Hanna Furmańczyk

EQUITABLE COLORING OF GRAPH PRODUCTS

Abstract. A graph is equitably k-colorable if its vertices can be partitioned into k independent sets in such a way that the number of vertices in any two sets differ by at most one. The smallest k for which such a coloring exists is known as the *equitable chromatic number* of G and denoted by $\chi_=(G)$. It is interesting to note that if a graph G is equitably k-colorable, it does not imply that it is equitably $(k+1)$-colorable. The smallest integer k for which G is equitably k'-colorable for all $k' \geq k$ is called *the equitable chromatic threshold* of G and denoted by $\chi_=^*(G)$. In the paper we establish the equitable chromatic number and the equitable chromatic threshold for certain products of some highly-structured graphs. We extend the results from [2] for Cartesian, weak and strong tensor products.

1. INTRODUCTION

A graph is a pair $G = (V, E)$, where V is a finite set of vertices and $E \subseteq \{\{u,v\}|u,v \in V, u \neq v\}$ is a set of edges. Hence graphs considered in this paper are undirected, finite and contain neither loops nor multiple edges. A graph product $G_1 \star G_2$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ most commonly means a graph with the vertex set $V_1 \times V_2$, while its edges can be determined in quite different ways. We will consider three of them: the *Cartesian product*, the *weak tensor product* and the *strong tensor product*.

 The Cartesian, weak tensor and strong tensor products of graphs G_1 and G_2 will be denoted by $G_1 \square G_2$, $G_1 \times G_2$ and $G_1 \boxtimes G_2$, respectively. Let $ij, kl \in V_1 \times V_2$. Then $\{ij, kl\}$ belongs to:

 $E(G_1 \square G_2)$ whenever $i = k$ and $\{j, l\} \in E_2$, or $j = l$ and $\{i, k\} \in E_1$;

 $E(G_1 \times G_2)$ whenever $\{i, k\} \in E_1$ and $\{j, l\} \in E_2$;

 $E(G_1 \boxtimes G_2)$ whenever $\{ij, kl\} \in E(G_1 \square G_2) \cup E(G_1 \times G_2)$.
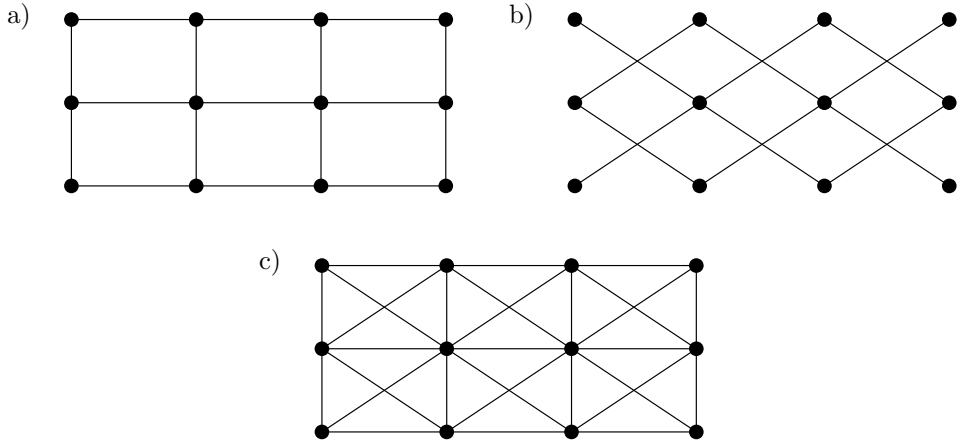
Fig. 1. Examples of graph products: a) $P_3 \square P_4$, b) $P_3 \times P_4$, c) $P_3 \boxtimes P_4$

Examples of graph products are given in Figure 1.

For a graph $G = (V, E)$ and $S \subseteq V$, the symbol $N(S) \subseteq V$ denotes the *neighborhood of S*, i.e., the set consisting of all vertices adjacent to the vertices in S. We denote the (maximum) degree of G, i.e., the maximum of the vertex degrees, by $\Delta = \Delta(G)$.

Definition 1. *A graph G is said to be equitably k-colorable if its vertices can be partitioned into k classes $V_1, V_2, \ldots, V_k$ such that each V_i is an independent set and the condition $||V_i| - |V_j|| \leq 1$ holds for every i, j. The smallest integer k for which G is equitably k-colorable is known as the equitable chromatic number of G and denoted by $\chi_=(G)$.*

Since an equitable coloring is a proper coloring, there is:

$$\chi_=(G) \geq \chi(G). \tag{1}$$

Applications of equitable coloring can be found in scheduling and timetabling. Consider, for example, a problem of constructing university timetables. As we know, we can model this problem as coloring the vertices of a graph G whose nodes correspond to classes, edges correspond to time conflicts between classes, and colors to hours. If the set of available rooms is restricted, then we may be forced to partition the vertex set into independent subsets of as near equal size as possible, since then the room usage is the highest. We can find another application of equitable coloring in transportation problems. Here, the vertices represent garbage collection routes and two such vertices are joined by an edge when the corresponding routes should not be run on the same day. The problem of assigning one of the six days of the work

week to each route becomes the problem of 6-coloring of G. On practical grounds it might also be desirable to have an approximately equal number of routes run on each of the six days, so we have to color the graph in the equitable way.

