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A NOTE ON SELF-COMPLEMENTARY HYPERGRAPHS

Abstract. In the paper we desribe all self-complementary hypergraphs. It turns out that such hypergraphs exist if and only if the number of vertices of the hypergraph is of the form $n = 2^k$. This answers a conjecture posed by A. Szymański (see [3]).

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Let us fix a set of vertices V, $|V| = n$. For $1 \leq k \leq n$ denote by $\binom{V}{k}$ the set of all k-elementary subsets of V. A *k-uniform hypergraph* X is any pair (V, E), where $V = V(X)$ is called a *vertex set* and $E = E(X) \subset \binom{V(X)}{k}$ is called an *edge set*.

Two k-uniform hypergraphs X and Y are called *isomorphic* if there is a one-to-one mapping $\theta \colon V(X) \mapsto V(Y)$, which induces the one-to-one mapping $E(X) \mapsto E(Y)$. The mapping θ is called an *isomorphism* between X and Y.

A k-uniform hypergraph X is called *self-complementary* if X is isomorphic to its complement $\overline{X}$. *The complement $\overline{X}$ of the hypergraph X* is defined as the pair $\Big(V(X), \binom{V(X)}{k} \setminus E(X)\Big)$.

Since any isomorphism of the k-uniform hypergraph X to its complement $\bar{X}$ is a one-to-one mapping of the set of vertices $V(X)$, we lose no generality assuming that $V(X) = \{1, \ldots, n\}$, so the one-to-one mapping is a permutation. We call such a permutation a *k-complementing permutation*.

Below we shall see that there is a very close correspondence between the class of self-complementary k-uniform hypergraphs and the set of k-complementing permutations. Namely, let θ be a permutation of the set $V = \{1, \ldots, n\}$. Then one may try to construct a self-complementary k-uniform hypergraph X induced by this permutation as follows.

Take any k-elementary subset $A_1 \subset \binom{V}{k}$. We define X (and simultaneously $\overline{X}$) as follows. The set $\theta^j(A_1)$ is from $E(X)$ iff j is even and the set $\theta^j(A)$ is from

$E(\overline{X})$ iff j is odd. Take an arbitrary element $A_2 \in \binom{V}{k} \setminus \bigcup_{j=0}^{\infty} \theta^j(A_1)$ (actually, the union here is a finite one) and we define elements from X and $\overline{X}$ as earlier (with A_1 replaced by A_2), i.e. $\theta^j(A_2) \in E(X)$ iff j is even and $\theta^j(A_2) \in E(\overline{X})$ iff j is odd. We proceed similarly till we exhaust all elements of $\binom{V}{k}$. Note that the presented procedure leads us to a well-defined self-complementary hypergraph X iff for any $A \in \binom{V}{k}$ the inequality $\theta^j(A) \neq A$ holds for any j odd. In such a case the permutation θ is a k-complementing permutation.

Remark that a permutation θ is a k-complementing permutation if and only if for any $A \in \binom{V}{k}$ the inequality $\theta^j(A) \neq A$ holds for any j odd.

Note that 2-uniform hypergraphs are simple graphs. So the problem of the description of 2-uniform self-complementary hypergraphs is the problem of the description of self-complementary graphs. There is a rich literature on this topic, see e.g. [1].

A full description of 3-uniform self-complementary hypergraphs was given in [2], and a full description of 4-uniform self-complementary hypergraphs was given in [3]. Both descriptions are given with the help of a characterization of complementing permutations.

We define *a hypergraph* X as a pair $(V(X), E(X))$, where $V(X) = V$ is a set with n elements called a *vertex set* and $E(X) \subset \bigcup_{k=1}^{n} \binom{V}{k}$. Denote $E_k(X) := E(X) \cap \binom{V}{k}$, $k = 1, \ldots, n$. Let us denote by X_k a k-uniform hypergraph $(V(X), E_k(X))$.

Then one may introduce a notion of a self-complementary hypergraph as a hypergraph isomorphic to its complement. In particular, it would imply that the k-uniform hypergraph X_k would be a self-complementary k-uniform hypergraph for any $k = 1, \ldots, n$. However, as one may easily verify there are no self-complementary hypergraphs in this sense. The reason why this is so is that the necessary condition for a k-uniform hypergraph to be self-complementary is that the set $\binom{V}{k}$ has even number of elements, $k = 1, \ldots, n$ – but $\left|\binom{V}{n}\right| = 1$ is never even.

Therefore, let us define the *complement* $\bar{X} = (V(\bar{X}), E(\bar{X}))$ of the hypergraph $X = (V(X), E(X))$ as a hypergraph such that $V(\bar{X}) = V(X)$ and $E(\bar{X}) = \bigcup_{k=1}^{n-1} \binom{V}{k} \setminus E(X)$. The hypergraph X is called *self-complementary* if X is isomorphic to $\bar{X}$, i.e. there is a one-to-one mapping $\theta \colon V(X) \mapsto V(\bar{X})$, which induces the one-to-one mapping $E(X) \mapsto E(\bar{X})$ – it implies, in particular that $E_n(X) = \emptyset$. The mapping θ is called an *isomorphism*.

