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INDEPENDENT SET DOMINATING SETS IN BIPARTITE GRAPHS

Abstract. The paper continues the study of independent set dominating sets in graphs which was started by E. Sampathkumar. A subset D of the vertex set $V(G)$ of a graph G is called a set dominating set (shortly *sd–set*) in G, if for each set $X \subseteq V(G) - D$ there exists a set $Y \subseteq D$ such that the subgraph $\langle X \cup Y \rangle$ of G induced by $X \cup Y$ is connected. The minimum number of vertices of an *sd–set* in G is called the set domination number $\gamma_s(G)$ of G. An *sd–set* D in G such that $|D| = \gamma_s(G)$ is called a γ_s-set in G. In this paper we study *sd–sets* in bipartite graphs which are simultaneously independent. We apply the theory of hypergraphs.

Keywords: set dominating set, set domination number, independent set, bipartite graph, multihypergraph..

Mathematics Subject Classification: 05C69, 05C65.

1. INTRODUCTION

All considered graphs are finite undirected graphs without loops and multiple edges. Some concepts which were introduced by E. Sampathkumar [1] are discussed for the case of bipartite graphs.

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The set domination number is a variant of the well-known domination number of a graph. A subset $D \subseteq V(G)$ is called a dominating set in G, if for each vertex $x \in V(G) - D$ there exists a vertex $y \in D$ adjacent to x. The minimum number of vertices of a dominating set in G is called the domination number $\gamma(G)$ of G.

In [1] E. Sampathkumar notes that each *sd–set* in G is a dominating set in G and consequently $\gamma(G) \leq \gamma_s(G)$. He also proves that if G is an independent *sd–set*, then $\operatorname{diam} G \leq 4$, where $\operatorname{diam} G$ denotes the diameter of G.

In the sequel we shall use the notation $N(x)$ for the open neighbourhood of a vertex x (the set of all vertices adjacent to x in G).

We prove a lemma.

Lemma 1.1. *Let G be a graph with the vertex set $V(G)$, let $D \subseteq V(G)$ be independent. Then the following two assertions are equivalent:*

(i) *For each independent subset $X^* \subseteq V(G) - D$ with $|X^*| \leq 2$ there exists a vertex $y \in D$ adjacent to all vertices of X^*.*

(ii) *The set D is an sd–set in G.*

Proof. (i) ⇒(ii). Let $X \subseteq V(G) - D$. By induction according to the number r of vertices of X we shall prove that there exists $Y \subseteq D$ such that $\langle X \cup Y \rangle$ is connected. For $r \leq 2$ the assertion is assumed. Suppose that it holds for the number $r-1$ and let $|X| = r$ (for $r \geq 3$). Let $a \in X$, $X_0 = X - \{a\}$. Then $|X_0| = r-1$ and there exists $Y_0 \subseteq D$ such that $\langle X_0 \cup Y_0 \rangle$ is connected. If a is adjacent to a vertex of $X_0 \cup Y_0$, we may put $X = Y_0$ and the assertion is true. If not, let $b \in X_0$; then the set $\{a, b\}$ is independent and has two elements. Therefore there exists a vertex $y \in D$ adjacent to both a, b. We may put $Y = Y_0 \cup \{y\}$ and the assertion is true. As X was chosen arbitrarily, (ii) holds.

(ii) ⇒(i). Let $X^* = \{u, v\} \subseteq V(G) - D$ and let u, v be non-adjacent. Then there exists $Y \subseteq D$ such that $\langle X \cup Y \rangle$ is connected. Let $Y_1 = Y \cap N(a)$, $Y_2 = Y \cap N(b)$. As D is independent in G, the graph $\langle X \cup Y \rangle$ may be connected if and only if $Y_1 \cap Y_2 \neq \emptyset$. Therefore the required vertex $y \in Y_1 \cap Y_2$. □

2. AUXILIARY CONSIDERATIONS ON HYPERGRAPHS

For the formulation of results concerning *sd–set*s we shall apply the hypergraph theory. More precisely, we treat objects which might be called "multihypergraphs" or hypergraphs with multiple edges. A multihypergraph H is an ordered pair $(V(H), \mathcal{E}(H))$ of sets, where $V(H)$ is a set of elements called vertices of H and $\mathcal{E}(H)$ is a family of elements called edges, while there exists a mapping $\mathcal{E}$ of $\mathcal{E}(H)$ into the family $\mathcal{P}(V(H))$ of all subsets of $V(H)$. If $\mathcal{E}$ is an injection, we may call H a hypergraph and identify edges with their images in $\mathcal{E}$. Then we say that an edge of H is a subset of $V(H)$.

To every multihypergraph H we may assign its total graph $TG(H)$. The vertex set of $TG(H)$ is $V(TG(H)) = V(H) \cup \mathcal{E}(H)$ and two vertices x, y of $TG(H)$ are adjacent if and only if either $x \in V(H)$, $y \in \mathcal{E}(H)$, $x \in \mathcal{E}(y)$, or $y \in V(H)$, $x \in \mathcal{E}(H)$, $y \in \mathcal{E}(x)$.

Now consider a complete graph K_r for a positive integer r. An edge-covering multihypergraph of K_r is a multihypergraph H with the vertex set $V(H) = V(K_r)$

and with the edge set $\mathcal{E}(H)$ with the property that each edge of K_r (as a two-element subset of $V(K_r)$) is contained as a subset in $\mathcal{E}(H)$ for some edge $E \in \mathcal{E}(H)$. The class of all edge-covering multihypergraphs of K_r will be denoted by $EC(r)$.

Now for our purposes we introduce a quite special concept. By $\Omega(r)$ we denote the class of graphs which are $TG(H)$ for $H \in EC(r)$.