The notion of equitable colorability was introduced by Meyer [10]. However, an earlier work of Hajnal and Szemerédi [5] showed that a graph G with degree $\Delta(G)$ is equitably k-colorable if $k \geq \Delta(G)+1$. In 1973, Meyer [10] formulated the following conjecture:

Conjecture 1 (Equitable Coloring Conjecture (ECC) [10]**).** *For any connected graph G, other than a complete graph or an odd cycle, $\chi_=(G) \leq \Delta(G)$.*

We also have a stronger conjecture:

Conjecture 2 (Equitable Δ-Coloring Conjecture [1]**).** *If G is a connected graph of degree Δ, other than a complete graph, an odd cycle or a complete bipartite graph $K_{2n+1,2n+1}$ for any $n \geq 1$, then G is equitably Δ-colorable.*

The Equitable Δ-Coloring Conjecture holds for some classes of graphs, e.g., bipartite graphs [9], outerplanar graphs with $\Delta \geq 3$ [13] and planar graphs with $\Delta \geq 13$ [14]. It is interesting to note that if a graph G is equitably k-colorable, it does not imply that it is equitably $(k+1)$-colorable. A counterexample is the complete bipartite graph $K_{3,3}$ which can be equitably colored with two colors, but not with three. The smallest integer k, for which G is equitably k'-colorable for all $k' \geq k$, is called the *equitable chromatic threshold* of G and denoted by $\chi_=^*(G)$.

The general problem of deciding if $\chi_=(G) \leq 3$ is NP-complete [4]. If, however, G has a regular or simplified structure we are sometimes able to provide a polynomial algorithm coloring it in the equitable way. In this paper we consider the equitable coloring of Cartesian, weak tensor and strong tensor products.

Graph products are interesting and useful in many situations. For example, Sabbidussi [11] showed that any graph has the unique decomposition into prime factors under the Cartesian product. Feigenbaum and Schäffer [3] showed that the strong tensor product admits a polynomial algorithm for decomposing a given connected graph into its factors. An analogous result with respect to weak tensor product is due to Imrich [7]. The complexity of many problems, also equitable coloring, that deal with very large and complicated graphs is reduced greatly if one is able to fully characterize the properties of less complicated prime factors.

This work presents some preliminary results of the problem of equitable coloring of graph products. We establish some exact values of the equitable chromatic number and the equitable chromatic threshold for the three above-mentioned products of some graphs with particular properties. Some bounds to these parameters are presented. In the next chapter, the complexity of the problem of coloring the Cartesian product of any two graphs is also established and proved. To our knowledge, this paper is the second one, after Chen's manuscript [2], concerning the equitable coloring of graph products.

2. EQUITABLE COLORING OF CARTESIAN PRODUCTS

Chen *et al.* in [2] obtained the following results.

Theorem 2.1 ([2]). *If G_1 and G_2 are equitably k-colorable, then so is $G_1 \square G_2$.*

As the corollary from this theorem we obtain the following.

Corollary 1. $\chi_=^*(G_1 \square G_2) \leq \max\{\chi_=^*(G_1), \chi_=^*(G_2)\}$.

Since $\chi_=^*(G_1) \leq \Delta(G_1) + 1$ and $\chi_=^*(G_2) \leq \Delta(G_2) + 1$, then $\chi_=^*(G_1 \square G_2) \leq \max\{\Delta(G_1) + 1, \Delta(G_2) + 1\}$. When both G_1 and G_2 have at least one edge, $\Delta(G_1 \square G_2) \geq \max\{\Delta(G_1)+1, \Delta(G_2)+1\}$. Therefore, $G_1 \square G_2$ is equitably $\Delta(G_1 \square G_2)$-colorable, i.e., the Equitable Δ-Coloring Conjecture holds. Sabidussi [11] proved the following result for the classical coloring of Cartesian products.

Theorem 2.2. $\chi(G_1 \square G_2) = \max\{\chi(G_1), \chi(G_2)\}$.

We know that the inequality in Corollary 1 cannot be replaced by equality. Let us consider the graph $K_{1,3} \square P_2$ (see Fig. 2). The equitable chromatic thresholds of the factors are 3 and 2, respectively, and the equitable chromatic threshold of $K_{1,3} \square P_2$ is equal to 2.
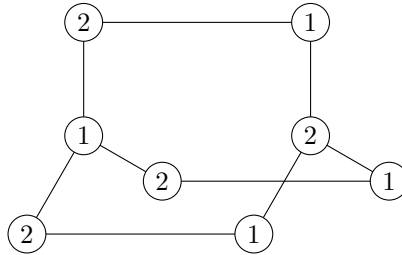

Fig. 2. The graph $K_{1,3} \square P_2$ and its equitable coloring

Unfortunately, the inequality $\chi_=(G_1 \square G_2) \leq \max\{\chi_=(G_1), \chi_=(G_2)\}$ does not hold for every pair of graphs. Chen *et al.* in [2] gave a counterexample where $G_1 = K_{3,3}$ and $G_2 = K_{2,1,1}$.

