begin{bmatrix}z\\\dfrac{1}{d^2}w''\\\psi\\\dfrac{1}{a^2}\phi''\end{bmatrix}, \qquad D(\Lambda_0) = \left\{\begin{bmatrix}w\\z\\\phi\\\psi\end{bmatrix} \in \mathrm{X}\oplus\mathrm{X} : \right.$$

$$\left. w,\phi\in\mathrm{H}^2(0,1),\ z,\psi\in\mathrm{H}_0^1(0,1),\ \phi(1)-w'(1)-gz(1)=0,\ \phi'(1)+b\psi(1)=0 \right\}.$$

The system operator corresponding to (5.6) is representable as

$$\Lambda\begin{bmatrix}w\\z\\\phi\\\psi\end{bmatrix} = \Lambda_0\begin{bmatrix}w\\z\\\phi\\\psi\end{bmatrix} + \Lambda_1\begin{bmatrix}w\\z\\\phi\\\psi\end{bmatrix}$$

where

$$\Lambda_1 \begin{bmatrix} w \\ z \\ \phi \\ \psi \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{d^2}\phi' \\ 0 \\ -\frac{K}{I_\rho}\phi + \frac{K}{I_\rho}w' \end{bmatrix}, \qquad \Lambda_1 \in \mathbf{L}(\mathrm{X} \oplus \mathrm{X}).$$

By Theorem 1.2 the operator $\Lambda_0$ generates a $C_0$-group on $\mathrm{X} \oplus \mathrm{X}$. Applying the standard perturbation result [28, Theorem 1.1, p. 76] we conclude that $\Lambda$ generates a $C_0$-group on $\mathrm{X} \oplus \mathrm{X}$. This result slightly improves that of Kim and Renardy [19].

**Remark 5.2.** *The more complicated boundary control system of the Timoshenko beam discussed in* [24] *also admit analysis by the methods presented above jointly with those of Section 4.2.*

## 6. APPLICATION TO A CANTILEVER BEAM

Bailey and Hubbard have described [1] (see also [22] for a related system) the distributed piezoelectric polymer active control of a cantilever beam. The dynamics of is governed by the equations

$$\left\{ \begin{array}{r} w_{tt} + w_{xxxx} = 0 \\ w(0,t) = 0 \\ w_x(0,t) = 0 \\ w_{xxx}(1,t) - w_{tt}(1,t) = 0 \\ w_{xx}(1,t) + w_{ttx}(1,t) + kw_{tx}(1,t) = 0 \end{array} \right\} \tag{6.1}$$

where $k \in \mathbb{R}$, $0 \le x \le 1$ and $t \ge 0$. The system (6.1) was analysed by Slemrod [34], and by Curtain and Oostveen [7]. Here we shall discuss the results of Slemrod [34] from the viewpoint of Skalikov's theory.

Substituting $w_t = \lambda w$ we get the Sturm–Liouville boundary value problem in which spectral parameter $\lambda$ enters polynomially the boundary conditions,

$$\left\{ \begin{array}{r} w''''(x) + \lambda^2 w(x) = 0 \\ w''(1) + \lambda^2 w'(1) + k\lambda w'(1) = 0 \\ w'''(1) - \lambda^2 w(1) = 0 \\ w'(0) = 0 \\ w(0) = 0 \end{array} \right\}. \tag{6.2}$$

Introducing the new spectral parameter $\mu$, $\lambda = \mu^2$ we transform (6.2) to the form discussed by Shkalikov,

$$\left\{ \begin{array}{r} w''''(x) + \mu^4 w(x) = 0 \\ w''(1) + \mu^4 w'(1) + k\mu^2 w'(1) = 0 \\ w'''(1) - \mu^4 w(1) = 0 \\ w'(0) = 0 \\ w(0) = 0 \end{array} \right\}. \tag{6.3}$$

Here the boundary conditions are ordered and have orders $\kappa_1 = 5$, $\kappa_2 = 4$, $\kappa_3 = 1$, $\kappa_4 = 0$, respectively. The total order of boundary conditions is $\kappa = 10$. We have $n = 4$, $p_1 = p_2 = p_3 = 0$, $p_4 = \mu^4$ and $p_{44} = 1 \neq 0$ in (3.1). By (3.3) the characteristic polynomial of the problem takes the form

$$\omega^4 + 1 = (\omega^2 - \sqrt{2}\omega + 1)(\omega^2 + \sqrt{2}\omega + 1).$$

Its roots are centers of sides of the polygon $\mathcal{M}$ being the square with edges at points $(\pm\sqrt{2}, 0)$, $(0, \pm\sqrt{2})$, see Figure 6. Assuming a solution of (6.3) in the form $w(x) = C_1 e^{\mu\omega_1 x} + C_2 e^{\mu\omega_2 x} + C_3 e^{\mu\omega_3 x} + C_4 e^{\mu\omega_4 x}$, after tedious but elementary calculations we find the characteristic determinant of the problem

$$\Delta(\mu) = \mu^6 \det \begin{bmatrix} \Phi_1 e^{\mu\omega_1} & \Phi_2 e^{\mu\omega_2} & \Phi_3 e^{\mu\omega_3} & \Phi_4 e^{\mu\omega_4} \\ \Psi_1 e^{\mu\omega_1} & \Psi_2 e^{\mu\omega_2} & \Psi_3 e^{\mu\omega_3} & \Psi_4 e^{\mu\omega_4} \\ \omega_1 & \omega_2 & \omega_3 & \omega_4 \\ 1 & 1 & 1 & 1 \end{bmatrix} =$$