Any permutation of the vertex set $V(X) = \{1, \ldots, n\}$ such that θ is an isomorphism of the hypergraph X to $\bar{X}$ is called a *complementing permutation*. Note that any permutation θ is a complementing permutation if and only if θ is a k-complementing permutation for any $k = 1, \ldots, n-1$.

It turns out that there is a complete description of self-complementary hypergraphs in terms of the number of vertices of the hypergraph and the hypergraphs are defined with the help of complementing permutations. We present below this description and we give its proof.

Theorem. *Let X be a self-complementary hypergraph with $V(X) = \{1, \ldots, n\}$. Then $n = 2^k$ for some $k = 1, 2, \ldots$.*

Moreover, the permutation given as a cycle $\theta := (1\,2\,\ldots\,2^k)$ induces a self-complementary hypergraph X for any $k = 1, 2, \ldots$. Conversely, any hypergraph with the vertex set equal to $\{1, \ldots, n\}$, where $n = 2^k$, is induced by a permutation which is a cycle of the length 2^k.

In other words, if $n = 2^k$ then the permutation θ is a complementing permutation if and only if θ is a cycle of the length 2^k.

The above theorem answers positively a conjecture posed by A. Szymański in [3], stating that a necessary condition for a self-complementary hypergraph is that its vertex set has 2^k elements for some k.

Proof. First we prove that the necessary condition for a hypergraph X to be self-complementary is that $n = 2^k$ for some k.

Suppose it does not hold. Then $2^k(2m+1) = n$ for some $m \geq 1$ and $k \geq 0$. Note that the necessary condition for a hypergraph to be self-complementary is that for any $1 \leq N \leq n-1$ the number of elements of $\binom{V(X)}{N}$, i.e. the number $\binom{n}{N}$ is even. It implies, in particular, substituting $N = 1$, that $k \geq 1$ and n is even. We show below that $\binom{n}{2^k}$ is odd, which gives the contradiction and finishes the proof. □

Note that
$$\binom{n}{2^k} = \frac{n(n-1) \cdot \ldots \cdot (n - 2^k + 1)}{1 \cdot 2 \cdot \ldots \cdot 2^k}.$$
To see that the above number is odd it is sufficient to show that for any even number $t = 2, \ldots, 2^k - 2$ the number $\frac{n-t}{t}$ is not even (either odd or not from $\mathbb{Z}$). So fix an even $2 \leq t \leq 2^k - 2$. Then $t = 2^r(2s+1)$, where $s \geq 0$, $1 \leq r \leq k-1$. Then

$$\frac{n-t}{t} = \frac{2^r(2^{k-r}(2m+1) - (2s+1))}{2^r(2s+1)},$$

which is trivially not even.

Now we show that for the fixed $k \geq 1$ the cycle $\theta = (1\ 2\ \ldots\ 2^k)$ is a complementing permutation, i.e. it is an l-complementing permutation for any $l = 1, 2, \ldots, 2^k - 1$.

For simplicity of the notation rewrite the set of vertices as the set $\mathbb{Z}_{2^k} = \{0, 1, \ldots, 2^k - 1\}$. We denote by $+_{2^k}$ the addition modulo 2^k.

To finish the proof of our claim it is sufficient to show that for any $\emptyset \neq A \subsetneq \mathbb{Z}_{2^k}$ the inequality $\theta^{2t+1}(A) \neq A$ holds for any $t = 0, 1, \ldots$.

Suppose the contrary. Therefore, there are some $1 \leq l \leq 2^k - 1$ and $A = \{j_1, \ldots, j_l\}$ such that $\theta^{2t+1}(A) = A$ for some $t \geq 0$. Since θ^{2t+1} is a one-to-one mapping of A, we conclude that $(\theta^{2t+1})^m(j_1) = j_1$ for some $1 \leq m \leq l \leq 2^k - 1$. Therefore,
$$j_1 = \theta^{(2t+1)m}(j_1) = j_1 +_{2^k} (2t+1)m.$$
Consequently, 2^k is a divisor of $(2t+1)m$, which contradicts the property $m < 2^k$.

In order to finish the proof it is sufficient to show that all self-complementary hypergraphs with 2^k vertices are induced by cycles of length 2^k.

Denote the set of vertices of the self-complementary hypergraph X by $\{1,\ldots,2^k\}$. Let σ be a complementing permutation of the hypergraph X. Certainly, $\sigma \neq \mathrm{id}$. We may present the permutation σ as follows:

$$\sigma = \sigma_1 \cdot \ldots \cdot \sigma_t,$$

where σ_j's are disjoint cycles, $j = 1,\ldots,t$, $1 \leq t \leq 2^k - 1$. Fix j. Let $\sigma_j = (v_{j,1} \ldots v_{j,k_j})$, $k_j \geq 1$, $j = 1,\ldots,t$. Since

$$\sigma(\{v_{j,1},\ldots,v_{j,k_j}\}) = \sigma_j(\{v_{j,1},\ldots,v_{j,k_j}\}) = \{v_{j,1},\ldots,v_{j,k_j}\}, \quad j = 1,\ldots,t,$$

the fact that σ is an l-complementing permutation for $l = 1,\ldots,2^k - 1$ implies that $k_j = 2^k$, $j = 1,\ldots,t$. Consequently, $t = 1$ and σ is a cycle of the length 2^k as claimed in the theorem. □

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