Due to the above theorems, the following corollary holds.

Corollary 2. *Let $G = G_1 \square G_2 \square \cdots \square G_n$, where each G_i is a path, a cycle, a hypercube or a complete graph. Then $\chi(G) = \chi_=(G) = \chi_=^*(G) = \max\{\chi(G_i) : i = 1, 2, \ldots, n\}$.*
$$\square$$

It is easy to give explicit formulas for the equitable chromatic number of Cartesian products of some classes of graphs. Chen *et al.* [2] proved the following.

Theorem 2.3 ([2]).

$$\chi_=(K_m \square K_n) = \chi_=^*(K_m \square K_n) = \max\{m, n\}. \tag{2}$$

$$\chi_=(C_m \square C_n) = \chi_=^*(C_m \square C_n) = \begin{cases} 2 & \text{if } m, n \text{ are even,} \\ 3 & \text{otherwise.} \end{cases} \tag{3}$$

Theorem 2.4. *Let $G_1(V_1, V_2)$ and $G_2(V_1', V_2')$ be any bipartite graphs such that one of them contains at least one edge and let $|V_1'| = |V_2'|$. Then*

$$\chi_=(G_1 \square G_2) = 2.$$

Proof. The graph $G_1 \square G_2$ is given in Figure 3. The polygons represent independent sets X_1, X_2, X_3 and X_4 of $|V_1||V_1'|$, $|V_1||V_2'|$, $|V_2||V_1'|$, $|V_2||V_2'|$ vertices, respectively. The lines show the possibilities of existing edges. If $|V_1'| = |V_2'|$, we can assign color 1 to the vertices of X_1 and X_4, and color 2 to the remaining vertices. The obtained coloring is equitable. $\square$
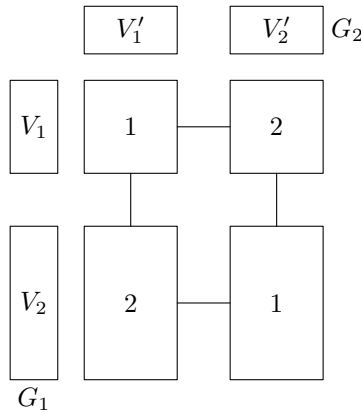

Fig. 3. The graph $G_1(V_1, V_2) \square G_2(V_1', V_2')$ and its equitable coloring. Polygons denote independent sets and numbers denote colors assigned to vertices

We observe an obvious fact:

Fact 3. *Let $G = (V, E)$ and $H = (V, E')$ be graphs with common vertex set such that $E \subseteq E'$. Then $\chi_=(G) \leq \chi_=(H)$.*

Corollary 4. *Let k, m, n and r be positive integers. Then the equitable chromatic numbers of the following graphs all equal to 2. $C_{2m} \square C_{2n}$, $P_m \square C_{2n}$, $Q_r \square C_{2n}$, $K_{k,m} \square C_{2n}$, $K_{1,m} \square C_{2n}$, $P_m \square P_{2n}$, $Q_r \square P_{2n}$, $K_{k,m} \square P_{2n}$, $K_{1,m} \square P_{2n}$, $Q_r \square Q_r$, where Q_d is a hypercube and $K_{k,m}$ is a complete bipartite graph.*

The result of Theorem 2.4 can be extended on r-partite graphs, $r \geq 2$.

Theorem 2.5. *Let $G_1(V_1, V_2, \ldots, V_r)$ and $G_2(V'_1, V'_2, \ldots, V'_r)$ be any r-partite graphs such that $|V'_1| = |V'_2| = \cdots = |V'_r|$. Then*

$$\chi_=(G_1 \square G_2) \leq r.$$

Proof. Use an $r \times r$ Latin square. $\square$

Moreover, it is interesting to note that the following theorem is also true.

Theorem 2.6. $\chi_=(G \square P_{3k}) \leq 3$ *if and only if G is a tripartite graph.*

Proof.

($\Leftarrow$) It simply follows from Theorem 2.5.

($\Rightarrow$) Since $G \subset G \square P_{3k}$, its chromatic number cannot be greater than the equitable chromatic number of $G \square P_{3k}$. $\square$

Theorem 2.7. *Let G, H be graphs. The problem of deciding whether $\chi_=(G \square H) \leq 3$ is NP-complete even if one of its factors is a path.*

Proof. This theorem immediately follows from Theorem 2.6 and the *NP*-completeness of the problem of 3-coloring in the classical sense. $\square$

Now we give further exact values of equitable chromatic numbers of some products.