$$\begin{aligned}
&= (\omega_3 - \omega_4)\Psi_2\Phi_1 e^{\mu(\omega_1+\omega_2)} + (\omega_2 - \omega_3)\Psi_4\Phi_1 e^{\mu(\omega_1+\omega_4)} + (\omega_4 - \omega_2)\Psi_3\Phi_1 e^{\mu(\omega_1+\omega_3)} - \\
&- (\omega_3 - \omega_4)\Psi_1\Phi_2 e^{\mu(\omega_2+\omega_1)} - (\omega_4 - \omega_1)\Psi_3\Phi_2 e^{\mu(\omega_2+\omega_3)} - (\omega_1 - \omega_3)\Psi_4\Phi_2 e^{\mu(\omega_2+\omega_4)} - \\
&- (\omega_4 - \omega_2)\Psi_1\Phi_3 e^{\mu(\omega_3+\omega_1)} - (\omega_1 - \omega_4)\Psi_2\Phi_3 e^{\mu(\omega_3+\omega_2)} - (\omega_2 - \omega_1)\Psi_4\Phi_3 e^{\mu(\omega_3+\omega_4)} - \\
&- (\omega_2 - \omega_3)\Psi_1\Phi_4 e^{\mu(\omega_4+\omega_1)} - (\omega_3 - \omega_1)\Psi_2\Phi_4 e^{\mu(\omega_4+\omega_2)} - (\omega_1 - \omega_2)\Psi_3\Phi_4 e^{\mu(\omega_4+\omega_3)} = \\
&= \{(\omega_4 - \omega_2)[\Psi_3\Phi_1 - \Psi_1\Phi_3] + (\omega_1 - \omega_3)[\Psi_2\Phi_4 - \Psi_4\Phi_2]\} + \\
&\quad + e^{-\mu\sqrt{2}}(\omega_1 - \omega_4)[\Psi_3\Phi_2 - \Psi_2\Phi_3] + e^{\mu\sqrt{2}}(\omega_2 - \omega_3)[\Psi_4\Phi_1 - \Psi_1\Phi_4] + \\
&\quad\quad + e^{-j\mu\sqrt{2}}(\omega_1 - \omega_2)[\Psi_4\Phi_3 - \Psi_3\Phi_4] + e^{j\mu\sqrt{2}}(\omega_3 - \omega_4)[\Psi_2\Phi_1 - \Psi_1\Phi_2]
\end{aligned}$$

where $\Phi_l := \left[\omega_l^2 + \omega_l(\mu^3 + k\mu)\right]$ and $\Psi_l := \omega_1^3 - \mu$, $l = 1, 2, 3, 4$. This yields

$$\begin{aligned}
\Delta(\mu) =& \mu^{10} \left\{ \left[ \frac{8}{\mu^4} - 8\left(\frac{k}{\mu^2} + 1\right) \right] + \right. \\
&+ e^{-\mu\sqrt{2}} \left[ \frac{2}{\mu^4} - \frac{2\sqrt{2}}{\mu^3} - 2\sqrt{2}\left(\frac{k}{\mu^3} + \frac{1}{\mu}\right) + 2\left(\frac{k}{\mu^2} + 1\right) \right] + \\
&+ e^{\mu\sqrt{2}} \left[ \frac{2}{\mu^4} + \frac{2\sqrt{2}}{\mu^3} + 2\sqrt{2}\left(\frac{k}{\mu^3} + \frac{1}{\mu}\right) + 2\left(\frac{k}{\mu^2} + 1\right) \right] + \\
&+ e^{-j\mu\sqrt{2}} \left[ \frac{2}{\mu^4} - \frac{2\sqrt{2}j}{\mu^3} + 2\sqrt{2}j\left(\frac{k}{\mu^3} + \frac{1}{\mu}\right) + 2\left(\frac{k}{\mu^2} + 1\right) \right] + \\
&\left. + e^{j\mu\sqrt{2}} \left[ \frac{2}{\mu^4} + \frac{2\sqrt{2}j}{\mu^3} - 2\sqrt{2}j\left(\frac{k}{\mu^3} + \frac{1}{\mu}\right) + 2\left(\frac{k}{\mu^2} + 1\right) \right] \right\}.
\end{aligned}$$

The dominating terms in square brackets are 2 and thus the problem (6.3) is regular.

Notice that $\mu$ is a root of the equation $\Delta(\mu)=0$ iff $\omega=(1\mp j)\mu/\sqrt{2}$ satisfies the equation

$$F(\omega)=\omega^4\cosh\omega\left\{\left[\frac{1}{\omega^4\cosh\omega}+\frac{\cos\omega}{\omega^4}+\frac{\cos\omega}{\omega^3}\tanh\omega-\frac{\sin\omega}{\omega^3}-\frac{\sin\omega}{\omega}-\frac{\cos\omega}{\omega}\tanh\right.\right.$$
$$\left.\left.+\frac{1}{\cosh\omega}-\cos\omega\right]\pm jk\left[\frac{\sin\omega}{\omega^3}+\frac{\cos\omega}{\omega^3}\tanh\omega-\frac{1}{\omega^2\cosh\omega}+\frac{\cos\omega}{\omega^2}\right]\right\}=0.$$