After these considerations in the next paragraph we return to *sd–sets* in graphs.

3. RESULTS CONCERNING INDEPENDENT *sd−sets*

Here we shall consider such *sd−sets* in bipartite graphs which are simultaneously independent.

Theorem 3.1. *Let r be a positive integer, let G be an arbitrary graph from $\Omega(r)$. Then G is a bipartite graph with bipartition classes A, B such that A is an sd−set in G.*

Proof. The graph G is in $TG(H)$ for some $H \in EC(r)$. Therefore if we put $A = \mathcal{E}(H)$, $B = V(H)$ in G, then G is a bipartite graph with the bipartition classes A, B. Let x, y be two vertices of $B = V(H)$. The vertices x, y are adjacent in K_r, therefore there exists $E \in \mathcal{E}(H)$ which contains both x, y and the vertex $E \in A$ is adjacent to both x, y. By Lemma 1.1 the set A is an *sd−set* in G. □

Theorem 3.2. *Let G be a bipartite graph with bipartition classes A, B and let A be an sd−set in G. Let $|B| = r$, where r is a positive integer. Then $G \in \Omega(r)$.*

Proof. Consider a multihypergraph H such that $V(H) = B$. For any $a \in A$ to the set $N(a)$ we assign the edge E with $\mathcal{E}(E) = N(a)$. In this way we obtain a multihypergraph $H \in EC(r)$. For all $a \in A$ we obtain all edges of H and $TG(H) = G$. □

Consider extremal cases of elements of $EC(r)$ which are hypergraphs (without multiple edges). One extremal case is a hypergraph with only one edge E with $\mathcal{E}(E) = V(H)$. To this case a star $K_{1,r}$ corresponds and its center evidently is an *sd−set* (with one element). Another extremal case is K_r itself considered as a hypergraph; its edges are considered as two-element sets of vertices. The corresponding bipartite graph has $|A| = \frac{1}{2}r(r-1)$, $|B| = r$. If in the first case we change a hypergraph for a multihypergraph with k edges $E_1, \ldots, E_k$ such that $\mathcal{E}(E_i) = V(H)$ for $i = 1, \ldots, k$; the corresponding bipartite graph is the complete bipartite graph $K_{k,r}$. This is also a bipartite graph in which both the bipartition classes are *sd−sets*. Other examples are finite geometries.

A finite geometry is an ordered pair $(P, \mathcal{L})$, where P is a set of elements called points and $\mathcal{L}$ is a family of subsets of P which are called lines, with the property that for any two points there exists exactly one line containing them and for any two lines there exists exactly one point common to them. Hence it is a particular case of a hypergraph and we may speak about the graph of a finite geometry which is the total graph of that hypergraph.

Theorem 3.3. *Let G be a graph of a finite geometry $(P, \mathcal{L})$. Then G is a bipartite graph with the bipartition classes P, $\mathcal{L}$ and with the property that both P, $\mathcal{L}$ are independent sd–sets in G.*

Proof. The assertion follows directly from Lemma 1.1. □

The simplest finite geometries are trivial ones. For each positive integer r we have a geometry $(P, \mathcal{L})$, where $P = \{1, 2, \dots, r\}$ and with the lines $\{1, k\}$ for $k = 2, \dots, r$ and $\{2, \dots, r\}$. In geometry and algebra Desarguesian finite geometries are studied; they may be constructed by means of Galois fields. The simplest of them has $P = \{1, 2, 3, 4, 5, 6, 7\}$ and edges $\{1, 2, 3\}$, $\{1, 4, 5\}$, $\{2, 4, 6\}$, $\{3, 5, 6\}$, $\{3, 4, 7\}$, $\{2, 5, 7\}$, $\{1, 6, 7\}$.

At the end we shall mention independent *sd−set*s in bipartite graphs which do not coincide with any bipartition class (evidently an *sd–set* cannot be a proper subset of a bipartition class).

Theorem 3.4. *Let G be a bipartite graph with bipartition classes A, B. Let D be an independent set in G such that $D \neq A$, $D \neq B$, $A_0 = A \cap D \neq \emptyset$, $B_0 = B \cap D \neq \emptyset$. The set D is an sd–set in G if and only if the following conditions are true.:*

(i) *G contains the complete bipartite graph on $A - A_0$, $B - B_0$ as its induced subgraph;*

(ii) *for any two vertices of $A - A_0$ there exists a vertex of B_0 adjacent to both of them;*

(iii) *for any two vertices of $B - B_0$ there exists a vertex of A_0 adjacent to both of them.*

Proof. Let D be an *sd−set* in G. Let $\{u, v\} = X \subseteq V(G) - D$. First suppose $X \subseteq A - A_0$. There exists $Y \subseteq D$ such that $\langle X \cup Y \rangle$ is connected. As u, v are not adjacent, they must be both adjacent to a vertex $y \in Y$. Obviously $y \in B$ and thus $y \in B_0$. We have proved that (ii) holds. The case $X \subseteq B - B_0$ is analogous, hence (iii) holds, too. Now suppose $u \in A - A_0$, $v \in B - B_0$. Again there exists $Y \subseteq D$ such that $\langle X \cup Y \rangle$ is connected. Let $Y_1 = Y \cap A$, $Y_2 = Y \cap B$. The vertex u is adjacent to $\bar{u} \in B_0$ and v is adjacent to $\bar{v} \in A_0$. The vertices $\bar{u}$, $\bar{v}$ are non-adjacent and therefore u, v must be adjacent. This implies the condition (i).

On the other hand, let (i), (ii), (iii) hold. Two vertices of $V(G) - D$ may be non-adjacent if and only if either are both in $A - A_0$ or they are both in $B - B_0$. The conditions (ii) and (iii) and Lemma 1.1 imply that D is an *sd–set* in G. □

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