Theorem 2.8. *Let m, n be positive integers, $m, n \geq 3$ and n is odd. Then*

$$\chi_=(K_{1,m} \square P_n) = 3. \tag{4}$$

Proof. For n odd, the graph $K_{1,m} \square P_n$ is a connected bipartite graph $(V_1 \cup V_2, E)$, where $|V_1| = (n-1)/2 + m(n+1)/2$ and $|V_2| = (n+1)/2 + m(n-1)/2$ (see Fig. 4). Since $|V_1| - |V_2| = m - 1$, an equitable 2-coloring is not possible for $m \geq 3$.
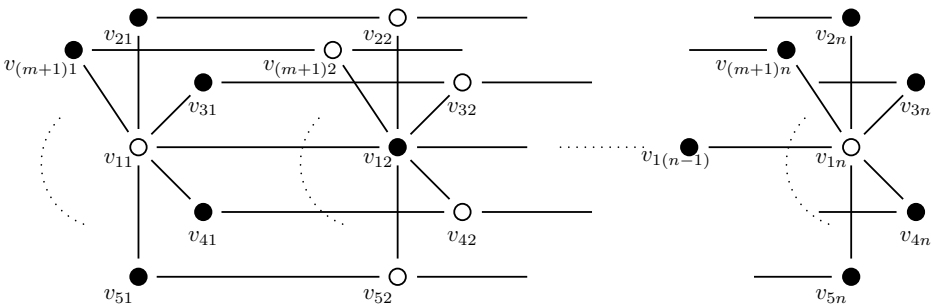

Fig. 4. A bipartite graph $(V_1 \cup V_2, E) = K_{1,m} \square P_n$ for odd n. Vertices marked with black circles belong to V_1, and the others to V_2

To show that the graph is equitably 3-colorable, we shall form an independent set $S \subset V_1 \cup V_2$ such that $|S| = \lceil n(m+1)/3 \rceil$, $|S \cap V_1| = |V_1| - \lceil (n(m+1)-1)/3 \rceil$ and

$|S \cap V_2| = |V_2| - \lceil (n(m + 1) - 2)/3 \rceil$, and then color the vertices of S with color 1, those in $V_1 \backslash S$ with color 2, and those in $V_2 \backslash S$ with color 3. The coloring is equitable.

Consider the sequence $v_{21}, v_{31}, \ldots, v_{(m+1)1}, v_{12}, v_{23}, \ldots, v_{(m+1)3}, v_{14}, \ldots, v_{2(n-2)}, \ldots, v_{(m+1)(n-2)}, v_{1(n-1)}, v_{2n}, \ldots, v_{(m+1)n}$ of vertices of V_1 (see Fig. 4). Notice that v_{21} has two neighbors in V_2 and adding each next element except one of $v_{2n}, v_{3n}, \ldots, v_{(m+1)n}$ increases the number of their neighbours in V_2 by one.

Now we choose the first $|V_1| - \lceil (n(m + 1) - 1)/3 \rceil$ elements of the sequence and put them into S. Since $\lceil (n(m+1)-1)/3 \rceil \geq m$ for $n \geq 3$, they will form the set $S \cap V_1$ such that $|N(S \cap V_1)| = |S \cap V_1| + 1$. The vertices from $S \cap V_1$ and $V_2 \backslash N(S \cap V_1)$ form an independent set of cardinality $|V_2| - 1$. Since $|V_2| > \lceil n(m + 1)/3 \rceil$ for every odd $n \geq 3$ and $m \geq 3$, so it is possible to select the remaining elements of the set S from $V_2 \backslash N(V_1 \cap S)$. $\qquad \square$

3. EQUITABLE COLORING OF WEAK TENSOR PRODUCTS

In 1966 Hedetniemi [6] formulated a conjecture concerning the classical coloring of weak tensor products. A good survey on this conjecture is given in [15].

Conjecture 3 ([6])**.** $\chi(G_1 \times G_2) = \min\{\chi(G_1), \chi(G_2)\}$.

If Conjecture 3 is true, then:

$$\chi_=(G_1 \times G_2) \geq \min\{\chi(G_1), \chi(G_2)\}. \tag{5}$$

There also holds the following

Lemma 5.
$$\chi_=(G_1 \times G_2) \leq \min\{|V(G_1)|, |V(G_2)|\}. \tag{6}$$

Proof. Let $V(G_1) = \{u_0, u_1, \ldots, u_l\}$ and let $U_i = \{u_i\} \times V(G_2)$, $i = 0, 1, \ldots, l$. Then U_i is an independent vertex set in the graph $G_1 \times G_2$ and $|U_i| = |V(G_2)|$, $i = 0, 1, \ldots, l$. So $\chi_=(G_1 \times G_2) \leq |V(G_1)|$. The inequality $\chi_=(G_1 \times G_2) \leq |V(G_2)|$ is proved in a similar way. $\qquad \square$

We know that the bound given in Lemma 5 on the equitable chromatic number is not a bound on the equitable chromatic threshold. Chen *et al.* [2] gave a counterexample of $K_2 \times K_n$. They suggest that the following holds.

Conjecture 4 ([2])**.**

$$\chi_=^*(G_1 \times G_2) \leq \max\{|V(G_1)|, |V(G_2)|\}. \tag{7}$$

Now we give some formulas involving the equitable chromatic number of weak tensor products for some graphs having particular properties.