This shows that the real asymptotic roots of $F$ coincide with roots of the trigonometric equation $\cos\omega=0$, whence $\omega_n\approx\dfrac{\pi}{2}+n\pi$, $n\in\mathbb{Z}$. Since

$$F'(\omega)=\omega^4\cosh\omega\left\{\left[2\frac{\cos\omega}{\omega^4}\tanh\omega-2\frac{\sin\omega}{\omega^4}-2\frac{\sin\omega}{\omega^3}\tanh\omega+4\frac{1}{\omega\cosh\omega}-6\frac{\cos\omega}{\omega}\right.\right.$$
$$\left.-3\frac{\sin\omega}{\omega^2}-3\frac{\cos\omega}{\omega^2}\tanh\omega+\sin\omega-\cos\omega\tanh\omega\right]$$
$$\left.\pm jk\left[\frac{\sin\omega}{\omega^4}+\frac{\cos\omega}{\omega^4}\tanh\omega-2\frac{1}{\omega^3\cosh\omega}+4\frac{\cos\omega}{\omega^3}-\frac{\sin\omega}{\omega^2}\right]\right\}$$

those roots of $F$ are single. From [33, Lemma 1.1] we know that the asymptotic roots of $F$ are located on the four rays going outside from the origin, and perpendicular to the sides of the square $\mathcal{M}$. This means that all asymptotic roots of $F$ are real and coincide with $\omega_n$. Consequently, the roots of the characteristic equation are asymptotically equally spaced and single (see Fig. 7),

$$\mu_n\approx\frac{1\mp j}{\sqrt{2}}\left(\frac{\pi}{2}+n\pi\right),\qquad n\in\mathbb{Z},$$

and the problem (6.3) is strictly regular.

For further investigations we represent $\ell(w,\mu)$ in the form (3.5),

$$\ell(w,\mu)=\sum_{k=0}^{3}\mu^k\ell_k(w)+\mu^4 w$$

where $\ell_0(w)=w''''$, $\ell_k(w)=0$, $k=1,2,3$. Here $n=4$, $r=0$ and thus

$$W_2^0:=\mathrm{H}^3(0,1)\oplus\mathrm{H}^2(0,1)\oplus\mathrm{H}^1(0,1)\oplus\mathrm{L}^2(0,1),$$
$$\tilde v=\begin{bmatrix}v_0\\ v_1\\ v_2\\ v_3\end{bmatrix},\quad H\tilde v=\begin{bmatrix}v_1\\ v_2\\ v_3\\ -v_0''''\end{bmatrix},\quad H^2\tilde v=\begin{bmatrix}v_2\\ v_3\\ -v_0''''\\ -v_1''''\end{bmatrix},\quad H^3\tilde v=\begin{bmatrix}v_3\\ -v_0''''\\ -v_1''''\\ -v_2''''\end{bmatrix}.$$

The results of interchanging terms in the boundary conditions according to the rule (3.6) are presented in Table 4.

**Table 4.** Results of applying the rule (3.6)

| Original term | $\nu$ | $k$ | $\nu+k<4+r$ | Replaced term |
|---|---|---|---|---|
| $\mu^4 w'(1)$ | 4 | 1 | not | $\mu^2 v_2'(1)$ |
| $\mu^2 w'(1)$ | 2 | 1 | yes | $v_2'(1)$ |
| $w''(1)$ | 0 | 2 | yes | $v_0''(1)$ |
| $\mu^4 w(1)$ | 4 | 0 | not | $\mu v_3(1)$ |
| $w'''(1)$ | 0 | 3 | yes | $v_0'''(1)$ |
| $w'(0)$ | 0 | 1 | yes | $v_0'(0)$ |
| $w(0)$ | 0 | 0 | yes | $v_0(0)$ |

The resulting boundary conditions are

$$\begin{aligned}
\tilde{U}_1(\tilde{v},\mu) &= [v_0''(1)+kv_2'(1)]+\mu^2[v_2'(1)]=0,\\
\tilde{U}_2(\tilde{v},\mu) &= v_0'''(1)-\mu v_3(1)=0,\\
\tilde{U}_3(\tilde{v},\mu) &= v_0'(0)=0,\\
\tilde{U}_4(\tilde{v},\mu) &= v_0(0)=0,
\end{aligned}$$

whence

$$\left\{\begin{array}{l} U_1^0(\tilde{v})=v_0''(1)+kv_2'(1)\\ U_1^1(\tilde{v})=0\\ U_1^2(\tilde{v})=v_2'(1)\end{array}\right\},\qquad \nu_1(0)=2,$$

$$U_2^0(\tilde{v})=v_0'''(1),\qquad U_2^1(\tilde{v})=-v_3(1),\qquad \nu_2(0)=1.$$

We have $\nu_3=\nu_4=0$, $q=2$, $N_0=3$ and by (3.8)

$$W_{2,U}^0=\left\{\tilde{v}\in\bigoplus_{j=0}^{3}\mathrm{H}^{3-j}(0,1):\ v_0(0)=0,\ v_0'(0)=0,\ v_1(0)=0,\ v_1'(0)=0,\ v_2(0)=0\right\}.$$

Indeed, if $k=0$ then the boundary conditions of order less than or equal to 2 should be taken into account, i.e.,

$$\tilde{U}_3(\tilde{v})=v_0'(0)=0,\qquad \tilde{U}_4(\tilde{v})=v_0(0)=0.$$

If $k=1$ then the boundary conditions of order less than or equal to 1 should be encountered into the definition of the state space, i.e.,

$$\tilde{U}_3(H^1\tilde{v})=v_1'(0)=0,\qquad \tilde{U}_4(H^1\tilde{v})=v_1(0)=0.$$

If $k=2$ then only null order boundary conditions should be taken into account, i.e.,

$$\tilde{U}_4(H^2\tilde{v})=v_2(0)=0.$$

From (3.10) and (3.8) we find the domain of $H_0$. This requires determination of $W^1_{2,U}$,

$$W^1_{2,U} = \left\{ \tilde{v} \in \bigoplus_{j=0}^{4} \mathrm{H}^{4-j}(0,1) : \ \tilde{U}_j(H^k \tilde{v}) = 0 \text{ for } 0 \le k \le 3 \right.$$
$$\left. \text{and all boundary conditions of order less than or equal to } \ 3-k \right\}.$$

If $k = 0$ then the boundary conditions of order less than or equal to 3 should be taken into account, i.e.,

$$\tilde{U}_3(\tilde{v}) = v_0'(0) = 0, \qquad \tilde{U}_4(\tilde{v}) = v_0(0) = 0.$$

If $k = 1$ then the boundary conditions of order less than or equal to 2 should be encountered, i.e.,

$$\tilde{U}_3(H\tilde{v}) = v_1'(0) = 0, \qquad \tilde{U}_4(H\tilde{v}) = v_1(0) = 0.$$

If $k = 2$ then the bounadry conditions of order less than or equal to 1 should be considered, i.e.,

$$\tilde{U}_3(H^2\tilde{v}) = v_2'(0) = 0, \qquad \tilde{U}_4(H^2\tilde{v}) = v_2(0) = 0.$$

If $k = 3$ the boundary conditions of null order should be encountered, which yields, $\tilde{U}_4(H^3\tilde{v}) = v_3(0) = 0$. Finally we obtain

$$W^1_{2,U} = \left\{ \tilde{v} \in \bigoplus_{j=0}^{4} \mathrm{H}^{4-j}(0,1) : \right.$$
$$\left. v_0(0) = 0,\ v_1(0) = 0,\ v_2(0) = 0,\ v_3(0) = 0,\ v_0'(0) = 0,\ v_1'(0) = 0,\ v_2'(0) = 0 \right\}.$$

Now

$$H_0 \begin{bmatrix} \tilde{v} \\ U_1^2 \\ z_{12} \\ U_2^1 \end{bmatrix} = H_0 \begin{bmatrix} \tilde{v} \\ v_2'(1) \\ z_{12} \\ -v_3(1) \end{bmatrix} = \begin{bmatrix} H\tilde{v} \\ z_{12} - U_1^1 \\ -U_1^0 \\ -U_2^0 \end{bmatrix} = \begin{bmatrix} H\tilde{v} \\ z_{12} \\ -v_0''(1) - kv_2'(1) \\ -v_0'''(1) \end{bmatrix},$$

$$D(H_0) = \left\{ \begin{bmatrix} \tilde{v} \\ v_2'(1) \\ z_{12} \\ -v_3(1) \end{bmatrix} : \ \tilde{v} \in W^1_{2,U}, \ z_{12} \in \mathbb{C} \right\},$$

or equivalently,

$$H_0 \begin{bmatrix} \tilde{v} \\ z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} H\tilde{v} \\ z_2 \\ -v_0''(1) - kz_1 \\ -v_0'''(1) \end{bmatrix},$$

$$D(H_0) = \left\{ \begin{bmatrix} \tilde{v} \\ z_1 \\ z_2 \\ z_3 \end{bmatrix} \in W^1_{2,U} \oplus \mathbb{C}^3 : \ z_1 = v_2'(1), \ z_3 = -v_3(1) \right\}.$$

Since the problem (6.3) is strictly regular and two of the boundary conditions are of order greater than $n+r-1=3$ the first assertion of Theorem 3.1 applies. Thus, the Shkalikov linearization of our boundary-value problem has a system of generalized eigenvectors (only finitely many of them are not eigenvectors) which constitutes a Riesz basis of the Shkalikov space $W^0_{2,U} \oplus \mathbb{C}^{N_0} = W^0_{2,U} \oplus \mathbb{C}^3$.

Since we made the substitution $\lambda = \mu^2$ the question is whether the above results apply to the original problem (6.1). Recall that $\mu$ is an eigenvalue of $H_0$, so $\lambda = \mu^2$ is an eigenvalue of $H_0^2$ while eigenvectors correspondig to these eigenvalues are the same. Thus eigenvectors of $H_0^2$ still generate a Riesz basis for $W^0_{2,U} \oplus \mathbb{C}^3$. This suggests to look at $H_0^2$,

$$H_0^2 \begin{bmatrix} \tilde{v} \\ z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} H^2\tilde{v} \\ -v_0''(1) - kz_1 \\ -v_1''(1) - kz_2 \\ -v_1'''(1) \end{bmatrix}, \ D\big(H_0^2\big) = \left\{ \begin{bmatrix} \tilde{v} \\ z_1 \\ z_2 \\ z_3 \end{bmatrix} \in D(H_0) : \ H_0 \begin{bmatrix} \tilde{v} \\ z_1 \\ z_2 \\ z_3 \end{bmatrix} \in D(H_0) \right\},$$