Theorem 3.1 ([2])**.**

$$\chi_=(K_m \times K_n) = \min\{m, n\}. \tag{8}$$

$$\chi_=(C_m \times C_n) = \chi_=^*(C_m \times C_n) = \begin{cases} 2 & \text{if } mn \text{ is even,} \\ 3 & \text{otherwise.} \end{cases} \tag{9}$$

Theorem 3.2. *Let G, H be graphs with at least one edge each and let $G = (V_1 \cup V_2, E)$ be a bipartite graph such that $|V_1| = |V_2|$. Then*

$$\chi_=(G \times H) = 2.$$

Proof. By the definition, the sets $V_1 \times V(H)$ and $V_2 \times V(H)$ are independent vertex sets. Since $|V_1| = |V_2|$, the sets have the same number of elements. We assign color number 1 to vertices from $V_1 \times V(H)$ and color number 2 to the remaining vertices. The coloring is an equitable 2-coloring. Since $E(G \times H) \neq \emptyset$, the coloring is optimal. $\qquad\square$

Using Theorem 3.2, we obtain simple corollaries.

Corollary 6. *Let H be any graph with at least one edge each and let d, m and n be positive integers. Then:*

$$\chi_=(C_n \times H) = 2 \text{ for even } n,$$
$$\chi_=(P_n \times H) = 2 \text{ for even } n,$$
$$\chi_=(Q_d \times H) = 2,$$
$$\chi_=(K_{m,n} \times H) = 2 \text{ for } m = n.$$

Corollary 7. *Let $G, H_1, H_2, \ldots, H_l$ be graphs with at least one edge and let $G = (V_1 \cup V_2, E)$ be bipartite graph such that $|V_1| = |V_2|$. Then*

$$\chi_=(G \times H_1 \times H_2 \times \ldots \times H_l) = 2.$$

The result of Theorem 3.2 can be extended.

Theorem 3.3. *Let H be any graph with at least one edge and let $G = (V_1 \cup V_2 \cup \cdots \cup V_r, E)$ be r-partite such that $|V_i| = |V_j|$ for any $i \neq j$. Then*

$$\chi_=(G \times H) \leq r.$$

Corollary 8. *Let H be any graph and let k and n be positive integers. Then:*

$$\chi_=(C_n \times H) \leq 3, \quad \text{for} \quad n = 3k,$$
$$\chi_=(K_n \times H) \leq n.$$

Now we give some exact values of the equitable chromatic number of products of some graphs.

Theorem 3.4. *Let m and n be positive integers. Then*

$$\chi_=(K_{1,m} \times K_{1,n}) = \frac{(m+1)(n+1)}{\max\{m,n\}+1} = \min\{m,n\}+1. \tag{10}$$

Proof. First, we notice that $K_{1,m} \times K_{1,n} = K_{1,m \cdot n} \cup K_{m,n}$. Without loss of generality, we can assume that $n \geq m$. The maximal size of color classes containing the universal vertex of $K_{1,m \cdot n}$, let us say x, is equal to $n + 1$. We assign color number 1 to vertex x and to n independent vertices of $K_{m,n}$. We have $mn + m = m(n + 1)$ uncolored independent vertices. We form m color classes of equal size. Hence

$$\chi_=(K_{1,m} \times K_{1,n}) \leq \frac{(m+1)(n+1)}{\max\{m,n\}+1} = \min\{m,n\} + 1.$$

Now we show that we cannot use smaller number of colors. Let us assume the contrary. Color 1 can be used at most $n+1$ times. We will try to divide the $m(n+1)$ uncolored independent vertices into l_1 classes of size $n+1$ and l_2 classes of size $n+2$, where $l_1 \geq 0$, $l_2 \geq 1$ and $l_1 + l_2 < m$. We have

$$m(n + 1) = l_1(n + 1) + l_2(n + 2).$$

Since $n + 1$ and $n + 2$ are relatively prime, then $m - l_1 = k(n + 2)$, $k \geq 1$. But we have assumed that $n \geq m$. It is a contradiction. We have proved that Formula (10) is true. □

Now we give a more general theorem.

Theorem 3.5. *Let m, n be positive integers, $m, n > 1$. Then*

$$\chi_=(K_n \times P_m) = \begin{cases} 2 & \text{if } m \text{ is even or } n = 2, \\ 3 & \text{otherwise.} \end{cases}$$

Proof. The graph $K_n \times P_m$ is a bipartite graph $(V_1 \cup V_2, E)$ such that every second column belongs to V_2, i.e., $|V_1| = \lceil m/2 \rceil n$ and $|V_2| = \lfloor m/2 \rfloor n$ (see Fig. 5).
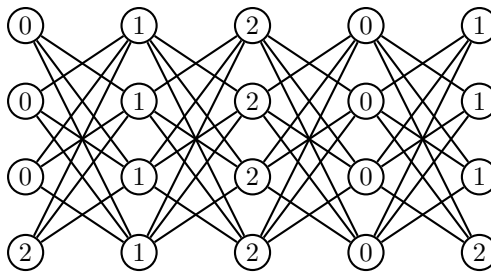

Fig. 5. The graph $K_4 \times P_5$ and its equitable 3-coloring

The case of m even or $n = 2$ was described earlier. We now have to prove the remaining one, mainly when m is odd and $n \geq 3$.

We first color the i-th column with $i \bmod 3$, then change the colors of some vertices in the first and last column in the following way. If $m = 6k + 1$, $k \geq 1$,

vertices in the first and last column are colored with 0. We change the color of $\lceil (n-2)/3 \rceil$ vertices in the first column to 2 and $\lceil (n-1)/3 \rceil$ vertices in the last column to 1. The case of $m = 3k$, $k \geq 1$, was described earlier, so it remains to consider the situation when $m = 6k - 1$, $k \geq 1$. Then we change the colors of $n - \lceil 2n/3 \rceil$ vertices from the first column and $n - \lceil (2n-1)/3 \rceil$ vertices from the last column to 2.

So the obtained colorings are equitable. Moreover, since m is odd, $|V_1| - |V_2| = n \geq 3$ and we cannot use less than three colors. □

Corollary 9. *Let G be a graph with n vertices and at least one edge and let m, n be positive integers. Then*

$$\chi_=(G \times P_m) \leq \begin{cases} 2 & \text{if } m \text{ is even,} \\ 3 & \text{otherwise.} \end{cases}$$

Corollary 10. *Let $m, n \geq 3$ be odd positive integers. Then*

$$\chi_=(P_m \times P_n) = 3. \tag{11}$$

Proof. Corollary 9 implies $\chi_=(P_m \times P_n) \leq 3$. The graph $P_m \times P_n$ is a disjoint union of two bipartite graphs $G_1(V_1, V_2)$ and $G_2(V_1', V_2')$ such that $|V_1| = \lceil n/2 \rceil \cdot \lceil m/2 \rceil$, $|V_2| = \lfloor n/2 \rfloor \cdot \lfloor m/2 \rfloor$, $|V_1'| = \lfloor n/2 \rfloor \cdot \lceil m/2 \rceil$ and $|V_2'| = \lceil n/2 \rceil \cdot \lfloor m/2 \rfloor$. Without loss of generality, assume $n \geq m$. Then $|V_1| \geq |V_2|$, $|V_1'| \geq |V_2'|$ and $|V_1| + |V_2'| - (|V_2| + |V_1'|) = m \geq 3$. So we cannot color this graph equitably with two colors. □

Corollary 11. *Let m, n be positive integers and $n \geq 2$. Then*

$$\chi_=(K_{1,m} \times P_n) = \begin{cases} 2 & \text{if } n \text{ is even or } m = 1, \\ 3 & \text{otherwise.} \end{cases} \tag{12}$$

Proof. The graph $K_{1,m} \times P_n$ consists of two components that are bipartite graphs $H_1 = (V_1 \cup V_2, E)$ and $H_2 = (V_1' \cup V_2', E')$, where $|V_1| = \lceil n/2 \rceil$, $|V_2| = \lfloor n/2 \rfloor \cdot m$, $|V_1'| = \lfloor n/2 \rfloor$ and $|V_2'| = \lceil n/2 \rceil \cdot m$ (see Fig. 6).
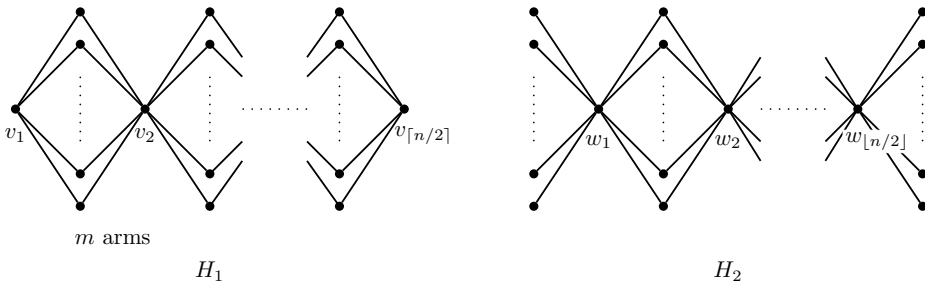

Fig. 6. A graph $K_{1,m} \times P_n$.

When n is even, an equitable 2-coloring is easy to achieve. We color all vertices in $V_1 \cup V_2'$ with color 1 and all vertices in $V_1' \cup V_2$ with color 2. We have used each color $n/2 + mn/2$ times, so the coloring is equitable. For $m = 1$, the graph $K_{1,m} \times P_n$ is a disjoint union of two paths of length $n - 1$, so it can be colored equitably with two colors.