or equivalently,

$$H_0^2 \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \\ z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} v_2 \\ v_3 \\ -v_0'''' \\ -v_1'''' \\ -v_0''(1) - kz_1 \\ -v_1''(1) - kz_2 \\ -v_1'''(1) \end{bmatrix}$$

with the domain

$$D\big(H_0^2\big) = \left\{ \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \\ z_1 \\ z_2 \\ z_3 \end{bmatrix} \in \left( \bigoplus_{j=0}^{3} \mathrm{H}^{5-j}(0,1) \right) \oplus \mathbb{C}^3 : \right.$$

$$\left.\begin{aligned}&v_0(0)=0,\ v_0'(0)=0,\ v_1(0)=0,\ v_2(0)=0,\ v_3(0)=0,\ v_1'(0)=0,\ v_2'(0)=0,\\&v_0''''(0)=0,\ v_3'(0)=0,\ v_0'''(1)=v_0''''(1), z_1=v_2'(1),\ z_2=v_3'(1),\ z_3=-v_3(1)\end{aligned}\right\}.$$

Now observe that $H_0^2$ splits into two operators. One of them,

$$H_B\begin{bmatrix}v_1\\v_3\\z_2\\z_3\end{bmatrix}=\begin{bmatrix}v_3\\-v_1''''\\-v_1''(1)-kz_2\\-v_1'''(1)\end{bmatrix},$$

$$D(H_B)=\left\{\begin{bmatrix}v_1\\v_3\\z_2\\z_3\end{bmatrix}\in \mathrm{H}^4(0,1)\oplus\mathrm{H}^2(0,1)\oplus\mathbb{C}^2:\right.$$

$$\left. v_1(0)=0,\ v_3(0)=0,\ v_1'(0)=0,\ v_3'(0)=0,\ z_2=v_3'(1),\ z_3=-v_3(1)\right\}$$

acts in the state space

$$\mathrm{H}_B:=\left\{\begin{bmatrix}v_1\\v_3\\z_2\\z_3\end{bmatrix}\in \mathrm{H}^2(0,1)\oplus\mathrm{L}^2(0,1)\oplus\mathbb{C}^2: v_1(0)=0,\ v_1'(0)=0\right\}.$$

Slemrod [34] and Curtain and Oostveen [7] have used this space as the underlying Hilbert space for the problem. The suboperator $H_B$ is a part of $H_0^2$ in $\mathrm{H}_B$ being an invariant subspace of $W^0_{2,U}\oplus\mathbb{C}^3$. Thus $H_B$ also possesses a system of generalized eigenvectors (only a finitely many of them are not eigenvectors) which forms a Riesz basis of the subspace $\mathrm{H}_B$. It is very easy to see that the eigenproblem for $H_B$ is equivalent to the eigenproblem for our original elastic system (6.2). $H_B$ splits into two parts,

$$H_Bx=PH_Bx+(I-P)H_Bx.$$

Here $P$ denotes the projector onto a finite dimensional eigenspace spanned by generalized eigenvectors corresponding to nonsimple eigenvalues,

$$Px=\frac{1}{2\pi j}\int\limits_{\partial\Omega^+}(\lambda I-H_B)^{-1}x\,d\lambda$$

where $\Omega$ is such a domain, with the boundary $\partial\Omega^+$ being a positively oriented Jordan curve, that all nonsimple eigenvalues are located in $\operatorname{Int}\Omega$. Hence, $PH_B$ is the finite dimensional part of $H_B$ while the complementary infinite dimensional part $(I-P)H_B$ is similar to a normal operator. Moreover, the spectrum of the latter is located in a vertical strip parallel to the $j\,\mathbb{R}$-axis because the eigenvalues of $H_B$ are among squares of $\mu_n$. By Theorem 1.1 the operator $H_B$ generates a $C_0$ − group on $\mathrm{H}_B$. The above example demonstrates that the state space assumed a priori in previous papers can be systematically generated using Shkalikov's theory.

Though, Theorem 1.2 applies to the system stability investigations, it is probably more convenient to use Slemrod's idea [34]. He has observed that the abstract model of the system in $\mathrm{H}_B$ reads as

$$\dot{x}(t) = H_B x(t) = \mathcal{A}_k x(t), \qquad \mathcal{A}_k := \mathcal{A} - kbb^*$$

where $\mathcal{A}$ equals $H_B$ for $k = 0$, $\mathcal{A} = -\mathcal{A}^*$ (i.e, $\mathcal{A}$ is a *skew-adjoint operator*), and $b^*x = \langle x, b\rangle_{\mathrm{H}_B} = z_2$. Now,

$$\frac{d}{dt}\|x(t)\|^2_{\mathrm{H}_B} = -2k\,|b^*x(t)|^2$$

for all initial conditions in $D(\mathcal{A})$. Hence, if $k > 0$ then the square of the norm is a Lyapunov functional for the system and all solutions are bounded. The pair $(\mathcal{A}, b^*)$ is *approximately observable* and by the *weak invariance principle* all solutions weakly tends to zero. But the resolvent of $\mathcal{A}$ is clearly compact, and therefore the weak stability implies the strong one.