In the case of n odd and $m \geq 2$, the following inequalities hold: $(|V_1| + |V_2'|) - (|V_1'| + |V_2|) > 1$ and $(|V_2| + |V_2'|) - (|V_1| + |V_1'|) > 1$. It is impossible to color $K_{1,m} \times P_n$ equitably with 2 colors. Formula (12) is true due to Corollary 9. $\qquad \square$

Corollary 11 implies that a theorem similar to Theorem 2.1 is not true for weak tensor products. We cannot say that if G_1 and G_2 are equitably k-colorable, then so is $G_1 \times G_2$. For example, $K_{1,2}$ and P_3 are equitably 2-colorable, but their weak tensor product is not.

4. EQUITABLE COLORING OF STRONG TENSOR PRODUCTS

We know some lower bounds for the equitable chromatic number of strong tensor products of graphs. They follow from Vesztergombi's [12] and Jha's [8] results concerning the proper coloring of this product.

Theorem 4.1 ([12])**.** *Let G_1, G_2 be graphs with at least one edge each. Then*

$$\chi_=(G_1 \boxtimes G_2) \geq \max\{\chi(G_1), \chi(G_2)\} + 2. \tag{13}$$

Theorem 4.2 ([8])**.** *Let G_1, G_2 be graphs with at least one edge each. Then*

$$\chi_=(G_1 \boxtimes G_2) \geq \chi(G_1) + \omega(G_2). \tag{14}$$

In the rest of this section, we collect several exact results.

Theorem 4.3. *Let m, n be positive integers and $m, n > 1$. Then*

$$\chi_=(P_m \boxtimes P_n) = \begin{cases} 4 & \text{if } nm \text{ is even,} \\ 5 & \text{otherwise.} \end{cases} \tag{15}$$

Proof. As we know, $E(P_m \boxtimes P_n) = E(P_m \square P_n) \cup E(P_m \times P_n)$. In the case of nm even, the coloring is straightforward. Let us assume that m is even. We color the vertices in odd rows with colors 1 and 2, alterately, but in rows $2k + 1$ for even k we start with color 1, and in rows $2k + 1$ for odd k we start with color 2. Even rows are colored with colors 3 and 4, alternately, but we start coloring the vertices in $2k$-th rows for odd k with color 3 and in others with color 4 (cf. Fig. 1). Each color is used $\left\lceil \frac{mn}{4} \right\rceil$ or $\left\lfloor \frac{mn}{4} \right\rfloor$ times and $\omega(P_m \boxtimes P_n) = 4$, so the obtained coloring is equitable and optimal.

In the second case, both m and n are odd. n might take one of five forms: $5k - 4$, $5k - 2$, $5k$, $5k + 2$, $5k + 4$ for some odd k.

Now we have to consider five cases.

1) $n = 5k$

This case is the simplest one. We color each row with colors $1, 2, 3, 4$ and 5, consecutively. In the i-th row, we start with color s, where

$$s = \begin{cases} 5 & \text{if } 2i - 1 \equiv 0 \pmod 5, \\ (2i - 1) \bmod 5 & \text{otherwise.} \end{cases} \tag{16}$$

We have used each color $mn/5$ times, so the coloring is equitable. Next two cases are based on this algorithm.

2) $n = 5k - 4$

We can change the form of n to $5l + 1$, for some even l. Then, we equitably color the graph $P_m \boxtimes P_{5l} \subset P_m \boxtimes P_{5l+1}$ in the way shown in Case 1. Vertices in the last column of $P_m \boxtimes P_{5l+1}$ are not colored. Each of these vertices (in the i-th row) gets color number s, where s is determined as above. Each color is used $\lceil mn/5 \rceil$ or $\lfloor mn/5 \rfloor$ times, so the coloring is equitable.

3) $n = 5k + 4$

First, we equitably color $P_m \boxtimes P_{5l}$, where $l = k + 1$. If we remove the vertices in the last column and edges incident to them, we will obtain a graph $P_m \boxtimes P_{5k+4}$ with proper equitable coloring. Each color is used $\lceil mn/5 \rceil$ or $\lfloor mn/5 \rfloor$ times.

4) $n = 5k + 2$

In this case, the equitable coloring is obtained as follows. We color the first row with colors 1, 2, 3, 4 and 5, consecutively. The vertex in the last column gets color number $2 \equiv 5k + 2 \pmod 5$. We continue coloring in the next row starting with color number $((5k + 2) \bmod 5) + 1$ and so on (see Fig. 7).

5) $n = 5k - 2$

This case is very similar to Case 4. We color the first row with colors 1, 2, 3, 4 and 5, consecutively. The vertex in the last column gets color number $3 \equiv 5k - 2 \pmod 5$. We continue coloring in the next row starting with color number $((5k - 2) \bmod 5) + 1$ and so on.
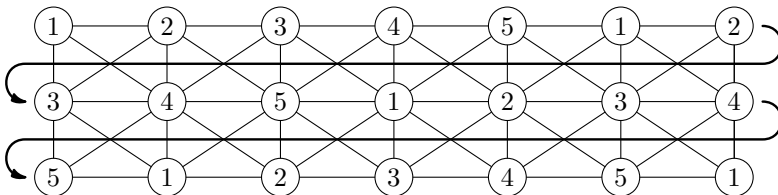

Fig. 7. The graph $P_3 \boxtimes P_7$ with its equitable coloring. The thick arrows show the order of color assignments.