The square of the norm is still a Lyapunov functional for the nonlinear Lur'e system of *direct control*,

$$\dot{x}(t) = \mathcal{A}x(t) - bf\left[b^*x(t)\right], \tag{6.4}$$

provided that $f$ is a locally Lipschitz scalar function satisfying the sector condition

$$yf(y) > 0 \qquad \forall y \neq 0, \quad f(0) = 0.$$

It enables us to prove both stability and global weak attractivity of all solutions. However, the author was only able to obtain the global strong asymptotic stability of the null equilibrium point under a stronger sector condition,

$$yf(y) \geq \varepsilon y^2 > 0 \qquad \forall y \neq 0, \quad f(0) = 0. \tag{6.5}$$

To be more precise, we have the following result.

**Theorem 6.1.** *The null equilibrium point is absolutely stable in the class of locally Lipschitz functions satisfying the sector condition* (6.5)*.*

*Proof.* It is not difficult to see that for all $x_0 \in D(\mathcal{A})$ we have

$$\frac{d}{dt}\|x(t)\|^2_{\mathrm{H}_B} = -2b^*xf\left[b^*x(t)\right] \leq -2\varepsilon\left(b^*x\right)^2.$$

Integrating both sides from 0 to $t$ we get

$$\|x(t)\|^2_{\mathrm{H}_B} - \|x_0\|^2_{\mathrm{H}_B} \leq -2\varepsilon\int\limits_0^t \left(b^*x(\tau)\right)^2 d\tau,$$

whence

$$\|x(t)\|_{\mathrm{H}_B} \leq \|x_0\|_{\mathrm{H}_B} \qquad \forall t \geq 0, \quad \forall x \in \mathrm{H}_B \tag{6.6}$$

and

$$\int_0^\infty (b^* x(t))^2 \, dt \leq \frac{1}{2\varepsilon} \|x_0\|^2_{\mathrm{H}_B} .$$

The latter implies $y \in \mathrm{L}^2(0,\infty)$, $y(t) = b^* x(t)$. Since $f$ is locally Lipschitz then by (6.6) there exists a positive constant $m$ such that

$$|f(b^* x)| = |f(b^* x) - f(0)| \leq m \, |b^* x| = m \, |y| ,$$

and thus $f[y(\cdot)] \in \mathrm{L}^2(0,\infty)$. Consequently, $u \in \mathrm{L}^2(0,\infty)$, $u(t) = \varepsilon y(t) - f[y(t)]$. The variation-of-constants formula for an equivalent form of (6.4),

$$\dot{x}(t) = \mathcal{A}_\varepsilon x(t) + bu(t), \qquad \mathcal{A}_\varepsilon := \mathcal{A} - \varepsilon b b^*$$

is

$$x(t) = e^{t\mathcal{A}_\varepsilon} x_0 + \left( e^{(\cdot)\mathcal{A}_\varepsilon} b \star u \right)(t). \tag{6.7}$$

The first term strongly tends to 0 as $t \to \infty$ by asymptotic stability of the linear semigroup $\left\{e^{t\mathcal{A}_\varepsilon}\right\}_{t\geq 0}$. Observe that $b^*$ is an infinite-time admissible observation functional with respect to the semigroup $\left\{e^{t\mathcal{A}^*_\varepsilon}\right\}_{t\geq 0}$ because $\mathcal{H} = \dfrac{1}{2\varepsilon} I$ is a unique bounded self-adjoint solution to the Lyapunov operator equation

$$\langle \mathcal{A}^*_\varepsilon x, \mathcal{H} x \rangle_{\mathrm{H}_B} + \langle x, \mathcal{H}\mathcal{A}^*_\varepsilon x \rangle_{\mathrm{H}_B} = - \left| b^* x \right|^2$$

for all $x \in D(\mathcal{A})$. By the duality theory [27, p. 9], $b$ is an admissible control vector with respect to the semigroup $\left\{e^{t\mathcal{A}_\varepsilon}\right\}_{t\geq 0}$. Employing the result of Curtain and Oostveen [27, Lemma 12] we establish that the convolution term in (6.7) also tends to 0 as $t \to \infty$. This means that the origin is globally attractive while its stability is a consequence of (6.6). □

**Remark 6.1.** *From* [34] *we know that for the saturation nonlinearity, for which* (6.5) *clearly does not hold, the null equilibrium point of* (6.4) *is still globally asymptotically stable.*

*Curtain and Oostveen* [7] *have proved that the null equilibrium point of the Lur'e* indirect control *system*

$$\left\{ \begin{array}{l} \dot{x}(t) = \mathcal{A}_k x - b\phi(\sigma) \\ \dot{\sigma}(t) = b^* x - \rho\phi(\sigma) \end{array} , \right\} \qquad \rho > 0, \quad k > 0$$

*is globally strongly asymptotically stable provided that $\phi$ is a locally Lipschitz function satisfying the sector conditions*

$$y\phi(y) > 0 \qquad \forall y \neq 0, \qquad \phi(0) = 0, \qquad \lim_{|\sigma|\to\infty} \int_0^\sigma \phi(\sigma) d\sigma = \infty.$$

## 7. CONCLUSIONS

The Shkalikov theory is a powerful tool in establishing the Riesz basis property of a system of eigenvectors even for complicated hybrid boundary control systems. It requires some computations to find the characteristic determinant of the associated boundary value problem arising from the eigenproblem. This is needed to verify the regularity of the boundary problem. Next, an asymptotic analysis of the roots of the characteristic determinant should be carried on to check the strict regularity of the boundary problem. Some regularity criteria of the algebraic type are known, see [35, 36]. Its seems that a great part of this task could be computerized. The results we get with the aid of Theorem 3.1 jointly with Theorems 1.1 and 1.2 are at least not worse than those obtained by applying the standard semigroups methods and/or the energy functional method. Frequently, the results are better and they provide a deep insight into spectral properties of the investigated system. Moreover, the procedure of finding the adequate state space for the problem is well-organized, contrary to the existing methods which require several trials.

All wave examples discussed in Section 4 asymptotically reduce to an appropriately modified Rideau's example. They can also be treated using the d'Alembert solution approach, i.e., they can be reduced to delay systems of the neutral type. The last approach applies not only for the lossless wave equations but also to the telegrapher's equations without distortion.

The example of the Timoshenko beam presented in Subsection 5.2 shows that sometimes it is convenient to combine the Shkalikov theory with the operator theoretic methods.

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## APPENDIX: DERIVATION OF THE CHARACTERISTIC DETERMINANT FOR RIDEAU'S SECOND PROBLEM

$$\Delta(\mu) = \mu^5 \det \begin{bmatrix} (\mu\omega_1^3 - k)e^{\mu\omega_1} & (\mu\omega_2^3 - k)e^{\mu\omega_2} & (\mu\omega_3^3 - k)e^{\mu\omega_3} & (\mu\omega_4^3 - k)e^{\mu\omega_4} \\ \omega_1^2 e^{\mu\omega_1} & \omega_2^2 e^{\mu\omega_2} & \omega_3^2 e^{\mu\omega_3} & \omega_4^2 e^{\mu\omega_4} \\ \omega_1 & \omega_2 & \omega_3 & \omega_4 \\ 1 & 1 & 1 & 1 \end{bmatrix} =$$

$$= (\mu\omega_1^3 - k)e^{\mu\omega_1} \left\{ \omega_2^2 e^{\mu\omega_2}\omega_3 + \omega_2\omega_4^2 e^{\mu\omega_4} + \omega_3^2 e^{\mu\omega_3}\omega_4 - \right.$$
$$\left. -\omega_3\omega_4^2 e^{\mu\omega_4} - \omega_4\omega_2^2 e^{\mu\omega_2} - \omega_2\omega_3^2 e^{\mu\omega_3} \right\} -$$
$$- \omega_1^2 e^{\mu\omega_1} \left\{ (\mu\omega_2^3 - k)e^{\mu\omega_2}\omega_3 + \omega_2(\mu\omega_4^3 - k)e^{\mu\omega_4} + \omega_4(\mu\omega_3^3 - k)e^{\mu\omega_3} - \right.$$