In all five cases we have obtained an equitable coloring with five colors. It remains to show that the strong tensor product of two odd paths does not have an equitable 4-coloring. Suppose that there is such a coloring. We claim that either exactly two colors appear alternately in every row or exactly two colors appear alternately in every column. Moreover, in, say odd, rows four distinct colors appear consecutively, say 1, 2, 3 and 4. In even rows, the pattern must be 3, 4, 1 and 2. It now follows that one of the colors appears in the first or last column too many times. It is a contradiction. □

Theorem 4.4. *Let l_1, l_2 be integers such that $l_1 \geq 0$ and $l_2 \geq 2$. Then*

$$\chi_=(C_{5(2l_1+1)} \boxtimes C_{2l_2+1}) = 5. \tag{17}$$

Proof. We start with an equitable coloring of $C_5 \boxtimes C_5$. It is shown in Figure 8, where the entry c_{ij} denotes the color of vertex v_{ij}.

1	3	5	2	4
2	4	1	3	5
3	5	2	4	1
4	1	3	5	2
5	2	4	1	3

Fig. 8. A coloring matrix C of the graph $C_5 \boxtimes C_5$.

For $C_5 \boxtimes C_{2l_2+1}$ ($l_2 > 2$), we color the first 5 columns in the same way as in $C_5 \boxtimes C_5$, and we repeat the colorings of the 4-th and 5-th column ($l_2 - 2$) times. This trivially gives an equitable coloring.

In the case of $C_{5(2l_1+1)} \boxtimes C_{2l_2+1}$, $l_1 > 0$, first we equitably color $C_5 \boxtimes C_{2l_2+1}$, then each i-th row, $i > 5$, is colored like the s-th one, where s

$$s = \begin{cases} 5 & \text{if } i \equiv 0 \pmod{5}, \\ i \bmod 5 & \text{otherwise.} \end{cases} \tag{18}$$

It is easy to see that the coloring is equitable. Since $\chi(C_{2l_1+1} \boxtimes C_{2l_2+1}) = 5$ ([12]), then our coloring is optimal. □

It would be interesting to give some relations between the equitable chromatic number or threshold of a graph product and equitable chromatic numbers or thresholds of its factors. The products of more than two graphs are also of interest.

REFERENCES

[1] Chen B.-L., Lih K.-W., *Equitable coloring of trees*, J. Combin. Theory, Ser. B **61** (1994), 83–87.

[2] Chen B.-L., Lih K.-W., Yan J.-H., *Equitable coloring of graph products*, manuscript, 1998.

[3] Feigenbaum J., Schäffer N. W., *Finding the prime factors of strong direct product graphs in polynomial time*, Discrete Math., **109** (1992), 77–102.

[4] Furmańczyk H., *Equitable coloring of graphs*, in: M. Kubale (ed.), Optymalizacja dyskretna. Modele i metody kolorowania grafów, WNT Warszawa 2002, 72–92 (in Polish).

[5] Hajnal A., Szemeredi E., *Proof of a conjecture of Erdös*, in: Combinatorial Theory and Its Applications, II 601–623, Colloq. Math. Soc. Jnos Bolyai, Vol. **4**, North-Holland, Amsterdam, 1970.

[6] Hedetniemi S., *Homomorphism of graphs and automata*, Univ. Michigan, Technical Report 03105-44-T, 1966.

[7] Imrich W., *Factoring cardinal product graphs in polynomial time*, Discrete Math. **192** (1998), 119–144.

[8] Jha P. K., *Hypercubes, Median graphs and products of graphs: Some algorithmic and combinatorial results*, Ph.D. Thesis, Iowa State Univ., 1990.

[9] Lih K.-W., Wu P.-L., *On equitable coloring of bipartite graphs*, Discrete Math., **151** (1996), 155–160.

[10] Meyer W. *Equitable coloring*, Amer. Math. Monthly **80** (1973), 920–922.

[11] Sabidussi G., *Graphs with given group and given graph-theoretical properties*, Canad. J. Math. **9** (1957), 515–525.

[12] Vesztergombi K., *Some remarks on the chromatic number of the strong product of graphs*, Colloq. Math. Soc. Janos Bolyai **25** (1981), 819–825.

[13] Yap H. P., Zhang Y., *The equitable Δ-coloring conjecture holds for outerplanar graphs*, Bulletin of the Inst. of Math. Academia Sinica **25** (1997), 143–149.

[14] Yap H. P., Zhang Y., *Equitable colorings of planar graphs*, J. Combin. Math. Combin. Comput. **27** (1998), 97–105.

[15] Zhu X., *A survey on Hedetniemi's conjecture*, Taiwanese J. Math. **2** (1998), 1–24.

Hanna Furmańczyk
hanna@math.univ.gda.pl

Gdańsk University
Mathematical Institute
Wita Stwosza 57, 80-952 Gdańsk, Poland