$$
\begin{aligned}
&-(\mu\omega_4^3-k)e^{\mu\omega_4}\omega_3-\omega_4(\mu\omega_2^3-k)e^{\mu\omega_2}-\omega_2(\mu\omega_3^3-k)e^{\mu\omega_3}\big\}+\\
&+\omega_1\left\{(\mu\omega_2^3-k)e^{\mu\omega_2}\omega_3^2e^{\mu\omega_3}+(\mu\omega_4^3-k)e^{\mu\omega_4}\omega_2^2e^{\mu\omega_2}+(\mu\omega_3^3-k)e^{\mu\omega_3}\omega_4^2e^{\mu\omega_4}-\right.\\
&\left.-(\mu\omega_4^3-k)e^{\mu\omega_4}\omega_3^2e^{\mu\omega_3}-(\mu\omega_2^3-k)e^{\mu\omega_2}\omega_4^2e^{\mu\omega_4}-(\mu\omega_3^3-k)e^{\mu\omega_3}\omega_2^2e^{\mu\omega_2}\right\}-\\
&-\left\{(\mu\omega_2^3-k)e^{\mu\omega_2}\omega_3^2e^{\mu\omega_3}\omega_4+(\mu\omega_4^3-k)e^{\mu\omega_4}\omega_2^2e^{\mu\omega_2}\omega_3+(\mu\omega_3^3-k)e^{\mu\omega_3}\omega_4^2e^{\mu\omega_4}\omega_2-\right.\\
&\left.-(\mu\omega_4^3-k)e^{\mu\omega_4}\omega_3^2e^{\mu\omega_3}\omega_2-(\mu\omega_2^3-k)e^{\mu\omega_2}\omega_4^2e^{\mu\omega_4}\omega_3-(\mu\omega_3^3-k)e^{\mu\omega_3}\omega_2^2e^{\mu\omega_2}\omega_4\right\}=\\
&=(\mu\omega_1^3-k)e^{\mu\omega_1}\left\{e^{\mu\omega_2}(\omega_2^2\omega_3-\omega_4\omega_2^2)+e^{\mu\omega_4}(\omega_2\omega_4^2-\omega_3\omega_4^2)+e^{\mu\omega_3}(\omega_3^2\omega_4-\omega_2\omega_3^2)\right\}+\\
&+(\mu\omega_2^3-k)e^{\mu\omega_2}\left\{e^{\mu\omega_1}(\omega_1^2\omega_4-\omega_1^2\omega_3)+e^{\mu\omega_4}(\omega_3\omega_4^2-\omega_1\omega_4^2)+e^{\mu\omega_3}(\omega_1\omega_3^2-\omega_3^2\omega_4)\right\}+\\
&+(\mu\omega_3^3-k)e^{\mu\omega_3}\left\{e^{\mu\omega_1}(\omega_1^2\omega_2-\omega_1^2\omega_4)+e^{\mu\omega_4}(\omega_1\omega_4^2-\omega_2\omega_4^2)+e^{\mu\omega_2}(\omega_4\omega_2^2-\omega_1\omega_2^2)\right\}+\\
&+(\mu\omega_4^3-k)e^{\mu\omega_4}\left\{e^{\mu\omega_1}(\omega_1^2\omega_3-\omega_2\omega_1^2)+e^{\mu\omega_2}(\omega_1\omega_2^2-\omega_2^2\omega_3)+e^{\mu\omega_3}(\omega_2\omega_3^2-\omega_1\omega_3^2)\right\}=\\
&=\mu\Big\{\omega_1^3e^{\mu(\omega_1+\omega_2)}\omega_2^2(\omega_3-\omega_4)+\omega_1^3e^{\mu(\omega_1+\omega_4)}\omega_4^2(\omega_2-\omega_3)+\omega_1^3e^{\mu(\omega_1+\omega_3)}\omega_3^2(\omega_4-\omega_2)+\\
&+\omega_2^3e^{\mu(\omega_2+\omega_1)}\omega_1^2(\omega_4-\omega_3)+\omega_2^3e^{\mu(\omega_2+\omega_4)}\omega_4^2(\omega_3-\omega_1)+\omega_2^3e^{\mu(\omega_2+\omega_3)}\omega_3^2(\omega_1-\omega_4)+\\
&+\omega_3^3e^{\mu(\omega_1+\omega_3)}\omega_1^2(\omega_2-\omega_4)+\omega_3^3e^{\mu(\omega_3+\omega_4)}\omega_4^2(\omega_1-\omega_2)+\omega_3^3e^{\mu(\omega_2+\omega_3)}\omega_2^2(\omega_4-\omega_1)+\\
&+\omega_4^3e^{\mu(\omega_4+\omega_1)}\omega_1^2(\omega_3-\omega_2)+\omega_4^3e^{\mu(\omega_4+\omega_2)}\omega_2^2(\omega_1-\omega_3)+\omega_4^3e^{\mu(\omega_4+\omega_3)}\omega_3^2(\omega_2-\omega_1)\Big\}-\\
&-k\Big\{e^{\mu(\omega_1+\omega_2)}\omega_2^2(\omega_3-\omega_4)+e^{\mu(\omega_1+\omega_4)}\omega_4^2(\omega_2-\omega_3)+e^{\mu(\omega_1+\omega_3)}\omega_3^2(\omega_4-\omega_2)+\\
&+e^{\mu(\omega_2+\omega_1)}\omega_1^2(\omega_4-\omega_3)+e^{\mu(\omega_2+\omega_4)}\omega_4^2(\omega_3-\omega_1)+e^{\mu(\omega_2+\omega_3)}\omega_3^2(\omega_1-\omega_4)+\\
&+e^{\mu(\omega_3+\omega_1)}\omega_1^2(\omega_2-\omega_4)+e^{\mu(\omega_3+\omega_4)}\omega_4^2(\omega_1-\omega_2)+e^{\mu(\omega_3+\omega_2)}\omega_2^2(\omega_4-\omega_1)+\\
&+e^{\mu(\omega_4+\omega_1)}\omega_1^2(\omega_3-\omega_2)+e^{\mu(\omega_4+\omega_2)}\omega_2^2(\omega_1-\omega_3)+e^{\mu(\omega_4+\omega_3)}\omega_3^2(\omega_2-\omega_1)\Big\}=\\
&=\mu\Big\{e^{\mu(\omega_1+\omega_2)}(\omega_1^3\omega_2^2-\omega_2^3\omega_1^2)(\omega_3-\omega_4)+e^{\mu(\omega_1+\omega_4)}(\omega_1^3\omega_4^2-\omega_4^3\omega_1^2)(\omega_2-\omega_3)+\\
&+e^{\mu(\omega_1+\omega_3)}(\omega_1^3\omega_3^2-\omega_3^3\omega_1^2)(\omega_4-\omega_2)+e^{\mu(\omega_2+\omega_4)}(\omega_2^3\omega_4^2-\omega_4^3\omega_2^2)(\omega_3-\omega_1)+\\
&+e^{\mu(\omega_2+\omega_3)}(\omega_2^3\omega_3^2-\omega_3^3\omega_2^2)(\omega_1-\omega_4)+e^{\mu(\omega_3+\omega_4)}(\omega_4^2\omega_3^3-\omega_4^3\omega_3^2)(\omega_1-\omega_2)\Big\}-\\
&-k\Big\{e^{\mu(\omega_1+\omega_2)}(\omega_2^2-\omega_1^2)(\omega_3-\omega_4)+e^{\mu(\omega_1+\omega_4)}(\omega_4^2-\omega_1^2)(\omega_2-\omega_3)+\\
&+e^{\mu(\omega_1+\omega_3)}(\omega_3^2-\omega_1^2)(\omega_4-\omega_2)+e^{\mu(\omega_2+\omega_4)}(\omega_4^2-\omega_1^2)(\omega_2-\omega_3)+\\
&+e^{\mu(\omega_2+\omega_3)}(\omega_3^2-\omega_2^2)(\omega_1-\omega_4)+e^{\mu(\omega_3+\omega_4)}(\omega_4^2-\omega_3^2)(\omega_1-\omega_2)\Big\}.
\end{aligned}
$$

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